# ELASTO-PLASTIC ANALYSIS OF FRAMES

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ABSTRACT

A simple technique for elasto-plastic analysis of plane frames is presented, in which the effects of change of geometry and large deformations on collapse loads and behavior of frames are included. The effect of axial force on plastic strength of cross-sections is also included. Numerical examples are solved and compared with published theoretical and experimental results.

*Keywords: Elasto-Plastic, Plane frame, Normal force, Change of Geometry, Large deformation, Collapse load, Stiffness matrix, Incremental load procedure, Secant iteration method, Modified Newton-Raphson method.*

*Notations*

| | |
|---|---|
| A | Cross section Area. |
| $a_o, a_1$ | Constants |
| $b_o, b_1, b_2, b_3$ | Constants |
| E | Modulus of Elasticity |
| F | Forces in frame axes |
| I | Second moment of Area (moment of Inertia) |
| i,j | Subscripts. |
| K | Stiffness matrix |
| L | Member length |
| M | Bending moment |
| P | Force in local member axes. |
| U,V | Member axes |
| u,v | Longitudinal and lateral displacements in member axes. |
| $\bar{U}$ | Strain Energy |
| X,Y | Frame axes |
| x,y | Horizontal and Vertical displacements in frame axes |
| $\bar{y}$ | Distance |
| $\epsilon, \epsilon^o, \epsilon^*$ | Strains |
| $\theta, \theta', \rho$ | rotations |

## 1. INTRODUCTION

Linear plastic analysis is well known and established [Refs. 4,6], in which the effects of axial forces and change of geometry are ignored. Including some or all of the above effects make the analysis nonlinear. In nonlinear plastic analysis, two approaches are available in the literature. In the first, which is termed plastic zone analysis, the members as well as the cross-sections are divided into submembers and fibers respectively [Ref. 7]. Although plastic zone analysis can predict accurately the inelastic behavior of systems but it is time consuming and computationally intensive since a fine discretization is needed through-out the cross-section and along the member length. Therefore, it is only attractive for simple structure and geometry. In the second approach, which is termed plastic hinge analysis, the inelastic behavior is contained within zero length plastic hinges, the hinges are concentrated at member ends and the member remains elastic between its ends, [Refs. 1,2,3,5,7,8,11,14]. Also, the sections are assumed elastic until the full cross-section plastic strength is achieved neglecting the strain hardening. Although plastic hinge analysis approximates the inelastic behavior, but it is practical and gives reasonable results.

In the present work, a plastic hinge analysis is employed with its assumptions above. The effects of axial forces on plastic strength of the cross-sections and on lateral stiffness of members are included in the analysis. A simple technique is incorporated into a computer program to carry-out the plastic analysis described later. The program can be run in different modes to show the above effects individually or altogether.

## 2. METHOD OF ANALYSIS

Consider a uniform member ij (Figure (1a)), with cross-sectional area A, Length L, and flexural stiffness EI.

The total strain, $\epsilon$ , can be written as:

$$\epsilon = \epsilon^o + \epsilon^a \quad (1)$$

where

$\epsilon^o$ is the initial strain

and $\epsilon^a$ is additional strain developed, which can be expressed as

$$\epsilon^a = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 - \bar{y}\frac{d^2v}{dx^2} \quad (2)$$

where u and v are the longitudinal and lateral displacements respectively, which can be expressed as:

$$u(x) = a_o + a_1 x \quad (3\text{-a})$$

$$v(x) = b_o + b_1 x + b_2 x^2 + b_3 x^3 \quad (3\text{-b})$$

where

$a_o, a_1, b_1, b_2, b_3$ are constants.

The total strain energy, $\bar{U}$, may be expressed as:

$$\bar{U} = \frac{1}{2}\int_v \sigma\epsilon\, dv = \frac{1}{2}\iiint \sigma\epsilon \; dx\; dy\; dz$$

$$= \frac{1}{2}\int_0^L AE\epsilon^2 dx \quad (4)$$

Substituting from eqs. 1,2, and 3 into eq.4 and performing the integration give the governing stiffness matrix k as

$$K = \begin{vmatrix} \frac{AE}{L} & & & & & \\ 0 & \frac{12EI}{L^3}+\frac{6P_x}{5L} & & & \text{Sym} & \\ 0 & \frac{6EI}{L^2}+\frac{P_x}{10} & \frac{4EI}{L}+\frac{2P_xL}{15} & & & \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & & \\ 0 & \frac{-12EI}{L^3}\frac{-6P_x}{5L} & \frac{-6EL}{L^2}-\frac{P_x}{10} & 0 & -\frac{12EI}{L^3}+\frac{6P_x}{5L} & \\ 0 & \frac{6EI}{L^2}+\frac{P_x}{10} & \frac{2EI}{L}-\frac{P_xL}{30} & 0 & -\frac{6EI}{L^2}-\frac{P_x}{10} & \frac{4EI}{L}+\frac{2P_xL}{15} \end{vmatrix} \quad (5)$$

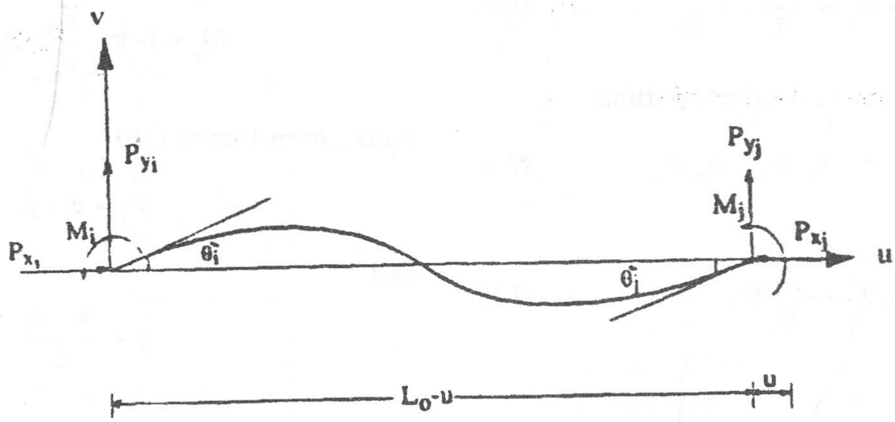



**Figure 1a.** Element deformation and forces in local coordinate (Euler formulation).

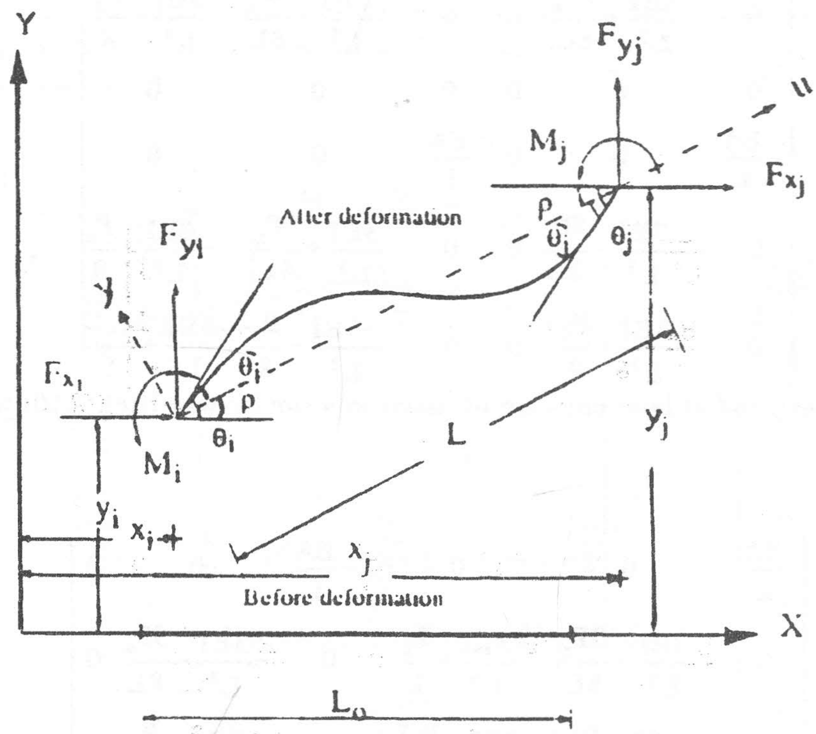



**Figure 1b.** Member in global coordinate system.

in which, $p_x$ is the axial force in the members.

The equilibrium equations for the two degrees of freedom member ij (Figure (1-a)) is written as:

$$\begin{bmatrix} M_i \\ M_j \end{bmatrix} = \begin{bmatrix} k_{ii} & k_{ij} \\ K_{ji} & k_{jj} \end{bmatrix} \begin{bmatrix} \theta'_i \\ \theta'_j \end{bmatrix} \qquad \text{(6-a)}$$

in which

$$k_{ii} = k_{jj} = \frac{4EI}{L} + \frac{2P_x L}{15} \qquad \text{(6-b)}$$

and

$$k_{ij} = k_{ji} = \frac{2EI}{L} - \frac{P_x L}{30} \qquad (6\text{-c})$$

if a hinge is formed at end i then, putting

$$M_i = 0 = k_{ii}\,\theta'_i + k_{ij}\,\theta'_j \qquad (7\text{-a})$$

then

$$\theta'_i = -\frac{k_{ij}}{K_{ii}}\theta'_j \qquad (7\text{-b})$$

and

$$M_j = (-\,k_{ji}\,\frac{K_{ij}}{k_{ii}} + k_{jj})\,\theta'_j \qquad (7\text{-c})$$

Since, from Figure (1b)

$$\theta'_j = \theta_j - \rho \qquad (7\text{-d})$$

and

$$\rho = \frac{v_j - v_i}{L} \qquad (7\text{-e})$$

Expanding the matrix for hinge at end i gives:

$$\begin{vmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{3EI}{L^3}+\frac{9P_x}{8L} & 0 & 0 & -\frac{2EI}{L^3}-\frac{9P_x}{8L} & \frac{3EI}{L^2}+\frac{P_x}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{3EI}{L^3}-\frac{9P_x}{8L} & 0 & 0 & \frac{3EI}{L^3}+\frac{9P_x}{8L} & -\frac{3EI}{L^2}-\frac{P_x}{8} \\ 0 & \frac{3EI}{L^2}+\frac{P_x}{8} & 0 & 0 & \frac{-3EI}{L^2}-\frac{P_x}{8} & \frac{3EI}{L}+\frac{P_x L}{8} \end{vmatrix} \qquad (8\text{-a})$$

Matrices for hinges at end j and at both ends are obtained in a similar way [Ref. 1,10] which are:

<u>For hinge at end j</u>

$$\begin{vmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{3EI}{L^3}+\frac{9P_x}{8L} & \frac{3EI}{L^2}+\frac{P_x}{8} & 0 & -\frac{3EI}{L^3}-\frac{9P_x}{8L} & 0 \\ 0 & \frac{3EI}{L^2}+\frac{P_x}{8} & \frac{3EI}{L}+\frac{P_x L}{8} & 0 & -\frac{3EI}{L^2}-\frac{P_x}{8} & 0 \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{3EI}{L^3}-\frac{9P_x}{8L} & -\frac{3EI}{L^2}-\frac{P_x}{8} & 0 & \frac{3EI}{L^3}+\frac{9P_x}{8L} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \qquad (8\text{-b})$$

For hinges at both ends

$$\begin{vmatrix} \frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & \frac{P_x}{L} & 0 & 0 & -\frac{P_x}{L} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{P_x}{L} & 0 & 0 & \frac{P_x}{L} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \qquad (8\text{-c})$$

The equilibrium equations of the structure are then established, and the boundary conditions are introduced. Solution of the equilibrium equations gives the displacements vector of the structure and the stresses are then calculated.

The effect of axial force on plastic moment (full strength) of section is explained and given in [Refs. 1, 12].

## 4- Numerical Examples

### *Example 1*

The frame shown in Figure (2a) with its geometry, properties of cross-section, and loadings. linear analysis (ISL=1) gives collapse load 0.6133 kips while large deformation* analysis (ISL=4) gives collapse load 0.545 kips with difference about 11%. The load deflection curves for different solutions are shown in Figure (2.b) and the corresponding loads are tabulated in table (1) later. Also, the failure mechanism and sequence of hinges formation are shown in the Figure. The complete solution (ISL =4) gives load deflection curve and collapse load very close to the experimental and theoretical values given in [Refs. 5,11].

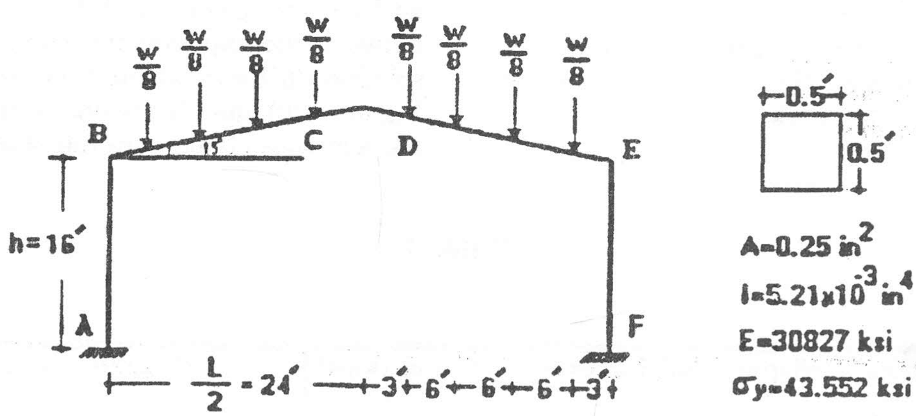



Figure 2a. Loading and dimensions.

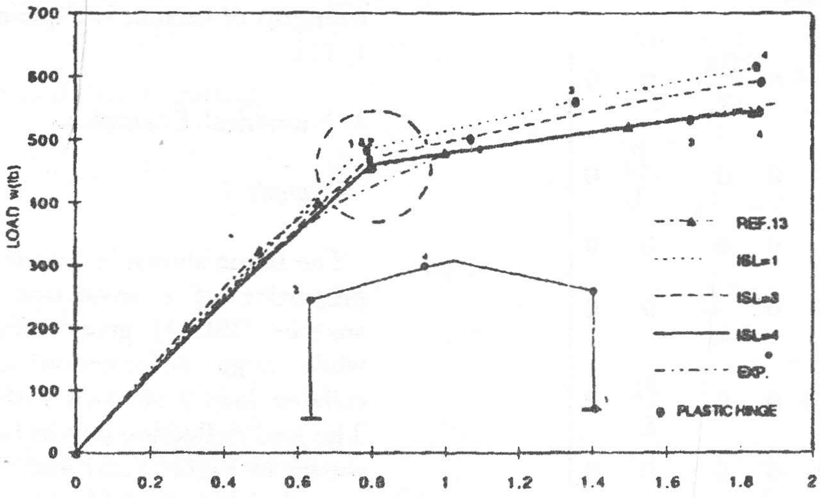



Figure 2b. Load-Deflection curves for Example (1).

*Example 2*

The frame with its loading and geometry is shown in Figure (3a).

The experimental results given in Ref [13] are compared with results obtained by the various analysis adopted in this work.

Figure (3b) shows the load-deflection curves. The numerical values for collapse load of different analysis are given in Table (1). The difference between the experimental curve and the complete solution (ISL=4) adopted in the zone circled on Figures (2b) and (3b) is due to the assumption that the section is elastic until full plastic moment occurs.

Table 1.

| Example | Plastic Collapse Load Under the action of (kips) | | | | % Reduction due to | | | |
|---|---|---|---|---|---|---|---|---|
| | ISL=1 | ISL=2 | ISL=3 | ISL=4 | ISL=2 | ISL=3 | ISL=4 | $\%P_x/P^o$ |
| 1 | 0.61332 | 0.613 | 0.5933 | 0.54548 | 0.05 | 3.269 | 11.06 | 2.8 |
| 2 | 16.6412 | 16.349 | 16.226 | 15.866 | 1.753 | 2.515 | 4.6547 | 13.65 |

ISL = 1 Linear Analysis
ISL = 2 effect of axial force only
ISL = 3 effect of axial force and change of geometry
ISL = 4 Complete solution (axial force + change of geometry + Large deflection)
$P^o$ Squash load = A $\sigma_y$

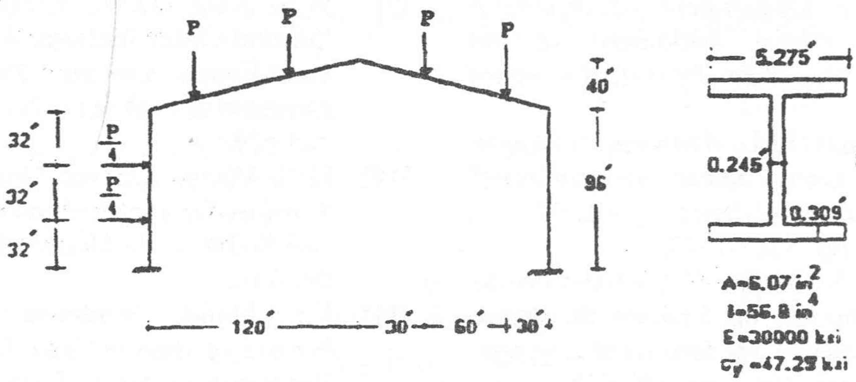
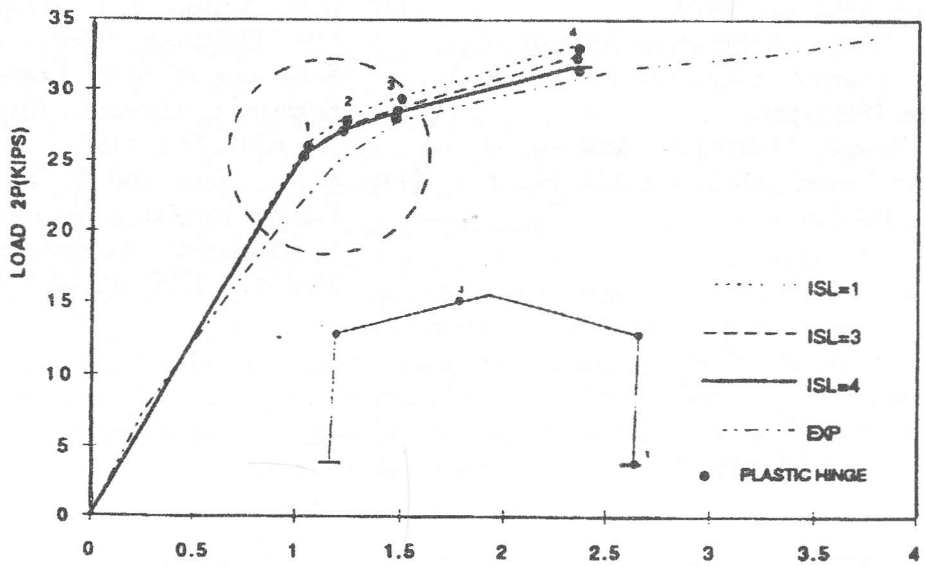



Figure 3a. Loading and dimensions.




Vertical Deflection At Ridge(in)
Vertical deflection at Ridge(in)

Figure 3b. Load-Deflection curves for example (2).

## 5. CONCLUSIONS

In the examples presented, the effect of axial force on full strength of members should be included in the analysis for member carrying heavy axial loads, also, the effect of large deformation should be included for slender structures. The inclusion of large deformation effect as well as the effect of axial force on full strength of section give realistic expression for collapse load and behaviour of structures, and they are very close to experimental results.

## 6. REFERENCE

[1] F.A.A. Al-Doubai, "Elasto-Plastic Analysis of Frames" A thesis submitted to the Department of Structural Engineering in Partial Fulfillment of the requirements for the *M.Sc Degree Alexandria University*, 1995.

[2] M.Z. Cohn and Tariq Rafay. "Collapse Load Analysis of Frames Considering Axial Forces", *Journal of the Engineering Mechanics Division*, vol, 100, EM 4, August, 1974, pp. 772-794.

[3] H. El-Houshi, "Plastic Analysis" A Thesis

submitted to the Department of Structural Engineering in Partial fulfillment of the requirements for the *M.Sc. Degree, Alexandria University*, 1979.

[4] D.E. Greirson and M.L. Graham, "Collapse Load Analysis Using linear Programming" *Journal of the Structural Division*, vol. 97, No. ST5, May, 1971, pp. 1561-1573.

[5] A. Jennings and K. Majid, "An Elastic-Plastic Analysis by Computer for Framed Structures Loaded Up to Cpllaps" *the Structural Engineer*, vol. 43, No. 12. Dec. 1965, pp. 407-412.

[6] H.G. Juhl, "Deformation of Structures at Ultimate Load" *J. Struct. Div, ASCE.*, vol. 88, No. ST 4, Aug. 1962, pp. 69-79.

[7] A. Kassimali, "Large Deformation Analysis of Elastic-Plastic Frames", *ASCE*. vol. 109, No. 8, Aug. 1983, pp. 1869-1886.

[8] T.Y. Kam "Large Deflection Analysis of Inelastic Plane Frame" *ASCE*, vol. 114, No. 1, Jan. 1988, pp. 184-197.

[9] W.S. King, D.W. White and W.F. Chen "Second-Order Inelastic Analysis Methods for Steel-Frame Design" *Journal of Structural Engineering*, vol. 118, No. 2, Feb. 1992, pp. 408-428.

[10] H.C. Martin "On the Derivation of Stiffness Matrices for the Analysis of Large Deflection and Stability Problems" *AFFDLTR*-66-80, pp. 697-716.

[11] K.I. Majid, Nonlinear Structures *Matrix Method of Analysis and Design by Computer*, University of Aston, Birmingham, 1972.

[12] B.G. Neal, *The Plastic Methods of Structural Analysis*, Chapman and Hall, London, 1963..

[13] H.M. Nelson, D.T. Wright, J.M. Asce and J.W. Dolphine "Demonstration of Plastic Behaviour of Steel Frames" *Journal of the Engineering Mechanics Division ASCE*, vol. 83, No. EM4, Oct. 1957.

[14] M.B. Wong and F. Tin-Loi, "Analysis of Frames Involving Geometrical and material Nonlinearities", *Computers and Structure*, vol. 34, No. 4, 1990, pp. 641-646.