NONLINEAR SUPERCRITICAL FREE SURFACE FLOW OVER TWO IDENTICAL TRIANGULAR OBSTACLES

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ABSTRACT

A free-surface flow past submerged two identical triangular obstacles at the bottom of a channel is considered. The flow is assumed to be steady, two-dimensional and irrotational; the fluid is treated as inviscid and incompressible. The problem is solved numerically by series truncation. It is shown that there are solutions for which the flow is supercritical both upstream and downstream. The results are plotted for different triangle heights, spacings of the obstacles and different values of the Froude number F>1. The effect of Froude number, the bottom height and the shape of the triangle on the free surface is discussed.

Keywords: Inviscid flow, Schwartz-Christoffel transformation, series truncation method.

1. INTRODUCTION

Fluid flow over various topographies has attracted considerable attention throughout the history of fluid mechanics.

Lamb [18] calculated the flow past a submerged semi-elliptical obstacle by an approximate linear theory. He obtained solutions with a train of waves downstream for subcritical flow and solutions without waves for supercritical flow. Recently Forbes and Schwartz [9] and Forbes [8] solved the corresponding exact problem numerically. Their results confirm and extend Lamb's solutions. Also we may mention in particular the work of Kreisel [17], Benjamin [4], Newman [20], Madison and Mei [19], Johnson [13], Hamilton [11], Smith and Lim [23], Boutros, Abd-el-Malek and Hanna [6], Abd-el-Malek, and Hanna [2], King and Bloor [15] and Abd-el-Malek, Hanna and Kamel [3].

For subcritical flows, there have been the papers of Salvesen and Von Kerczek [21,22], Korving and Hermans [16] and Dias and Vanden-Broeck [7].

For critical flows, there have been the papers of Forbes [10] and Hanna [12].

In this paper we calculate the flow past a submerged two triangular obstacles by a series truncation procedure. This technique has been used successfully by Birkhoff and Zarantonello [5], Vanden-Broeck and Keller [24], Dias and Vanden-Broeck [7] and Hanna [12] to calculate nonlinear free-surface flows.

Solutions including waves downstream are not considered in this paper. Therefore, we assume that the flow approaches a uniform stream with constant velocity and constant depth far downstream.

The type of solutions for which the flow is supercritical both upstream and downstream is referred to as " supercritical flow ".

In this paper we numerically solve the case of a nonlinear supercritical flow of an ideal fluid over two identical triangular obstacles by specifying the uniform flow upstream and downstream.

In section 2, we formulate the problem. Solutions for the two identical triangular obstacles are presented in section 3. The results are presented and discussed in section 4.

2. *Formulation of the problem*

Consider a steady, two-dimensional, incompressible and irrotational flow of an ideal fluid over two identical triangular obstacles having 6 corners and 5 straight segments, placed at the bottom of an open channel at a distance 2L apart, Figure (1a).

The channel is assumed to be of constant width and depth in both directions faraway from the obstacle.

A cartesian coordinate system is selected as follows: the x-axis coincides with the stream bed before and after the obstacle, the origin is placed at the mid point between the two triangles and the y-axis pointing vertically upwards. Fluid flows through the channel in the positive x-direction, with speed U and depth H far upstream.

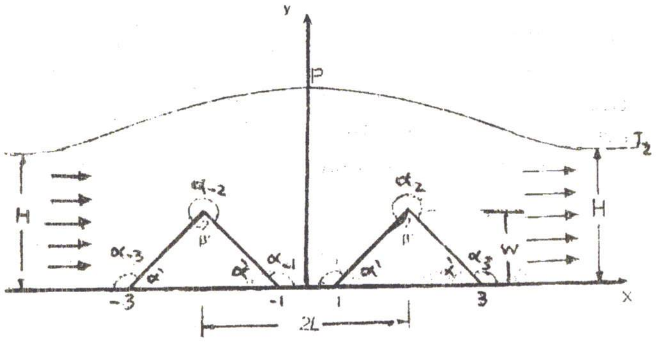

Figure 1-a. Physical plane of the flow and of the coordinates.

Relative to the coordinate axes, the flow is steady and is subject to the acceleration of gravity g in the negative y-direction.

Since the flow is irrotational and the fluid is incompressible, a velocity potential f and a stream function y exist, in terms of which the horizontal and vertical components, u and v, of the fluid velocity vector may be expressed as

$$u = \phi_x = \psi_y ; v = \phi_y = -\psi_x . \tag{2.1}$$

Equations (2.1) show that a complex potential exists, namely,

$$\chi = \phi + i\psi . \tag{2.2}$$

Since ϕ and ψ are conjugate solutions of Laplace equation, $\chi(z)$ is an analytic function of z within the region of flow, with complex conjugate velocity

$$\xi = \frac{d\chi}{dz} = u(x,y) - iv(x,y) = qe^{-i\theta} \tag{2.3}$$

The dynamic boundary condition on the free surface $\psi = Q = U H$ is

$$\frac{1}{2} |\nabla\phi|^2 + gy = \text{constant} \tag{2.4}$$

We introduce the dimensionless variables by taking $(Q^2/g)^{\frac{1}{3}}$ as the unit length and $(Qg)^{\frac{1}{3}}$ as the unit velocity.

The dimensionless discharge is now equal to one. Hence, the free surface is a streamline on which $\psi=1$. In terms of the dimensionless variables, the condition (2.4) becomes,

$$|\nabla\phi|^2 + 2y = \text{constant on } \psi = 1. \tag{2.5}$$

The complex potential χ maps the flow domain conformally onto an infinite strip of height unity as shown in Figure (1b).

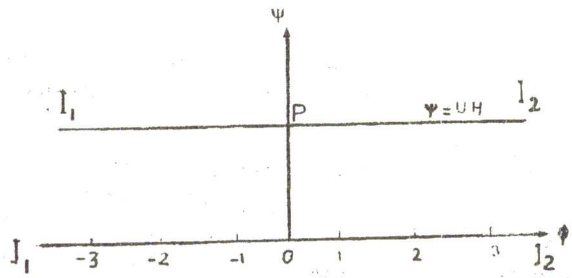

Figure 1-b. The complex potential plane, χ plane.

In dimensionless form,(2.3) becomes

$$\xi = \frac{d\chi}{dz} = qe^{-i\theta} \tag{2.6}$$

Let

$$\omega = \ln \xi = \ln q - i\,\theta . \tag{2.7}$$

where ω is the socalled logarithmic hodograph variable.

Then, from (2.6) and (2,7) we get

$$z = \int e^{-\omega} d\chi \tag{2.8}$$

Using a suitable Schwartz-Christoffel transformation, we map the infinite strip onto the upper half of the unit disk with I_1 and I_2 corresponding to the points -1 and 1, respectively as shown in Figure (1c); the solid boundary goes onto the real diameter and the free surface onto the upper half of the unit circle. The images of the points [-3,-2,-1,1,2,3] are $[-t_3,-t_2,-t_1,t_1,t_2t_3]$ respectively.

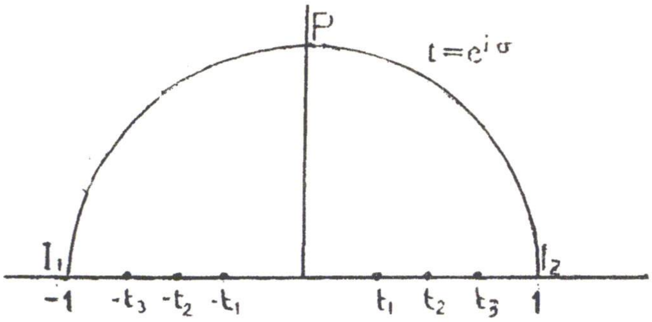

Figure 1-c. The complex t-plane.

The mapping is given by

$$\chi = \frac{2}{\pi}\ln\frac{1+t}{1-t}, \quad |t| \leq 1. \tag{2.9}$$

Our goal is to find ξ as an analytic function of t satisfying the boundary condition

$$|\xi|^2 + 2y = \text{constant on } |t| = 1 \tag{2.10}$$

and the boundary condition on the real diameter $|t| \leq 1$.

3. SOLUTION

Let us consider here the steady flow over a symmetric two- triangular obstacles consisting of a raising-up inclined segment [-3 to -2] of inclination angle α', Figure (1a), a steeping down segment [-2 to -1] of inclination angle β', a horizontal segment [-1 to 1] of the stream bed, a raising-up segment [1 to 2] of inclination angle α' and a steeping down segmen [2 to 3] of inclination angle β'. Since the two triangles are identical, then

$$\alpha_{-3} = \alpha_3 = \pi - \alpha' = \pi\,(1-\alpha);$$

$$\alpha_{-2} = \alpha_2 = 2\pi - \beta' = \pi + 2\alpha' = \pi\,(1+2\alpha);$$

$$\alpha_{-1} = \alpha_1 = \pi - \alpha' = \pi\,(1-\alpha),$$

where

$$\alpha = \frac{\alpha'}{\pi}.$$

The first step is to remove the strong singularities from the complex velocity ξ which occur at the corners of the triangles, namely, at the points

$$t = \pm\, t_i, \qquad i = 1, 2, 3. \tag{3.1}$$

Those singularities are

$$\xi \sim (t \pm t_i)^{1-\frac{\alpha_i}{\pi}} \quad t \to \mp t_i; i = 1,2,3. \tag{3.2}$$

where the angles α_i satisfy the relation

$$\sum_{\substack{i=-3 \\ i\neq 0}}^{3} \alpha_i = 6\pi \quad . \tag{3.3}$$

As $\phi \to +\infty$ the flow approaches a uniform supercritical state. Therefore the asymptotic form of ξ as $\phi \to +\infty$ is obtained by linearizing the equations around a uniform stream.

Following Lamb [18] and Abd-el-Malek and Masoud [1] we solve the resulting equations by separation of variables.

Hence, we get

$$\xi \sim \xi(1) + A'e^{-\lambda x}, \text{ as } \phi \to +\infty,$$

where A′ is a constant and λ is the smallest positive root of the equation

$$F^2\lambda - \tan\lambda = 0, \tag{3.4}$$

F being the Froude number defined by

$$F = \frac{U}{\sqrt{gH}} \quad . \tag{3.5}$$

We now define the function $\Omega(t)$ by the relation

$$\xi(t) = \left(\frac{t-t_3}{1-tt_3}\right)^{\alpha}\left(\frac{t-t_2}{1-tt_2}\right)^{2\alpha-1}\left(\frac{t-t_1}{1-tt_1}\right)^{\alpha}\left(\frac{t+t_1}{1+tt_1}\right)^{\alpha}\left(\frac{t+t_2}{1+tt_2}\right)^{2\alpha-1}\left(\frac{t+t_3}{1+tt_2}\right)^{\alpha} e^{\Omega(t)} \tag{3.6}$$

The function $\Omega(t)$-see Birkhoff and Zarantonello [5]- is analytic and continuous for $|t| \leq 1$.

The boundary condition (2.10) implies that the expansion of $\Omega(t)$ in powers of t has real coefficients. Also since the flow is symmetric with respect to the y-axis only even powers of t should be present in this expansion. Therefore ,the expansion of $\Omega(t)$ can be written as

$$\Omega(t) = A(1-t^2)^{2\lambda/\pi} + \sum_{j=0}^{\infty} a_j t^{2j}, \qquad (3.7)$$

where λ is determined from (3.4).

If we use the relation $t = |t|e^{i\sigma}$, then the points on the free surface are given by $t = e^{i\sigma}$, $0 < \sigma < \pi$. Using (2.9) and the identity

$$\frac{\partial x}{\partial \phi} + i\frac{\partial y}{\partial \phi} = \frac{1}{\xi}, \qquad (3.8)$$

we obtain after some algebraic manipulations,

$$\frac{dy}{d\sigma} = -\frac{2}{\pi}\frac{v}{u^2+v^2}\frac{1}{\sin\sigma} \qquad (3.9)$$

Differentiating (2.10) with respect to σ and using (3.9) we obtain

$$\frac{d}{d\sigma}[u^2(\sigma)+v^2(\sigma)] - \frac{4}{\pi}\frac{v(\sigma)}{u^2(\sigma)+v^2(\sigma)}\frac{1}{\sin(\sigma)} = 0 \qquad (3.10)$$

Upon substituting $t = e^{i\sigma}$, in (3.6) and (3.7) we get

$$\xi(\sigma) = e^{i\theta + AL_R + S_1}, \qquad (3.11)$$

where

$$\theta = \alpha(\theta_{-3}+\theta_3+\theta_{-1}+\theta_1) - \beta(\theta_{-2}+\theta_2) + +A\, L_I + S_2,$$

$$L_I = [2\sin\sigma]^{\lambda}\sin\lambda\left(\sigma - \frac{\pi}{2}\right),$$

$$L_R = [2\sin\sigma]^{\lambda}\cos\lambda\left(\sigma - \frac{\pi}{2}\right),$$

$$\theta_{\pm i} = \tan^{-1}\frac{(1-t_i^2)\sin\sigma}{(1+t_i^2)\cos\sigma \mp (2t_i)}, \quad i = 1,2,3,$$

$$S_1 = \sum_{j=0}^{n} a_j \cos 2j\sigma,$$

$$S_2 = \sum_{j=1}^{n} a_j \sin 2j\sigma,$$

$$Y(\sigma) = |\xi|^3 \left[A\lambda\left(\frac{L_R}{\tan\sigma} - L_I\right) - S_3\right] + \frac{2\sin\theta}{\pi\sin\sigma} = 0, \qquad (3.12)$$

$$S_3 = \sum_{j=1}^{n} 2j a_j \sin 2j\sigma.$$

The resulting equation will be used to determine the coefficients a_j.

We solve for the a_j's numerically by truncating the infinite series in (3.7) at j = n, where n = N- 2; N is the number of unknown coefficients, namely, A, a_0, $a_1, \ldots, a_n$.

We fix the geometry of the polygon by specifying the values for t_1 ,t_2 , t_3, $\alpha' = \pi/4$.

The N unknowns A, a_0, $a_1, \ldots ,a_n$ are determined by collocation. To do so we introduce the N-1 mesh points on the free surface

$$t_M = e^{(i\sigma_M)}, \text{ where}$$

$$\sigma_M = \frac{\pi}{2(N-1)}\left(M - \frac{1}{2}\right); M = 1,2,3,4,\ldots(N-1) \qquad (3.13)$$

These points are equally spaced on the upper half of the unit circle. using (3.11) , (3.12) we obtain N-1 nonlinear algebraic equations of N uknowns. The remaining equation is obtained by relating F to the dimensionless velocity downstream $\xi(1)$, namely,

$$F^2 = |\xi(1)|^3 \qquad (3.14)$$

This system of N nonlinear equations of N unknowns is solved by Newton's method . Inverting (2.6) yields

$$\frac{dz}{d\chi} = \frac{1}{\xi}$$

and thus

$$\frac{dz}{dt} = \frac{1}{\xi}\frac{d\chi}{dt} \tag{3.15}$$

Once we have solved the system of equations (3.12),(3.14) we calculate the height W of the obstacle by numerical integration of (3.15) along the real axis. This leads to

$$W = \sin\alpha \int_{t_1}^{t_2} \left(\frac{d\chi}{dt}\right)\frac{1}{\xi}\,dt \tag{3.16}$$

As a check the integration of the same relation between t_2 and t_3 should give the same W, i.e.,

$$W = \sin\alpha \int_{t_2}^{t_3} \left(\frac{d\chi}{dt}\right)\frac{1}{\xi}\,dt \tag{3.17}$$

In order to calculate the profile shape we first calculate the elevation Z_p of the fluid above the obstacle at $x = 0$. To do so we integrate (3.15) along the vertical axis of the t-plane from $t = 0$ to $t = i$, that is,

$$Z_p = \frac{1}{i}\int_0^i \frac{d\chi}{dt}\frac{1}{\xi}\,dt. \tag{3.18}$$

Now by integrating (3.15) along the upper half of the unit circle, we obtain the shape of the free surface, namely,

$$x + iy = i\,Z_p - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\sigma} \frac{1}{\xi(s)\sin s}\,ds, 0<\sigma<\pi. \tag{3.19}$$

4. NUMERICAL RESULTS AND DISCUSSION

A numerical method based on series truncation is presented to solve the problem of an irrotational, inviscid, incompressible and steady flow over a two-dimensionl two-triangular obstacle.

The system of N nonlinear algebraic equations of N unknowns is solved by Newton's method for given values of Froude number F.

The coefficients a_j are found to decrease rapidly as the index n increases which means rapid convergenc. For example in Table (1) we give the last coefficient $|a_n|$ for values of $n = 5$ to 30 in steps of 5 for a small obstacle $W = .05$ and for a large obstacle $W = .32$ when $F = 2.0$.

Most of the calculations are performed with $N = 15$, and the nonlinear equations are solved with relative error of order 10^{-6}.

We noticed that , by increasing the value of F we need less terms of the infinite series due to rapid convergence.

In our study we investigated the effect of the obstacle height W, the obstacle width 2L and the Froude number F .

Table 1. values of $|a_n|$ for the case $F = 2.0$, $\alpha' = 45°$ $2L = 2.35$ for small obstacle $W = .05$ and for large obstacle $W = 0.32$.

n	5	10	15	20	25	30
W=.05	4.07×10^{-5}	4.69×10^{-6}	1.32×10^{-6}	3.02×10^{-7}	5.56×10^{-8}	2.66×10^{-9}
W=.32	8.88×10^{-4}	1.09×10^{-4}	3.41×10^{-5}	1.01×10^{-5}	3.07×10^{-6}	9.54×10^{-7}

4.1 Effect of the obstacle height W :

The elevation of the free surface η increases by the increase of the obstacle height. In Figure (2) we present η for different values of W from 0.05 to 0.32 for $F = 2$, $2L = 2.35$ and $\alpha' = 45°$.

4.2 Effect of the obstacle width 2L :

The elevation of the free surface decreases by increase of the obstacle width. When the obstacle width is further increased the elevation of the free surface takes the same form as the obstacle . Figure (3) represents h for different values of 2L from 0.8 to 2.37 and for $F = 2$, $\alpha' = 45°$ and $W = .25$.

4.3 Effect of the Froude number F :

We found that the elevation of the free surface falls down by the increase of the Froud number. The computed free surface elevation h is displayed in Figure (4) for different values of F vfrom 1.5 to 3.5 and for $W=.05$, $2L=0.8$ and $\alpha' = 45°$.

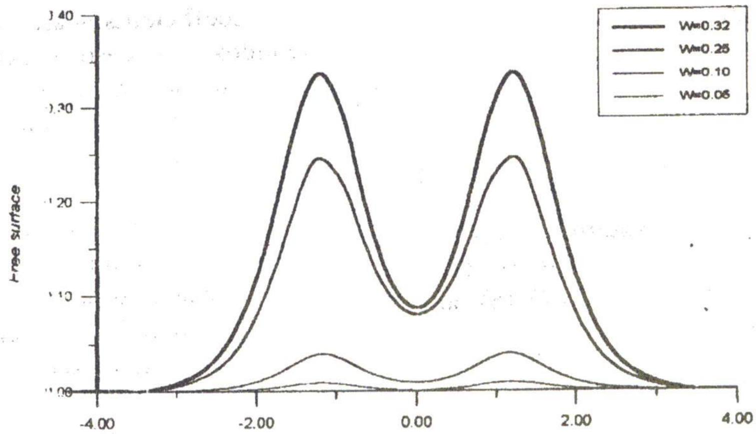

Figure 2. Effect of the obstacles height W for fixed F=2.0 and 2L = 2.35.

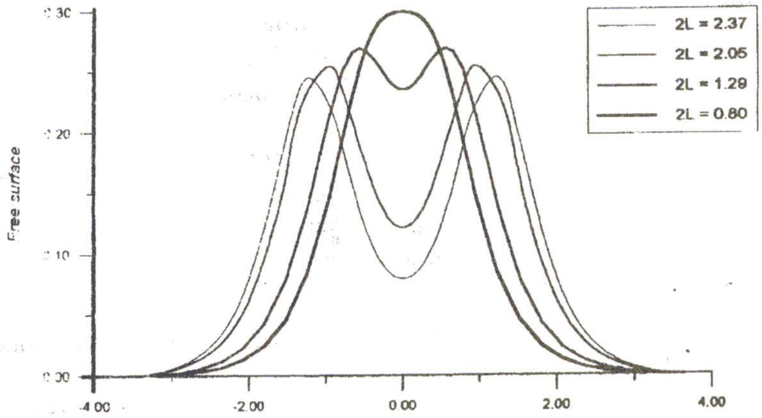

Figure 3. Effect of the obstacles width 2L for fixed F=2 and W = 0.25.

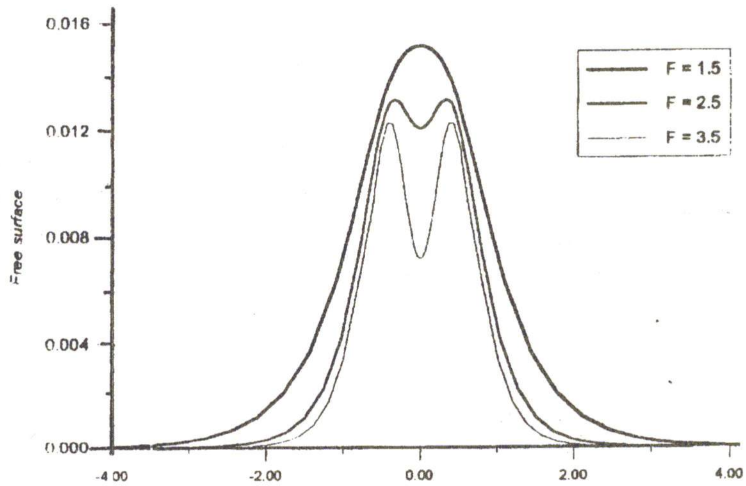

Figure 4. Effect of the Froude number F for fixed W=0.05 and 2L = 0.8.

REFERENCES

[1] M.B. Abd-el-Malek and S.Z. Masoud, Linearized solution of a flow over a ramp, *Appl. Math. Modelling*, 12, pp. 406-410, 1988.

[2] M.B. Abd-el-Malek and S.N. Hanna, Approximate solution of a flow over a ramp for large Froude number, *J. Comput. Appl. Math.* 28, pp. 105-117, 1989.

[3] M.B. Abd-el-Malek, S.N. Hanna and M.T. Kamel, Approximate solution of gravity flow from a uniform channel over triangular bottom for large Froude number, *Appl. Math. Modelling*, 15, pp. 25-32, 1991.

[4] T.B. Benjamin, On the flow in channels when rigid obstacles are placed in the stream. *J. Fluid Mech.* 1, pp. 227-248., 1956.

[5] G. Birkhoff and E.H. Zarantonello, Jets, wakes and cavities (*Academic, New York*, 1957).

[6] F.Z. Boutros, M.B. Abd-el-Malek and S.Z. Masoud, Hilbert's method for numerical solution of flow from a uniform channel over irregular bottom topographies. *Comput. Math. Appl. Mech. Engng.* 65, pp. 215-228, 1987.

[7] F. Dias and J.M. Vanden-Broeck, Open channel flow with submerged obstruction, *J. Fluid Mech.* 206, pp. 155-170, 1989.

[8] L.K. Forbes, On the wave resistance of a submerged semi-elliptical body. *J. Engng. Math.*, 15, pp. 287-298, 1981.

[9] L.K. Forbes and L.M. Schwartz, Free-surface flow over a semi- circular obstruction, *J. Fluid Mech.* 114, pp. 299-314, 1982.

[10] L.K. Forbes, Critical free-surface flow over a semi-circular obstruction, *J. Engng. Maths.* 22, pp. 3-13, 1988.

[11] J. Hamilton, Differential equations for long-period gravity waves of a fluid of rapidly varying depth, *J. Fluid Mech.* 83, pp. 289-310, 1977.

[12] S.N. Hanna, Critical free-surface flow over polygonal obstacle, *Alex. Engng. J.* 33-1, pp. 27-36, 1994.

[13] R.S. Johnson, On the development of a solitary wave moving over an uneven bottom, *Proc. Camb. Phil. Soc.* 73, pp. 183-203, 1973.

[14] C.H. Von Kerczek and N. Salvesen, Numerical solutions of two- dimensional nonlinear wave problems. O.N.R. 10th Symposium on Navel Hydrodynamics. *Massachusetts Institute of Technology, Cambridge*, Mass., pp. 649-666, 1974.

[15] A.C. King and M.I.G. Bloor, Free-surface flow over a step, *J. Fluid Mech.* 184, pp. 193-208, 1987.

[16] C. Korving and A.J. Hermans, The wave resistance for flow problems with a free-surface. *2nd Int. Conf.Numerical ship Hydrodynamics*. pp. 285-291, 1977.

[17] G. Kreisel, Surface waves. Q. *Appl. Math.* 7, pp. 21-24, 1949.

[18] *H. Lamb, Hydrodynamics*, (Dover, New York, 6th ed., 1945).

[19] O.S. Madison and C.C. Mei, The transformation of a solitary wave over an uneven bottom. *J. Fluid Mech.* 39, pp. 781-791, 1969.

[20] J.N. Newman, Propagation of water waves past long two-dimensional obstacles. *J. Fluid Mech.* 23, pp. 23-29, 1965.

[21] N. Salvesen and C.H. Von Kerczek, Numerical solutions of two-dimensional nonlinear body-wave problems. *1st Int. Conf. Numerical Ship Hydrodynamics*, Gaithersburg, Maryland, pp. 279-293, 1975.

[22] N. Salvesen and C.H. Von Kerczek, Comparison of numerical and perturbation solutions of two dimensional nonlinear water-wave problems. *J. Ship Res.* 20, pp. 160-170, 1976.

[23] A.C. Smith and T.H. Lim, Symmetric supercritical free- surface flow over a polygonal obstacle. *Int. J. Engng. Sci.*, 23, no.3, pp. 289-306, 1985.

[24] J.M. Vanden-Broeck and J.B. Keller, Weir flows, *J. Fluid Mech.* 176, pp. 283-293, 1987.