EFFICIENT USE OF OPTICAL OVERLAPPING PPM CHANNELS UNDER PRACTICAL CONSTRAINTS

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ABSTRACT

Direct detection self-noise-limited optical channel is considered. Overlapping-pulse-position modulation (OPPM) with at most two positions per pulsewidth is assumed. The depth of the overlap is $r\tau$, where τ is the pulsewidth and $r \in [0,0.5]$ is the overlapping index. Tight lower bounds on the capacity of this channel are derived. It is shown that, under peak power and pulsewidth constraints, there exist an overlapping index $r \leq 0.5$ and a pulse-position multiplicity $M> 3$ that minimize the OPPM energy required to transmit data at a rate equal to the maximum throughput attainable by conventional PPM, which minimum energy is less than half that for conventional PPM. It is also shown that an overlapping index r and a pulse-position multiplicity M exist so that OPPM has a greater throughput than PPM for a given efficiency.

Keywords: overlapping PPM, optical channel capacity, ambiguity and erasure channel.

I. INTRODUCTION

A common signaling format in direct-detection pulsed optical communication links is pulse-position modulation (PPM) in which a laser pulse is transmitted in one of a finite set of possible disjoint time intervals. The capacity and cutoff rate of the self-noise-limited channel (i.e., the channel with negligible background and thermal noise) have been examined by many authors [1-6]. It has been shown that a signaling rate of about 3 nats/photon can be obtained with practical coding schemes. Some of these authors [4-6] have examined the optimization of channel performance under pulsewidth and peak (or average) power constraints.

In Overlapping-Pulse-Position Modulation (OPPM) [7-10], the possible time positions of the laser pulse are allowed to overlap. The overlap between two adjacent positions will be denoted by $r\tau$, where $r\in [0,1)$ is the overlapping index and τ is the pulsewidth. If r=0, we have the ordinary disjoint PPM (called DJPPM in [7]).

Our model for OPPM with overlapping index r is as follows. A rectangular laser pulse is transmitted in one of M possible positions $\{1,2,...,M\}$ within a time frame of duration T.A pulse of width τ is said to be in position m, $m \in\{1,2,...,M\}$ if it extends over the subinterval beginning at time $(m-1)(1-r)\tau$ and ending at time $(m(1-r)+r)\tau$. The relation between T, r, M, and τ is

$$T= (M(1-r)+r)\ \tau.$$

Previous interest [7-9] was given to the special cases $r=1-\frac{1}{n}$, n=1,2, ... (i.e., $r\in\{0,\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},...\}$). Error probabilities for this signaling format have been given in [8,9] where it has been shown that OPPM has a worse performance than DJPPM. Trellis coded modulation was applied to OPPM [9] in order to improve both the performance and throughput of the system without expanding the bandwidth. Lower bounds for capacity and cutoff rate have been derived in [7], where it has been shown that OPPM, with r restricted as above, outperforms DJPPM in terms of both capacity and cutoff rate under pulsewidth and power constraints. For large n, the increase in capacity approaches that for continuous PPM [7].

We notice that if $r \in (0, \frac{1}{2}]$, only two laser pulse positions can overlap at any time. If $r \in (\frac{1}{2}, \frac{2}{3}]$, three pulse positions can overlap at a time, and so on. Increasing the number of pulse positions that can overlap adds some complexity to the detection system and enforces more refined timing to be used. Furthermore, ambiguities and erasures become more frequent and more complex error-correcting codes will be required to obtain an acceptable error rate. For these reasons, we allow at most two pulse positions to overlap, i.e., we assume $r \in [0, \frac{1}{2}]$. The performance (in terms of the bit error rate) of this channel under some communication constraints has been studied previously in [10]. In that paper we have showed that under pulsewidth and throughput constraints, uncoded OPPM (with overlapping index not exceeding 0.3) is superior to ordinary PPM in terms of bit error rate. In other words, we can decrease the energy required to transmit a given amount of information without sacrificing either the bit error rate or the throughput. The overlapping index r that offers the minimum energy was shown to vary significantly with the throughput constraint and should be chosen carefully from characteristic curves as those given in [10]. The main objectives of this paper are:

1) To derive tight lower bounds on the capacity of the OPPM channel with $r \in [0,0.5]$.
2) To show that, under pulsewidth and peak power constraints,
   a) there exist a pulse-position multiplicity M> 3 and an overlapping index $r \in [0,0.5]$ that minimize the energy required by an OPPM system such that its throughput capacity is at least equal to maximum DJPPM throughput--this yields more than a 100 percent increase in OPPM efficiency (as a measure of energy saving) over DJPPM for moderate numbers of photons per pulse; and
   b) there exist a pulse-position multiplicity M and an overlapping index r that maximize OPPM throughput under the constraint that its efficiency is at least equal that of DJPPM with $M=M^*$, where M^* denotes the pulse-position multiplicity corresponding to maximum DJPPM throughput (M^* is known to be three)-this yields a 50 percent increase in the throughput.

The paper is organized as follows. The channel model, as well as capacity bounds, are given in Section II. Section III is devoted to the efficiency maximization problem, while the throughput maximization problem is treated in Section IV. Concluding remarks are given in Section V.

II. LOWER BOUNDS ON OPPM CHANNEL CAPACITY

Define the following subintervals within the time frame (0,T):

$$J_o(m) = (m(1-r)\tau, (m(1-r)+r)\tau),$$

$$J_1(m) = (((m-1)(1-r)+r)\tau, m(1-r), \tau).$$

The demodulator output corresponding to transmitting a pulse in position m, $m \in \{2,...,M-1\}$ will be one of the following:

(i) m: if photons are detected in subinterval $J_1(m)$ or in both $J_o(m-1)$ and $J_o(m)$.
(ii) an ambiguity a(m-1,m) between positions m-1 and m:if photons are detected only in subinterval $J_o(m-1)$.
(iii) an ambiguity a(m,m+1) between positions m and m+1:if photons are detected only in subinterval $J_o(m)$.
(iv) an erasure e: if no photons are detected through the frame.

The resulting OPPM channel model (ambiguity and erasure channel) for $r \in [0,0.5]$ is illustrated in Figure (1). The input random variable X denotes the position of the transmitted pulse, whereas the output random variable Y denotes the demodulator output. To calculate the transition probabilities in this case, we denote by N(I), the photon count observed in the subinterval I. Since we are dealing with a self-noise-limited optical channel, each of these counts is a Poisson random variable. Denote by Q the average photon count per pulse. It is obvious that Q is proportional to the peak power for fixed τ. Assuming that a pulse is transmitted in position m, m $\varepsilon\{2, ..., M-1\}$, we obtain

$$P_{Y|X}(e|m)$$

$$= \Pr\{N((J_o(m-1) \bigcup J_1(m) \bigcup J_o(m)) = 0\}$$

$$= \exp[-Q],$$

$$P_{Y|X}(a(m-1, m)\,|m) = \Pr\{N((J_o(m-1)) \neq 0,$$

$$N(J_1(m) \bigcup J_o(m)) = 0\}$$

$$= (1 - \exp[-Qr]) \exp[-Q(1-r)],$$

and

$$P_{Y|X}(a(m,m+1)|m = P_{Y|X}(a(m-1,m)\,|m).$$

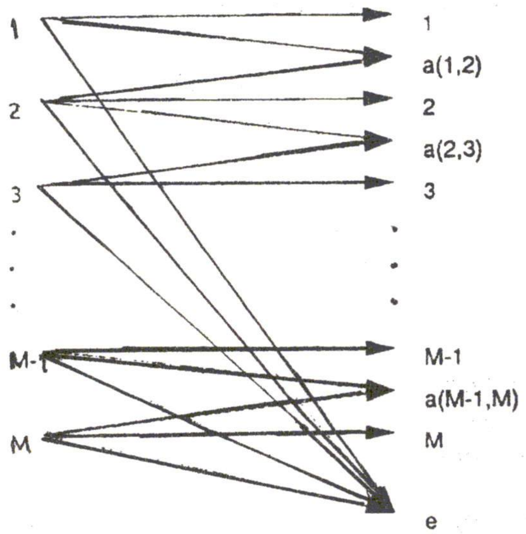

Figure 1. Ambiguity and erasure channel.

Thus, we can write

$$P_{Y|X}(y|m) = \begin{cases} 1-2e^{Q(1-r)}+e^{-Q}; \text{if } y=m, \\ e^{-Q}(1-r)-e^{-Q}; \text{if } y=a(m-1,m) \\ \quad \text{or } a(m,m+1), \\ e^{-Q}; \text{if } y=e, \\ 0; \text{otherwise}, \end{cases}$$

For X=1 or M, there are only three possible channel outputs. The transition probabilities for m=1 are given by

$$P_{Y|X}(y|1) = \begin{cases} 1-e^{Q(1-r)}; \text{if } y=1, \\ e^{-Q(1-r)}-e^{-Q}; \text{if } y=a(1,2), \\ e^{-Q}; \text{if } y=e, \\ 0; \text{otherwise}, \end{cases}$$

The transition probabilities for m=M are similar.

The mutual information for our channel model is given by

$$I(X \wedge Y) = \sum_{r,y} P_{XY}(x,y) \log \frac{P_{Y|X}(y|x)}{P_Y(y)}$$

$$= -(e^{-Q(1-r)}-e^{-Q}) \,.\, \sum_{x=1}^{M-1} (P_X(x)+P_X(x+1))$$

$$.\, \log(P_X(x)+P_X(x+1))$$

$$-(1-e^{-Q(1-r)})\{P_X(1)\log P_X(1)+P_X(M)\log P_X(M)\}$$

$$-(1-2e^{-Q(1-r)}+e^{-Q}) \sum_{x=2}^{M-1} P_X(x)\log P_X(x). \quad (1)$$

To find capacity, we want to maximize $I(X \wedge Y)$ over the input distribution P_X. The symmetry of the channel ensures that the maximizing input distribution satisfies

$$P_X(i) = P_X(M-i+1), \quad i=1,2,...,[\frac{M}{2}].$$

For the special case of M=2, it follows that

$P_X(1)=P_X(2)=\frac{1}{2}$. By substitution in (1), we get the corresponding capacity

$$C_2(r)= (1-e^{-Q(1-r)}) \log 2 \text{ nats/channel use.}$$

An explanation to this simple formula can be seen by noticing that for M=2, the channel reduces to the familiar binary erasure one because $P_{Y|X}(a(1,2)|1)=P_{Y|X}(a(1,2)|2)$, so that a(1,2) can be combined with the erasure symbol e to give the above formula.

For $M \geq 3$, we derive lower bounds on capacity.

First, we define the following function:

$$f(p,r,M) = -2\,(e^{-Q(1-r)} - e^{-Q})$$

$$\cdot\left\{\frac{1+(M-4)p}{M-2}\log\frac{1+(M-4)p}{M-2}\right.$$

$$\left. +(M-3)\frac{1-2p}{M-2}\log\frac{2(1-2p)}{M-2}\right\}$$

$$-2\,(1-e^{-Q(1-r)})\, p \log p$$

$$-(1-2e^{-Q(1-r)}+e^{-Q})(1-2p)$$

$$\log\frac{1-2p}{M-2},$$

where $0 \leq p \leq 1/2$.

Theorem 1: If Q is the average photon count per pulse, then the capacity of the optical OPPM channel with M pulse positions and overlapping index $r \in [0, 0.5]$ is lower bounded by

$$C_M(r) \geq I_M(r),$$

where $I_M(r) = f(p^*,r,M)$ and p^* is the solution of $\frac{\partial f}{\partial p}(p,r,M)= 0$.

Proof:

$$C_M(r)= \max_{P_x} I(X \wedge Y) \geq I_L,$$

where I_L is the value of $I(X \wedge Y)$ for

$$P_X(x) = \begin{cases} p; & \text{if } x = 1 \text{ or } M, \\ \frac{1-2p}{M-2}; & \text{otherwise,} \end{cases}$$

with $0 \leq p \leq 1/2$. Substituting in (1) yields $I_L = f(p,r,M)$. Hence

$$C_M(r) \geq \max_{p\in[0,1/2]} f(p,r,M)=f(p^*,r,M),$$

where p^* is the solution of the equation

$$\frac{\partial f}{\partial p}(p,r,M)=0.$$

III. EFFICIENT USE OF OPPM UNDER PULSEWIDTH, PEAK POWER, AND THROUGHPUT CONSTRAINTS

In this section we aim at minimizing the OPPM energy required to transmit data at the maximum DJPPM throughput, given pulsewidth and peak power constraints. We study two different cases (A and B). In case A we assume that the peak power is fixed through the optimization, whereas in case B, the peak power is allowed to vary in a way such that it does not exceed a maximum value.

The capacity per second (throughput) and capacity per photon (efficiency) are given by

$$C_T(r) = \frac{C_M(r)}{T} = \frac{C_M(r)}{(M(1-r)+r)\tau} \text{ nats/s}$$

and

$$C_{ph}(r) = \frac{C_M(r)}{Q} = \frac{C_M(r)}{\lambda\tau} \text{ nats/photon}$$

respectively, where λ is the peak source intensity and $Q=\lambda\tau$ is the average photons per pulse. Setting r=0 in the above equations yields the capacities for DJPPM, which are known to be

$$C_M(0) = (1-e^{-Q}) \log M \text{ nats/channel use,}$$

$$C_T(0) = (1-e^{-Q}) \frac{\log M}{M\tau} \text{ nats/s}$$

and

$$C_{ph}(0) = (1-e^{-Q}) \frac{\log M}{Q} \text{ nats/photon} .$$

For fixed pulsewidth and peak power, let $C_T^m(r)$ denote the maximum throughput given $r\varepsilon$ [0,0.5], i.e.,

$$C_T^m(r) = \max_M C_T(r).$$

It is well known [7] and easy to see that, for DJPPM (r=0), the maximum throughput (given Q) is attained for M=3. That is

$$C_T^m(0) = (1-e^{-Q}) \frac{\log 3}{3\tau} = \frac{0.366}{\tau}(1-e^{-Q}). \quad (2)$$

Define the following lower bounds on the capacities:

$$I_T(r) = \frac{I_M(r)}{T} = \frac{I_M(r)}{(M(1-r)+r)\tau} \text{ nats/s}$$

and

$$I_{ph}(r) = \frac{I_M(r)}{Q} = \frac{I_M(r)}{\lambda\tau} \text{ nats/photon,}$$

where $I_M(r)$ is given in Theorem 1. Clearly (cf. Theorem 1) $C_{ph}(r) \geq I_{ph}(r)$ and $C_T(r) \geq I_T(r)$.

Case A. Optimization under fixed peak power

We consider the following optimization problem. Fix the pulsewidth and peak power of the OPPM system. Vary M and r ε [0,0.5] so as to maximize the OPPM efficiency under the constraint that the throughput is at least equal to $C_T^m(0)$. In other words, we have the following maximization problem:

$$\max_{\substack{M,r \\ C_T(r) \geq C_T^m(0)}} C_{ph}(r)$$

for fixed peak power and pulsewidth. Instead of the above (time consuming and complex) optimization we study the following simpler problem:

$$\max_{\substack{M,r \\ I_T(r) \geq C_T^m(0)}} I_{ph}(r) \quad (3)$$

for fixed peak power and pulsewidth. Clearly,

$$\max_{\substack{m,r \\ C_T(r) \geq C_T^m(0)}} C_{ph}(r) \geq \max_{\substack{M,r \\ I_T(r) \geq C_T^m(0)}} I_{ph}(r)$$

because $C_{ph}(r) \geq I_{ph}(r)$ and $\{M,r: I_T(r) \geq C_T^m(0)\} \subset \{M,r: C_T(r) \geq C_T^m(0)\}$. If we define

$$I_{ph}^m = \max_{\substack{M: \\ I_T(r) \geq C_T^m(0)}} I_{ph}(r)$$

then (3) is equivalent to

$$\max_{r\varepsilon[0,0.5]} I_{ph}^m(r).$$

$I_{ph}^m(r)$ is plotted in Figure (2) for different values of Q. From the figure we can see that $I_{ph}^m(r)$ attains its maximum at some point r<0.5. We notice that, during some intervals of overlapping indices, the curve decreases as r increases. This is because increasing r causes the ambiguities to be more frequent which leads to a decrease in the efficiency. Obviously the throughput increases with r, but trying to increase M (in an attempt to raise the efficiency) during these intervals will cause a drop in the throughput below the constraint $C_T^m(0)$. Fortunately, increasing r more and more will lead to enough rise in the throughput and there exist instants where we can increase M without disturbing the constraint.

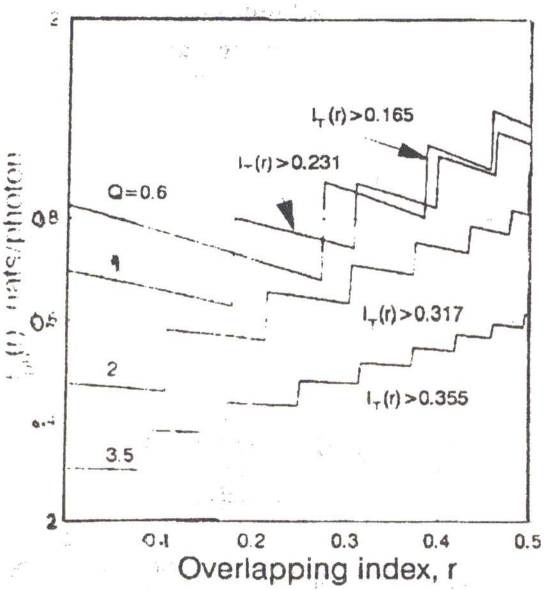

Figure 2. $I_{ph}^{m}(r)$ versus the overlapping index for fixed pulsewidth and peak power.

The sharp jumps in the curve correspond to these instants. The behavior of the curve after a jump, can be estimated by noticing that immediately after a jump we can write $I_T(r) \approx C_T^m(0)$, whence

$$I_{ph}^{m}(r) = \frac{I_T(r)(M(1-r)+r)}{\lambda}$$

$$\approx \frac{C_T^m(0)(M(1-r)+r)}{\lambda}$$

This quantity decreases as r increases, as long as M is constant.

It is seen from Figure (2) that for Q=0.6 (photons/pulse) the maximum increment of OPPM efficiency over DJPPM is about 23% (achieved for $r \approx 0.46$). This increment is about 40% for Q=1, 72% for Q=2, and 101% for Q=3.5. The percentile efficiency increment increases $\frac{\log 13 - \log 3}{\log 3} \times 100 \approx 133.47\%$ as Q increases, and approaches infinity. Indeed, we have

$$\frac{\max\limits_{\substack{M,r:\\ C_T(r)\geq C_T^m(0)}} C_{ph}(r) - C_{ph}^{*}(0)}{C_{ph}^{*}(0)} = \frac{\max\limits_{\substack{M,r:\\ C_T(r)\geq C_T^m(0)}} C_M(r) - C_M^{*}(0)}{C_M^{*}(0)} \geq \frac{\max\limits_{\substack{M,r:\\ I_T(r)\geq C_T^m(0)}} I_M(r) - C_M^{*}(0)}{C_M^{*}(0)}$$

Taking the limit in (2) as $Q \rightarrow \infty$ yields $C_T^{m}(0) \rightarrow \frac{\log 3}{3\tau}$ and $C_M^{*}(0) \rightarrow \log 3$. Applying Theorem 1 for $Q \rightarrow \infty$, we obtain

$$I_M(r) \rightarrow \log M$$

and

$$I_T(r) \rightarrow \frac{\log M}{(M(1-r)+r)\tau}. \tag{4}$$

Since $I_T(r) \geq C_T^{m}(0)$, then

$$\frac{\log M}{M(1-r)+r} > \frac{\log 3}{3}.$$

In view of $C_M(r) \rightarrow \log M$, we seek the maximum M satisfying the last inequality. Since $r \in [0,0.5]$, M satisfying the last inequality increases as r increases. Hence we seek the maximum M satisfying

$$\frac{\log M}{0.5M+0.5} > \frac{\log 3}{3}$$

This yields M=13. Thus $C_M(r) \rightarrow \log 13$ and

$$\frac{\max\limits_{\substack{M,r:\\ I_T(r)\geq C_T^m(0)}} I_M(r) - C_M^{*}(0)}{C_M^{*}(0)} \rightarrow$$

$$\frac{\log 13 - \log 3}{\log 3}.$$

Case B. Optimization under upper bound on peak power

We notice that we can increase the efficiency of OPPM by decreasing the peak power (Q) in addition to increasing M. Decreasing Q, however, leads to a degradation in the system performance because in this case the transmitted energy per pulse becomes less. Error correcting codes are thus necessary and the system is more complex than in case A. The maximization problems in case A are now modified to:

$$\max_{\substack{M,r,Q:\\ C_T(r)\geq C_T^m(0),\,Q\leq Q_o}} C_{ph}(r)$$

and

$$\max_{\substack{M,r,Q:\\ I_T(r)\geq C_T^m(0),\,Q\leq Q_o}} I_{ph}(r)$$

for fixed pulsewidth. Here Q_o corresponds to the ultimate energy per pulse. $I_{ph}{}^m(r)$ is thus modified to

$$I_{ph}{}^m(r) = \max_{\substack{M,Q:\\ I_T(r)\geq C_T^m(0),\,Q\leq Q_o}} I_{ph}(r)$$

$I_{ph}{}^m(r)$ is evaluated numerically, according to the last optimization problem, and plotted in Figure (3) versus r. We found that for small values of Q_o, the maximum efficiency is achieved when $Q \approx Q_o$ (independent of r); which indicates that cases A and B are equivalent as long as $Q_o \leq 1$. On the other hand as Q_o increases, Q achieving the best efficiency depends on r and is always less than Q_o. This obviously means that we have a larger gain in efficiency than in case A. Indeed, for Q_o=3.5 the gain now is 132% instead of 101% in case A and for Q_o =2 the gain is 74% instead of 72%. If, however, Q_o =1 the gain is the same as in case A, i.e., 40%.

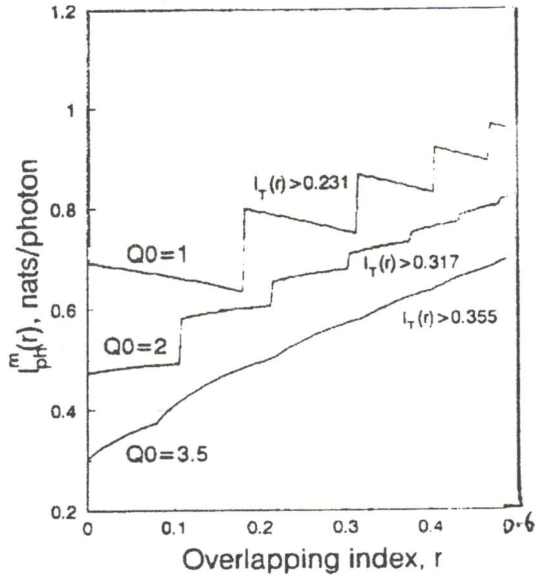

Figure 3. $I_{ph}{}^m$ (r) versus the overlapping index under peak power constraint.

IV. MAXIMUM OPPM THROUGHPUT UNDER PULSEWIDTH, PEAK POWER, AND EFFICIENCY CONSTRAINTS

The objective in this section is to find maximum OPPM throughput (under peak power and pulsewidth constraints) such that the efficiency is not less than $C_{ph}{}^*(0)$. This is useful when high data rates are desired. We thus consider the optimization problem:

$$\max_{\substack{M,r\\ C_{ph}(r)\geq C_{ph}^*(0)}} C_T(r)$$

and

$$\max_{\substack{r:\\ C_{ph}(r)\geq C_{ph}^*(0)}} C_T^m(r).$$

As argued above we consider instead the lower bound

$$\max_{\substack{r: \\ I_{ph}(r) \geq C^{*}_{ph}(0)}} I_T^m(r),$$

where $I_T^m(r) = \max_M I_T(r)$. $I_T^m(r)$ is plotted in Figure (4) along with the corresponding $I_{ph}(r)$.

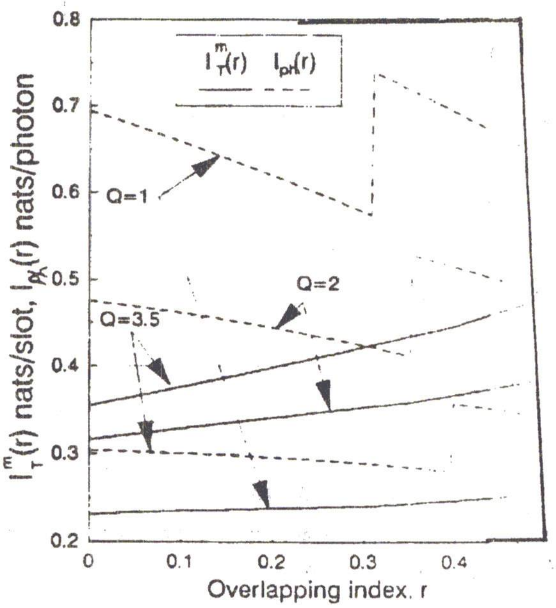

Figure 4. $I_T^m(r)$ and corresponding $I_{ph}(r)$ versus the overlapping index for fixed pulsewidth and peak power.

We can see only one jump occurring in $I_{ph}(r)$ which indicates that M increases from 3 to 4 . For Q=1 the above maximization is attained for $r \approx 0.4$, where we have an OPPM throughput advantage of about 6.5% over DJPPM. The advantage is about 21.5% if Q=2 and 34.78% if Q=3.5. The value of the advantage increases as λ increases (always achieved for M=4) and approaches

$$\frac{\dfrac{\log 4}{4-(4-1)x0.5} - \dfrac{\log 3}{3}}{\dfrac{\log 3}{3}} x100 \approx 51.42\%$$

as λ goes to ∞. Indeed in the first term in the numerator we substitute M=4, r=0.5 in the limiting expression for $I_T(r)$ as given in (4). In the second term, however, we substitute the maximum throughput for PPM. OPPM loses its advantage over DJPPM for small values of Q (Q<0.2).

V. OCCLUDING REMARKS

We restricted our study to an overlapping index $r \in [0,0.5]$ because as the number of pulse positions that are allowed to overlap increases the complexity of the system increases, more refined timing will be required, and final error rate increases.

Our results in Section III were obtained under the requirement that the throughput capacity of OPPM should exceed the maximum throughput attainable by DJPPM. These results would still be valid if we demanded that the throughput should exceed some fixed (not necessarily the maximum) quantity. This last conclusion also applies in a similar way to the results in Section IV.

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