# ACCURATE CALCULATIONS OF THE ELECTROSTATIC RESISTANCE AND CAPACITANCE OF GROUNDING CYLINDRICAL CONDUCTOR

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ABSTRACT

The moment method, which has been used in antenna impedance calculations, is applied to calculate the resistance or capacitance of a cylindrical conductor driven in a two-layer earth. The conventional point matching moment method (PM-MM) and the Galerkin's moment method (G-MM) for resistance calculations are explained. The error reduction property of the G-MM is verified analytically as well as numerically.

*Keywords: Electrostatic resistance, Capacitance, Moment method.*

## 1. INTRODUCTION

The Galerkin's moment method is a very powerful method which requires few segments but still gives accurate answers to the integrated results such as resistance and capacitance in electrostatics. The reason for this is that it has a built in error reduction property in the formulation of the matrix elements of the moment method.

An error reduction technique was actually used long time ago by Sunde [1] through averaging the voltage over a long segment. Sunde did not know, however, that the formulation has a universal error reduction property. *As a result his formulation has not made a significant impact.* In previous papers [2-7], we have used this property to analyze the grounding resistance and surface potentials of different grounding structures.

In this paper, an explanation for the error reduction property of the Galerkin's moment method, a formulation, a mathematical proof, and a numerical verification are introduced. The explanation starts with the formulation of the problem and then points out the difference between the PM-MM and the voltage averaging of G-MM. Followed by a proof of the error reduction property and ends with numerical verifications by applying the G-MM to a single driven rod in a two-layer earth. The moment method (MM) formulation is summarized from [8]. The variational (error reduction) aspect of the G-MM is summarized from [9].

## 2. FORMULATION OF THE MOMENT METHOD

The moment method [8] is a numerical procedure to solve a general inhomogeneous equation of the form

$$g = L(f) \tag{1}$$

where L is a linear operator, g is a known function and f is the unknown function to be determined. This can readily be identified with the integral equation used for a grounding electrode of surface area S, that is,

$$V = \int_S \frac{\rho \overline{J_s}}{4\pi r} \cdot \overline{dS} \tag{2}$$

where V is the voltage everywhere and is known on S. V corresponds to the unknown function g of (1), $J_S$ is the vector current density over the electrode surface S corresponding to the unknown function f, and the reset of the expression of (2) is the linear operator L where $\rho$ is the medium resistivity (Ω.m).

For capacitance calculations, the charge density and the dielectric constant are used instead of $J_S$ and $1/\rho$, respectively.

In general, the MM defines

$$f = \sum_{n=1}^{N} \alpha_n f_n \quad (3)$$

where $f_n$ are a set of N *chosen basis functions* that, with proper coefficients $\alpha_n$, closely represents f. The coefficients $\alpha_n$ at this point are unknowns. Substitution (3) into (1) using the linearity of L, we get

$$g = \sum_{n=1}^{N} \alpha_n L(f_n) \quad (4)$$

To proceed further with (4), we give two more definitions. The first is the inner product <f,g>. For the integration (2), the inner product for the problem under consideration is identified as:

$$<f,g> = \int_S \frac{\rho}{4\pi r} f g dS = \int_S \frac{\rho}{4\pi r} V \overline{J_S} \cdot \overline{dS} \quad (5)$$

The second is a set of N weighing functions $W_m$. The choice of $W_m$ results in different types of MM, e.g. point matching MM, Galerkin's MM, etc. Details of the choice are to be discussed later. With $W_m$ chosen, we have the following inner product,

$$<W_m,g> = \sum_{n=1}^{N} \alpha_n <W_m,Lf_n> \quad (6)$$

with m = 1 ....... N. We can thus form a matrix equation

$$[g_m] = [l_{mn}] [\alpha_n] \quad (7)$$

The current on the segment n is given by

$$I_n = \frac{1}{\Delta \ell_n} \int_S \overline{J_s} \cdot \overline{dS} \quad (8)$$

## 3. PM-MM AND G-MM FOR A SINGLE DRIVEN ROD IN TWO-LAYER EARTH

Figure (1) shows a driven rod in a homogeneous earth and its single image divided into N segments.

For two-layer analysis, Figure (2) shows the multiple images, in this case, for a segment n in the upper layer with current of $I_n$. The field point is considered in the upper layer. Figure (3) shows the images and reflections for a source in the lower layer and the field point in the lower layer as well. Figure 4 shows the images and reflections for a source in the lower layer while the field point is in the upper layer.

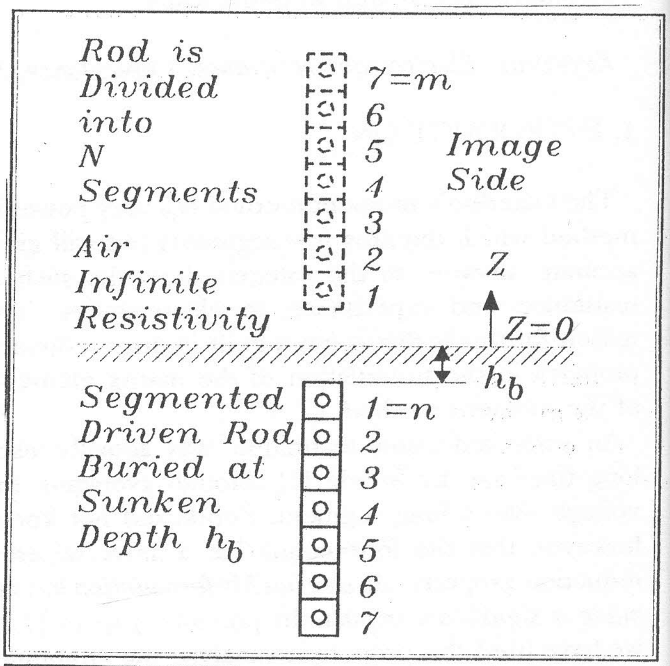


Figure 1. A single driven rod into a homogenous earth and its images divided into N segments.

In these figures we used K to represent the reflection factor which is defined as

$$K = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \quad (9)$$

where $\rho_1$ is the resistivity of the upper layer (Ω.m) and $\rho_2$ is the resistivity of the lower layer (Ω.m).

The matrix equation (7) of the MM relating the known $[V_m]$ to the unknown current on the electrode. The convergence of the solution of the matrix equation depends on the choice of the weight function $W_m$. Also the choice of both $W_m$ and the basis function $f_n$ depends on the geometry of the electrode to be solved. We used a simple geometry of a single conductive rod of radius a, length $\ell$ in a homogeneous and two-layer lossy earth of resistivities $\rho_1$ and $\rho_2$, respectively.

We choose step pulse basis function for the uniform current over the segment $\Delta\ell_n$ i.e.

$$f_n = 1 \text{ on } \Delta\ell_n$$

$$f_n = 0 \text{ on } \Delta\ell_m , \quad m \neq n \tag{10}$$

For the usual PM-MM, we choose the weighing functions;

$$W_m = \delta(z - z_m) \tag{11}$$

where $z_m$ is the mid point of $\Delta\ell_m$. This results in the matrix (7) with $I_n$ the current considered uniform over the nth segment and

$$R = \frac{1}{I^2}\int_\ell V_o I_\ell(z)\,dz = \frac{V_o}{I^2}\int_\ell J_\ell(z)\,dz = \frac{V_o}{I} \tag{12}$$

$$R_{mn} = \frac{\rho}{2\pi\Delta\ell_n}\ln\left(\frac{\Delta\ell_n}{a}\right) \tag{13}$$

This usual PM-MM is well known and gives slow convergence. With G-MM (the weighing functions are equal to the basis functions), i.e.

$$W_m = f_m \tag{14}$$

This results in

$$V_m = \frac{1}{\Delta\ell_m}\int_{\Delta\ell_m} V(z)\,dz \tag{15}$$

Eqn. (15) is the *average* of the voltage over the segment $\Delta\ell_m$.

Such averaging reduces the error in the voltage when compared to the PM-MM. With step pulse basis functions as well as the weighing functions of (11), Eqn. (15) leads to

$$R_{mn} = \frac{\rho}{4\pi\Delta\ell_n\Delta\ell_m}\int_{\Delta\ell_m}\left[\int_{\Delta\ell_n}\frac{dz'}{\sqrt{(z-z')^2+a^2}}\right]dz \tag{16}$$

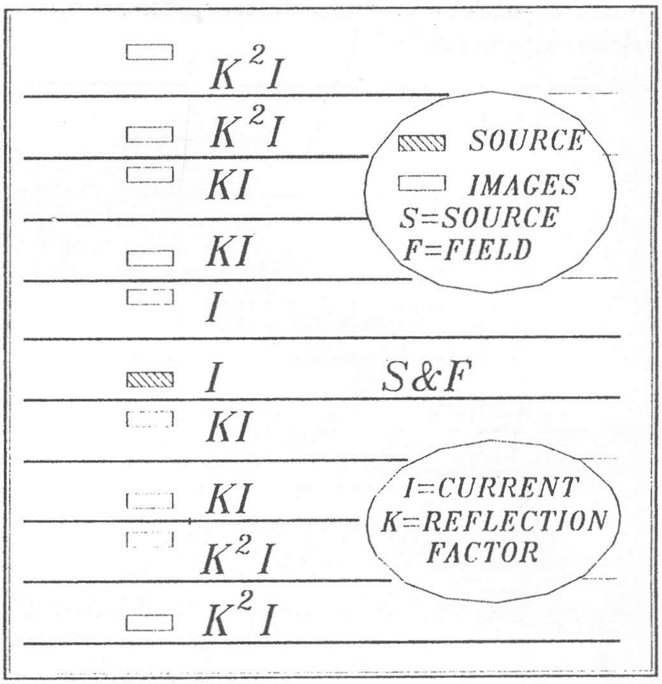


**Figure 2.** The source and field points in the upper layer.

For all m and n including the self term of m = n. The above $R_{mn}$ formula is analytically integrable and results are discussed in previous papers [2-7]. Eqn.(16) is known as the *variational* form.

## 4. PROOF OF THE ERROR REDUCTION PROPERTY OF THE G-MM

The essence of the error reduction property is in the averaging of the potential over the whole segment even if the segment covers the whole conductor. This is due to the fact that the total current on the conductor is the same as the exact one, however, the resulted *current distribution* on the conductor differs from the exact one specially at both ends of the conductor. Since we are calculating the conductor resistance and the value of the resistance depends on the conductor total current and does not depend on its distribution, the error in calculating the resistance is very small. This will not be valid in the case of calculating the surface potential because it does depend on the current distribution. We have discussed a novel technique for satisfying the boundary conditions at the conductor ends so that the surface potential can be calculated accurately [7].

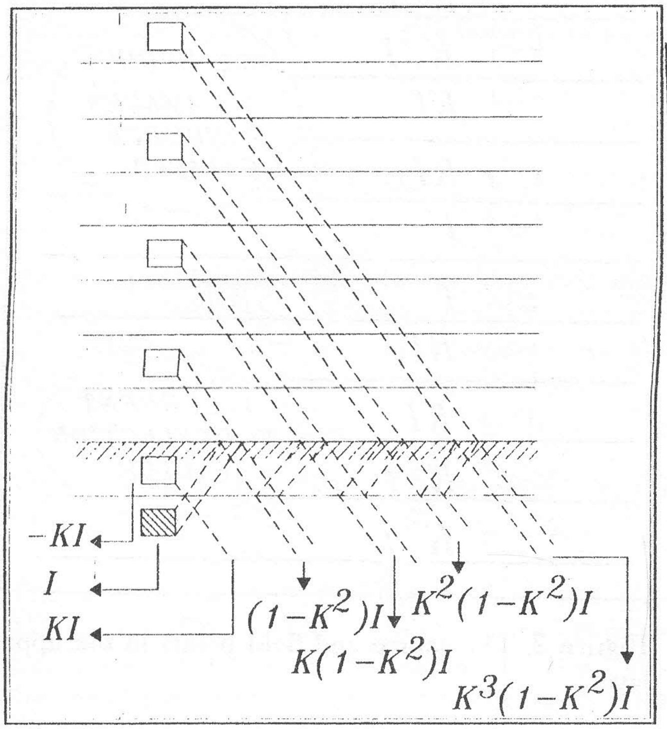


Figure 3. Source and field points in the lower layer.

Such desirable property (i.e. error reduction property) has been considered in various texts [9]. We shall apply it to the grounding resistance. Without the approximation of uniform current in the one segment over the whole conductor, Eqn. (16) becomes,

$$R = \frac{\rho}{4\pi I^2}\int_\ell\left[\int_\ell G(z-z')\, J_\ell(z)J_\ell(z')dz'\right]dz \qquad (17)$$

where

$$G(z-z') = \frac{1}{\sqrt{(z-z')^2 + a^2}} \qquad (18)$$

and

$$I = \int_\ell J_\ell(z)\, dz \qquad (19)$$

If $J_\ell$ is exactly known, Eqn. (17) can be written as he following two equations

$$V(z) = \frac{\rho}{4\pi}\int_\ell G(z-z')J_\ell(z')dz' \qquad (20)$$

where $V(z) = V_o$ a constant over the rod $\ell$, and

$$R = \frac{1}{I^2}\int_\ell V_o J_\ell(z)dz = \frac{V_o}{I^2}\int_\ell J_\ell(z)dz = \frac{V_o}{I} \qquad (21)$$

Now assume $J_\ell$ is in error to become $J_\ell + \delta J_\ell$, there should be an error $\delta R$ added to R in (17). The relationship between $\delta J_\ell$ and $\delta R$ can be obtained by applying a differential to both sides of (17)

$$\delta\left[\left(\int_\ell J_\ell(z)dz\right)^2 R\right] = \frac{\rho}{4\pi}\delta\left(\int_\ell\left[\int_\ell G(z-z')J_\ell(z)\, J_\ell(z')\, dz'\right] dz\right) \qquad (22)$$

Because of the symmetry of the function G(z-z'), the above equation becomes

$$\left[\int_\ell J_\ell(z)dz\right]^2 \delta R + 2R\int_\ell J_\ell(z)dz\int_\ell J_\ell(z')dz' = \frac{\rho}{4\pi}\left[2\int_\ell\int_\ell G(z-z')J_\ell(z')\delta J_\ell dz dz'\right] \qquad (23)$$

with (19) and (21), this gives,

$$I^2\delta R = 2\int_\ell \delta J_\ell \left[ -V_o + \frac{\rho}{4\pi}\int_\ell G J_\ell(z') dz' \right] dz \qquad (24)$$

Because of (20) the bracket in the integrand of (24) is zero and the finite integral of (24) gives

$$I^2\ \delta R = 0, \text{ i.e. } \delta R = 0 \qquad (25)$$

Eqn. (25) showed that for a first order error $\delta J_\ell$ in the current distribution, the first order error in the resistance $\delta R$ of one segment rod is zero. This means that only the second order error remains. Numerically it means that for a 20% error in the current distribution, the second order error in R was only 4%.

## 5. RESULTS AND DISCUSSION

It is clear that, if N, the number of segments, is large, Eqns. (16), (12), and (13) approach each other and there is no difference in accuracy of the G-MM and PM-MM versions. If N is small, without averaging of (15), the point matching results of resistance deteriorate badly and cannot be used. With averaging, the G-MM results remain quite accurate, with a resistance error less than 3%. This means that the whole rod can be represented by one segment.

Although the exact current distribution on the rod has singularities (one at each conductor tip for sunken depth ≠ 0 and only one singularity for sunken depth = 0) while the assumed current distribution is a uniform step pulse (the whole rod is taken as one segment), the numerical calculations showed that resistance error is less than 3%. When three segments are used over the rod, two small ones at the two ends and a bigger one in the middle, the resistance error is less than (3/9)%.

To demonstrate the error reduction property presented in this paper, a single rod of radius 0.01m and length 10 m is buried in two-layer earth with the upper layer resistivity of 100 Ω.m. The resistance of the rod is calculated for different values of the sunken depth, the reflection factor, and the height of the upper layer. The calculations of the rod resistance for each set of these parameters are repeated for two cases : N = 1 and N = 20. Table I contains the results of such calculations. It is clear that the maximum percentage difference in the resistance values between N = 1 and N = 20 is less than 1% for any set of rod parameters.

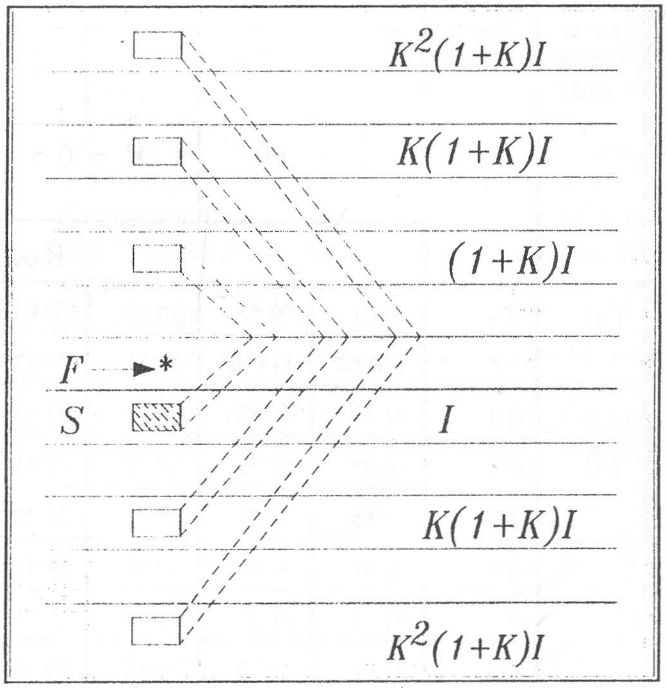


**Figure 4.** Source in the lower layer and the field point in the upper layer.

## 6. CONCLUSION CONCERNING THE CALCULATIONS OF THE RESISTANCE

The Galerkin's moment method is compared with the point matching moment method. Applying PM-MM will achieve a high accuracy in resistance calculation, however it needs a large computer resources and is time consuming. This is due to the large number of segments needed to achieve adequate accuracy. Using the G-MM one can take the whole rod as one segment if buried into a homogeneous earth and two segments if buried in two-layer earth. This makes the computation time short which will help in a preliminary quick design. However, calculating the surface potential needs more than one segment per rod. It was found [7] that three segments is enough to achieve accurate values of the surface potential.

Table I. Resistance calculated for the whole rod as one segment and as 20 equal segments per rod. Rod length = 10 m, rod radius = 0.01 m, and upper layer resistivity = 100 Ω.m.

| Upper Layer Height (m) | Sunken Depth (m) | Number of Segments per Rod | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 20 | 1 | 20 | 1 | 20 | 1 | 20 | 1 | 20 |
| | | K = 0.9 | | K = 0.5 | | K = 0.0 | | K = - 0.5 | | K = - 0.9 | |
| | | Rod Resistance Values (Ω) | | | | | | | | | |
| 1.0 | 0.0 | 99.62 | 99.62 | 30.18 | 30.85 | 11.55 | 11.56 | 3.99 | 4.11 | 0.66 | 0.66 |
| | 0.5 | 116.32 | 115.72 | 30.54 | 30.31 | 11.35 | 11.35 | 3.85 | 3.97 | 0.63 | 0.63 |
| 2.0 | 0.0 | 62.58 | 62.47 | 26.77 | 27.02 | | | 4.29 | 4.36 | 0.73 | 0.73 |
| | 0.5 | 68.29 | 67.94 | 27.19 | 27.45 | Same | | 4.12 | 4.20 | 0.69 | 0.69 |
| | 1.0 | 79.55 | 79.03 | 28.10 | 28.45 | 11.23 | 11.24 | 3.98 | 4.06 | 0.66 | 0.66 |
| 5.0 | 0.0 | 29.99 | 29.97 | 19.49 | 19.47 | | | 5.41 | 5.42 | 1.06 | 1.05 |
| | 0.5 | 31.14 | 31.07 | 19.78 | 19.75 | Same | | 5.15 | 5.16 | 0.98 | 0.98 |
| | 1.0 | 32.91 | 32.79 | 20.32 | 20.28 | | | 4.94 | 4.94 | 0.92 | 0.91 |
| | 2.0 | 38.20 | 38.00 | 21.78 | 21.74 | 11.07 | 11.10 | 4.56 | 4.58 | 0.81 | 0.81 |
| | 2.5 | 42.12 | 41.88 | 22.69 | 22.66 | 11.00 | 11.05 | 4.40 | 4.42 | 0.77 | 0.76 |
| | 3.0 | 47.48 | 47.27 | 23.74 | 23.74 | .10.95 | 11.01 | 4.25 | 4.27 | 0.73 | 0.73 |
| | 4.0 | 67.52 | 66.96 | 26.40 | 26.48 | 10.85 | 10.95 | 3.98 | 4.00 | 0.66 | 0.66 |
| 8.0 | 0.0 | 19.96 | 19.96 | 15.17 | 15.16 | | | 7.41 | 7.38 | 2.04 | 2.02 |
| | 0.5 | 20.44 | 20.42 | 15.37 | 15.34 | | | 6.90 | 6.87 | 1.75 | 1.73 |
| | 1.0 | 21.44 | 21.10 | 15.72 | 15.68 | Same | | 6.49 | 6.46 | 1.54 | 1.53 |
| | 2.0 | 22.98 | 22.91 | 16.61 | 16.55 | | | 5.84 | 5.82 | 1.25 | 1.24 |
| | 3.0 | 25.54 | 25.43 | 17.73 | 17.65 | | | 5.33 | 5.31 | 1.05 | 1.05 |
| | 4.0 | 29.19 | 29.04 | 19.01 | 19.02 | | | 4.90 | 4.89 | 0.91 | 0.91 |
| | 5.0 | 34.72 | 34.51 | 20.79 | 20.72 | 10.88 | 10.87 | 4.55 | 4.54 | 0.81 | 0.81 |
| | 6.0 | 44.14 | 43.81 | 22.95 | 22.89 | 10.83 | 10.83 | 4.24 | 4.24 | 0.73 | 0.73 |
| | 7.0 | 64.25 | 63.67 | 25.82 | 25.80 | 10.80 | 10.80 | 3.97 | 3.98 | 0.66 | 0.66 |

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