FREE-SURFACE FLOW OVER A PERIODIC IRREGULAR BOTTOM

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ABSTRACT

Free-surface flow past a submerged periodic bottom of different shapes is considered. The two-dimensional flow is assumed to be steady, irrotational, inviscid and incompressible. Following the method suggested by Thomson and Lamb the free-surface profile is obtained for the supercritical and subcritical cases. The effect of the surface tension is taken into account for the two kinds of flow. The parameters governing the flow such as the Froude number F, the periodic length L and the shape of the bottom are discussed in both cases of the presence or absence of the surface tension.

Keywords: Two-dimensional flow, Froude number, Surface tension.

1. INTRODUCTION

Fluid flow over various bottom topographies has attracted considerable attention throughout the history of fluid mechanics.

The fluid is assumed to be inviscid and incompressible and the flow to be steady and irrotational. In 1900, Wien [1] assumed a bottom of the form

$$Y(x) = h + a\cos(kx) \tag{1.1}$$

where h is the upstream depth of uniform flow and a is the amplitude of unevenness of the bottom. He chose the origin in the undisturbed surface, and supposed that the solution was a linear combination of trigonometric functions, under the assumption that the amplitude of unevenness of the bottom was small compared with the depth of the uniform flow. Eventually, he found that the shape of the free surface is

$$\eta(x) = \frac{a\cos(kx)}{\cosh(kh)\left[1 - g\dfrac{\tanh(kh)}{ku^2}\right]} \tag{1.2}$$

in which u is the mean-stream velocity. An interesting consequence is that the free-surface wave and the bed wave are in phase or out of phase according as $u^2/(g/k)$ is greater or less than tanh (kh respectively. However, when $u^2=(g/k)\tanh(kh)$, the theory fails and linearised theory can not be applied. The flow takes one of two possible forms, depending on the value of the upstream Froude number F, which is the ratio of the phase speed of the fluid infinitely far upstream to the speed at which a small disturbance would travel in the fluid. If $F^2<1$, linearized theory predicts a region of uniform flow far ahead of the obstruction, followed by a train of downstream waves. When $F^2>1$, a wave-free solution is obtained, in which the fluid surface simply rises over the obstacle, before returning to the undisturbed level downstream.

In 1932, Lamb [2] presented a general lineralized theory for flow over stream beds of arbitrary shape. Lamb's theory was reviewed by Wehausen and Laitone [3] in 1960; they also discussed the free-surface flow over a step discontinuity in the stream bed.

In infinite depth, Kochin, Kiebel, and Rozen [4], in

1964 treated the problem of a point vortex, a point source, and a dipole moving beneath the free surface, but the method does not work conveniently in the case of finite depth. This is due to the problems arising from the formulation of a new boundary condition, namely the vanishing of the normal component of velocity at the bottom. Mei [5] in 1969, considered a linear solution of a steady free-surface flow over a wavy bed. He applied a perturbation technique to a third-order approximation and found that by carrying out a higher-order analysis, the singularity will be removed by the nonlinearity of the free-surface condition. In 1973, Gazdar [6] replaced the obstacle by an equivalent system of singularities. He was able to clear away the difficulties faced by Kochin by properly choosing the perturbation potential. Gazdar found that for a value of the parameter $gh/u^2>1$, the wavelength of the wave does not change with different obstacles, but the amplitude does. The dependence of the amplitude on the different shapes of obstacles leads to an interesting situation: it has been observed that for certain shapes no wave-like motion occurs even when $gh>u^2$ a condition where wave motion is always expected to take place. Forbes [7], in 1981, investigated the flow over submerged semielliptical body. In 1981, Abd-El-Malek [8] treated the nonlinear problem of a flow over a ramp by applying Hilbert's transformation. In 1982, Forbes and Schwartz [9] considered a flow over a semicircular body, and 1983, Forbes [10] studied the effect of gravity and surface tension in his previous work. Boutros, Abd-El-Malek, and Masoud [11] in 1986, considered a flow over a triangular obstruction at the bottom and studied the effect of Froude number, bottom height, and slope of the triangle sides.

Recently, a considerable amount of work has been done by King and Bloor [12,13], Abd-El-Malek and Hanna [14], Faltas, Hanna and Abd-El-Malek [15].

2. FORMULATION OF THE PROBLEM

Consider the steady, two-dimensional flow of an ideal fluid in an infinite open channel with a nonuniform bottom, as shown in Figure (1).

The depth and speed of the flow far upstream being h and u, respectively.

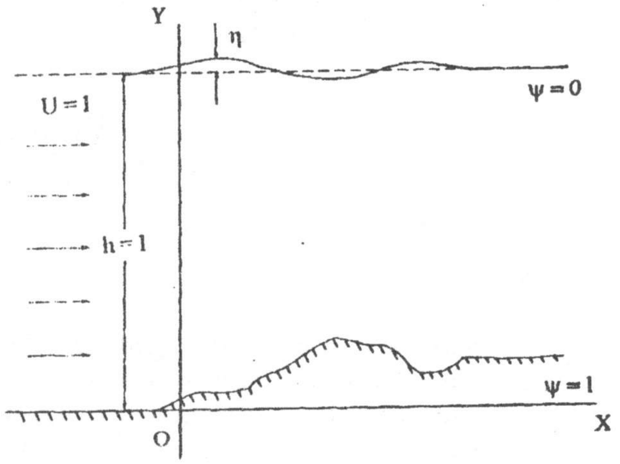

Figure 1. Physical configuration of a flow over an obstacle.

The surface tension of the fluid is t and g is the downward acceleration due to gravity. In the presence of surface tension a wave train may appear upstream; in this case the reference speed, and depth h are defined at points of zero curvature of the free surface upstream (the effect of surface tension on fluid at this point vanishes).

Let the velocity potential be stream ϕ and the stream function be Ψ, so the bottom line is $\Psi = -u\, h$ and the free surface $\Psi=0$.

Thus in the case of a simple harmonic corrugation in the bed

$$y = - h + \varepsilon \cos kx, \tag{2.1}$$

with wave number k and small amplitude ε, the origin being in the undisturbed surface, we assume

$$\phi = - x\, u + u\, (a \cosh ky + b \sinh ky) \sin kx, \tag{2.2}$$

and

$$\Psi = - y\, u + u\, (a \sinh ky + b \cosh ky) \cos kx, \tag{2.3}$$

where a and b are constants to be determined. The condition that (2.1) should be a stream-line is,

$$\varepsilon = - a \sinh kh + b \cosh kh. \tag{2.4}$$

At the free surface of the fluid $y = \eta$, the pressure-formula is given by,

$$\frac{p_o}{\rho} = const. - g\eta + \frac{t}{R\rho} + ku^2(a\cosh k\eta + b\sinh k\eta)\cos kx + O(\varepsilon^2) \qquad (2.5)$$

where p_o is the atmospheric pressure, ρ is the density of the fluid and R is the local radius of curvature of the free-surface defined by

$$R = \left[1 + (\frac{d\eta}{dx})^2\right]^{\frac{3}{2}} / (d^2\eta/dx^2) \simeq 1/[d^2\eta/dx^2] \qquad (2.6)$$

At the free surface from (2.3) and under the condition $\varepsilon << 1$ we get,

$$\eta = b\cos kx \qquad (2.7)$$

Hence the curvature is,

$$\frac{1}{R} = -bk^2\cos kx \qquad (2.8)$$

From (2.5) and (2.8) we get,

$$\frac{p_o}{\rho} = const. - gb\cos kx - (tb\, k^2\cos kx)/\rho + ku^2\, a\cos kx + o(\varepsilon^2). \qquad (2.9)$$

So the condition for a free surface gives,

$$ku^2\, a - b(g + t\, k^2/\rho) = 0. \qquad (2.10)$$

The equations (2.4) and (2.10) determine a and b. The profile of the free surface is given by,

$$\eta = \frac{\varepsilon\cos kx}{\cosh kh - \frac{g}{ku^2}\left[1 + \frac{tk^2}{g\rho}\right]\sinh kh} \qquad (2.11)$$

Introducing the dimensionless parameters of the problem, namely, the Froude number F,

$$F^2 = u^2/gh \qquad (2.12)$$

and surface tension number,

$$T = t/gh\rho \qquad (2.13)$$

and introducing the dimensionless quantities,

$$k' = kh,\ \eta' = \eta/h,\ y' = y/h,\ x' = x/h,\ \varepsilon' = \varepsilon/h, \qquad (2.14)$$

we get from (2.11) the normalized shape of the free surface. Having done so, we proceed to drop the primes so that hence forward all variables will be dimensionless.

Hence for the simple harmonic corrugation,

$$y = -1 + \varepsilon\cos kx \qquad (2.15)$$

The free surface profile is

$$\eta = \frac{\varepsilon\cos kx}{\cosh k - \frac{(1+Tk^2)\sinh k}{F^2 k}} \qquad (2.16)$$

This equation is invalid in the case k=0. It could be shown that as $k \to 0$, $y \to -1 + \varepsilon$ and $\eta \to \varepsilon$.

The profile disregarding the influence of surface tension can thus be deduced by putting T = 0 in equation (2.16), namely,

$$\eta = \frac{\varepsilon\cos kx}{\cosh k\left[1 - \frac{\tanh k}{F^2 k}\right]} \qquad (2.17)$$

3. EXAMPLES

We shall now consider two cases of bottom irregularities. In each case the function representing the bottom is first expanded in a Fourier series of the form

$$s(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L)$$

then the free surface profile η (x) is calculated according to equation (2.16)

CASE (1)

The bottom shape function, Figure (2), is given by

$$f(x) = \begin{bmatrix} -\varepsilon(x+L-d)/d & -L<x<-L+d \\ 0 & -L+d<x<L-d \\ \varepsilon(x-L+d)/d & L-d<x<L \end{bmatrix} \quad (3.1)$$

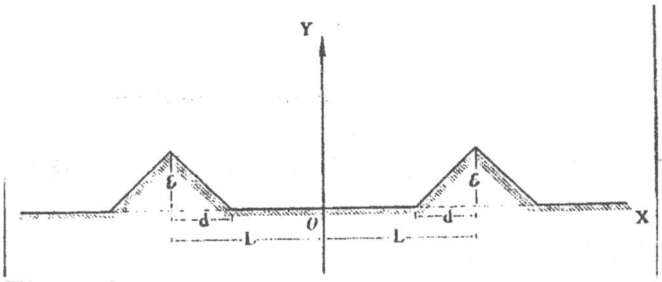

Figure 2. Bottom profile of case 1: (d≠L).

where $f(x+2L) = f(x)$. The expansion of $f(x)$ is

$$s(x) = \varepsilon d/2L +$$

$$\sum_{n=1}^{\infty} \frac{2\varepsilon L}{n^2\pi^2 d}[\cos n\pi - \cos(n\pi(L-d)/L)] \cos(\frac{n\pi x}{L})$$

and the free - surface profile is

$$\eta(x) = \varepsilon d/2L +$$

$$\sum_{n=1}^{\infty} \frac{2\varepsilon L}{n^2\pi^2 d}[\cos n\pi - \cos(n\pi(L-d)/L)]$$

$$\left[\frac{\cos\frac{n\pi x}{L}}{\cosh\frac{n\pi}{L} - (\sinh\frac{n\pi}{L})(1+\frac{Tn^2\pi^2}{L^2})/(\frac{F^2 n\pi}{L})}\right]$$

The results are illustrated in Figures (4),(5),(6) and (7).

A special interesting case is that when L = d. The bottom shape function has then the form, Figure (3),

$$f(x) = \begin{bmatrix} -\varepsilon x/L & -L<x<0 \\ \varepsilon x/L & 0<x<L \end{bmatrix} \quad (3.2)$$

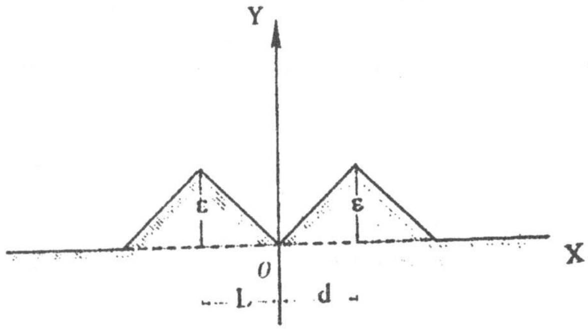

Figure 3. Bottom profile of case 1: (d=L).

where $f(x+2L) = f(x)$. The expansion of $f(x)$ is

$$s(x) = \varepsilon/2 + \sum_{n=1}^{\infty}\left[\frac{2\varepsilon}{n^2\pi^2}(\cos n\pi - 1)\right]\cos(\frac{n\pi x}{L})$$

and the free- surface profile is

$$\eta(x) = \varepsilon/2 + \sum_{n=1}^{\infty}\left[\frac{2\varepsilon}{n^2\pi^2}(\cos n\pi - 1)\right]$$

$$\left[\frac{\cos\frac{n\pi x}{L}}{\cosh\frac{n\pi}{L} - (\sinh\frac{n\pi}{L})(1+\frac{Tn^2\pi^2}{L^2})/(\frac{F^2 n\pi}{L})}\right]$$

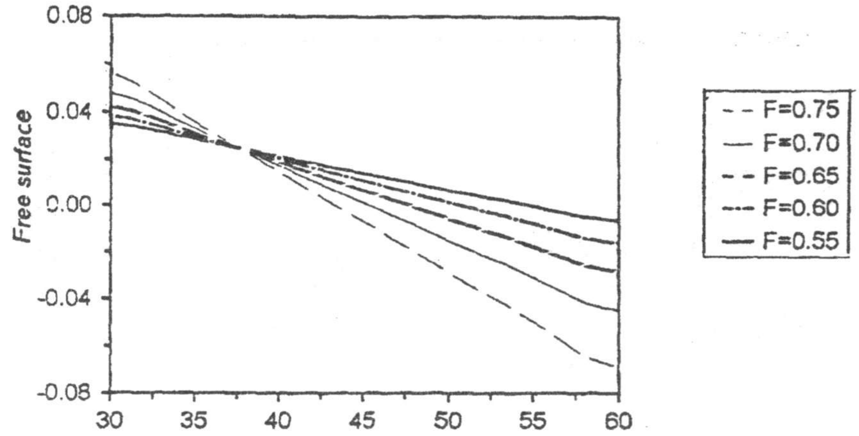

Figure 4. Effect of Froude number F on the free surface above ascending part of the irregularity. Case 1: (L=60, d=30, T=0, ϵ=.1).

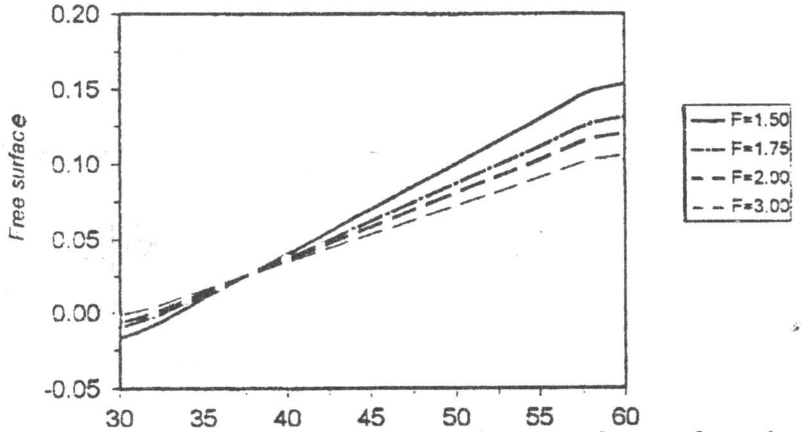

Figure 5. Effect of Froude number F on the free surface above ascending part of the irregularity. Case 1: (L=60, d=30, T=0,ϵ=.1).

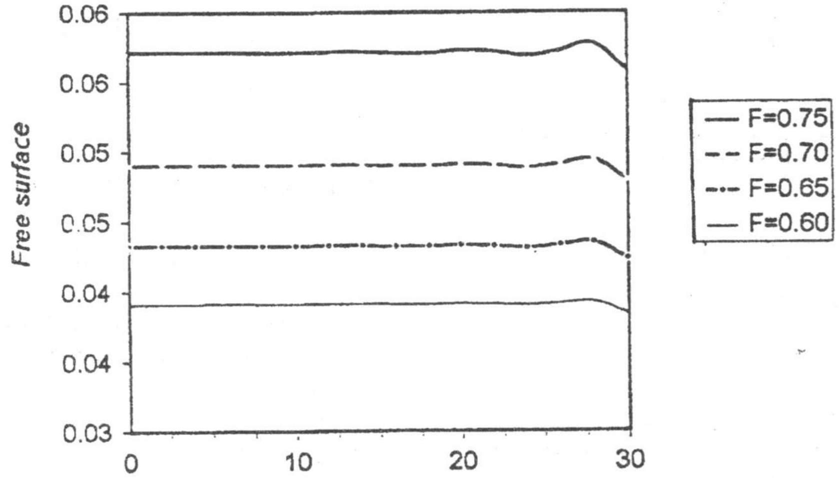

Figure 6. Effect of Froude number F on the free surface in the region between two humps. Case 1: (L=60, d=30, T=0,ϵ=.1).

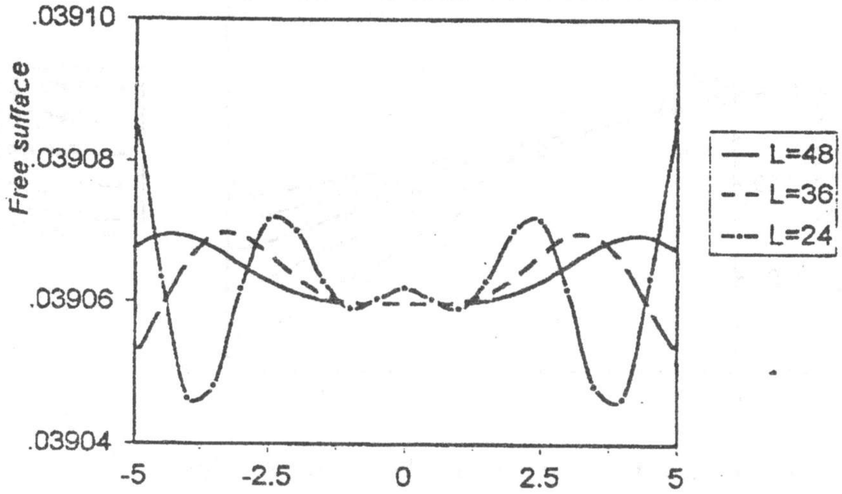

Figure 7. Effect of the periodic length L on the free surface in the region between two humps. Case 1: (d=L/2. T=0, F=0.6, ε=0.1).

CASE (2)

The bottom shape function is of the form, Figure (8)

$$f(x)=\begin{bmatrix} 0 & -L<x<-d \\ \varepsilon\,(1+\cos(\pi x/d))/2 & -d<x<d \\ 0 & d<x<L \end{bmatrix} \qquad (3.3)$$

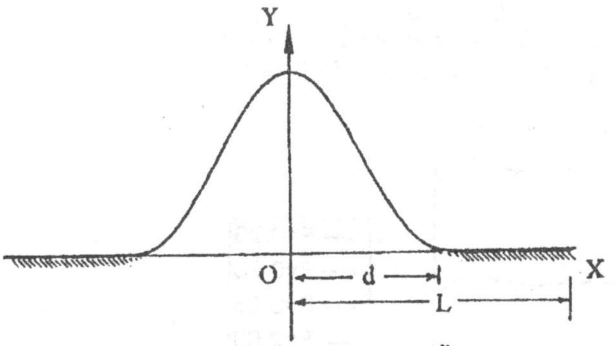

Figure 8. Bottom profile of case 2: (d"L).

where $f\,(x+2L)\; = f(x)$. The expansion of $f\,(x)$ is

$$s(x)=\varepsilon\, d/2L+\sum_{n=1}^{\infty}\left[\frac{\varepsilon}{n\pi}\sin(n\pi\, d/L)+\frac{\varepsilon\, d}{2\pi\,(n\,d-L)}\right.$$

$$\left.\sin(\pi\,(nd-L)/L)+\frac{\varepsilon\, d}{2\pi\,(nd+L)}\sin(\pi\,(n\,d+L)/L)\right].$$

$$\cos(n\pi\, x/L).$$

The corresponding free-surface profile is

$$\eta\,(x)=\varepsilon\, d/2L+\sum_{n=1}^{\infty}\left[\frac{\varepsilon}{n\pi}\sin(n\pi\, d/L)+\frac{\varepsilon\, d}{2\pi\,(nd-L)}\right.$$

$$\left.\sin(\pi\,(nd-L)/L)+\frac{\varepsilon\, d}{2\pi\,(nd+L)}\sin(\pi(n\,d+L)/L)\right].$$

$$\left[\frac{\cos\dfrac{n\pi x}{L}}{\cosh\dfrac{n\pi}{L}-(\sinh\dfrac{n\pi}{L})(1+\dfrac{Tn^2\pi^2}{L^2})/(\dfrac{F^2 n\pi}{L})}\right]$$

The results are illustrated in Figures (9) through (16).

SUMMARY AND DISCUSSION

The steady, two-dimensional free surface flow of an inviscid incompressible and irrotational fluid over a periodic irregular bottom has been investigated. The free surface profile is obtained, for supercritical and subcritical cases in the presence of surface tension. Two cases are considered, first the flow over irregular bottom of periodic triangular form and second the flow over a periodic smooth irregular bottom.

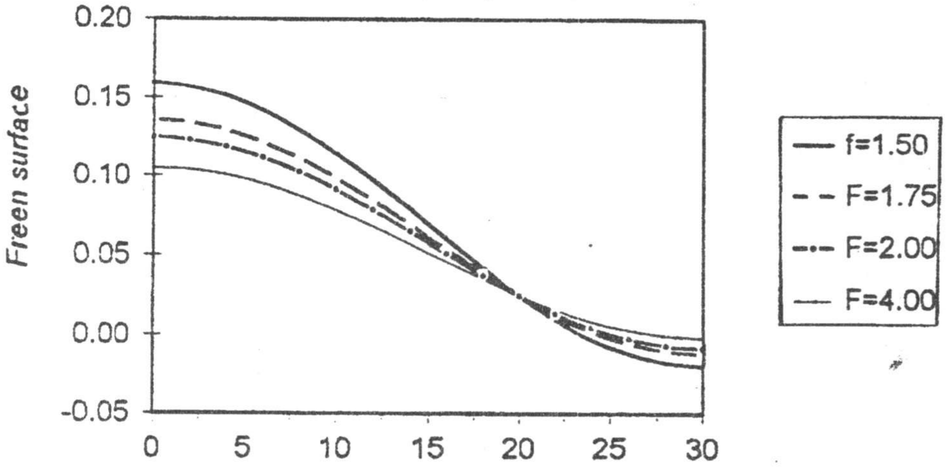

Figure 9. Effect of Froude number F on the free surface above ascending part of the irregularity. Case 2: (L=60, d=30, T=0,ϵ=0.1).

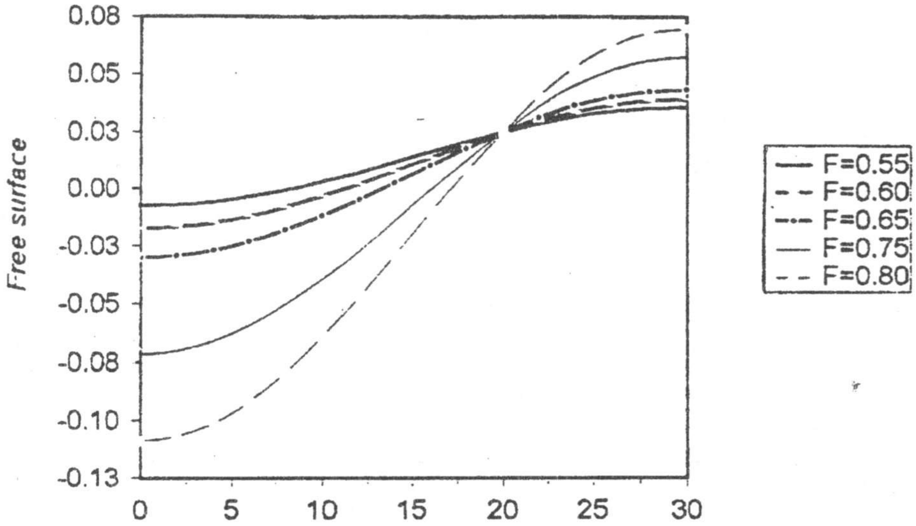

Figure 10. Effect of Froude number F on the free surface above of the irregularity. Case 2: (L=60, d=30, T=0,ϵ=0.1).

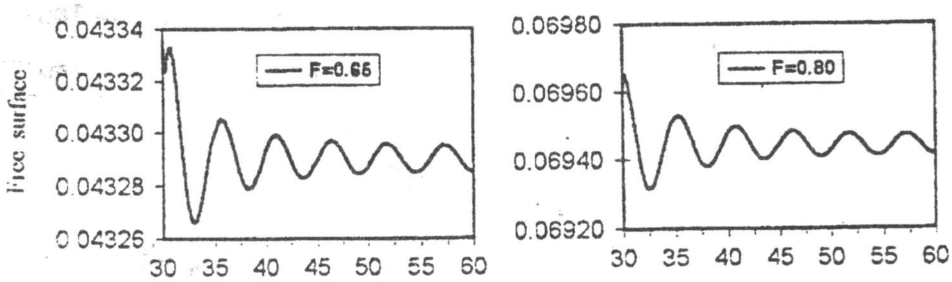

Figure 11. Effect of Froude number F on the free surface in the region between two humps. Case 2: (L=60, d=30, T=0,ϵ=.1).

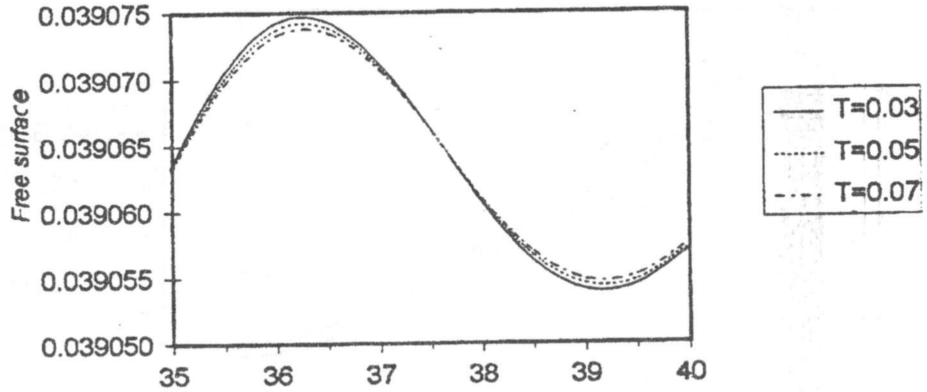

Figure 12. Effect of the surface tension T on the free surface in the region between two humps. Case 2. (L=60, d=30, F=0.6, ϵ=0.1).

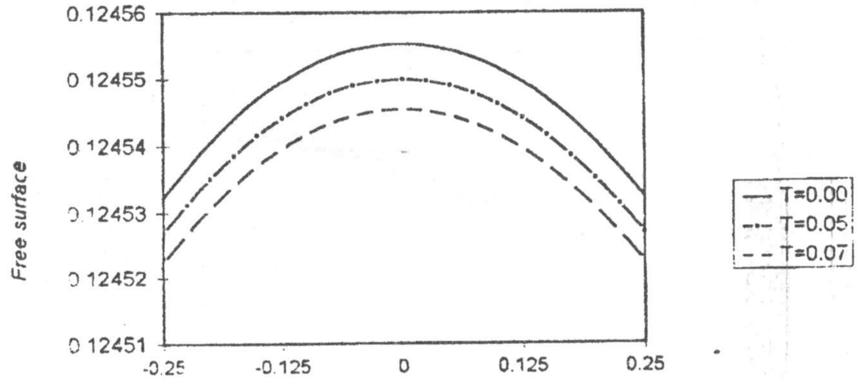

Figure 13. Effect of the surface tension T on the free surface just above the irregularity. Case 2: (L=60, d=30, F=2.0, ϵ=0.1).

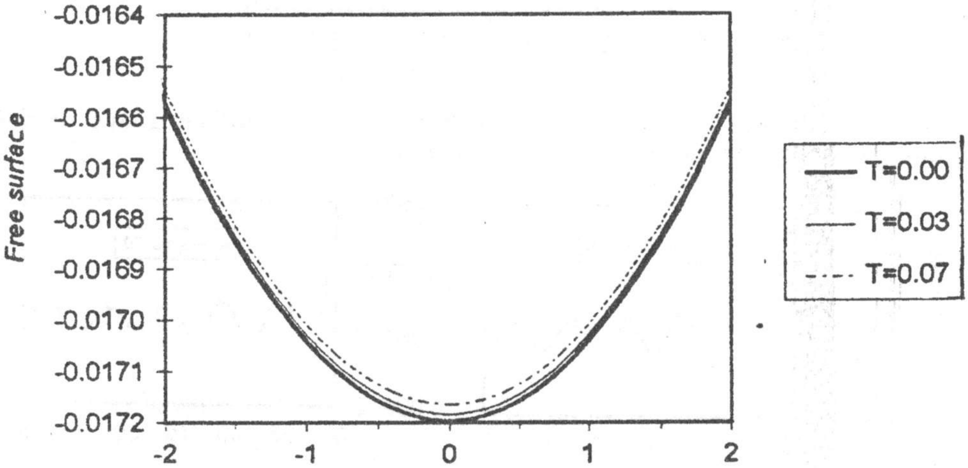

Figure 14. Effect of surface tension T on the free surface just above the irregularity. Case 2: (L=60, d=30, F=0.6,ϵ=0.1).

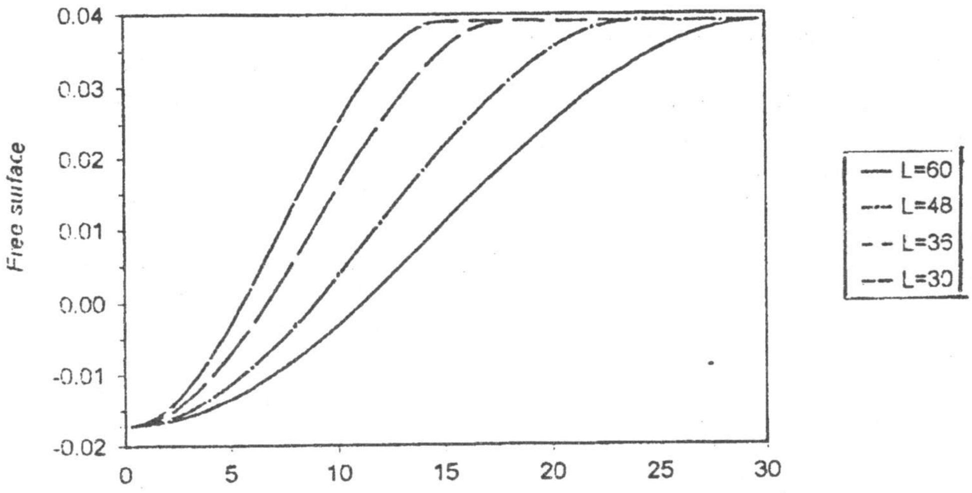

Figure 15. Effect of the periodic length L on the free surface above descending part of the irregularity. Case 2: (d=L/2, T=0,F=0.6, ϵ=0.1).

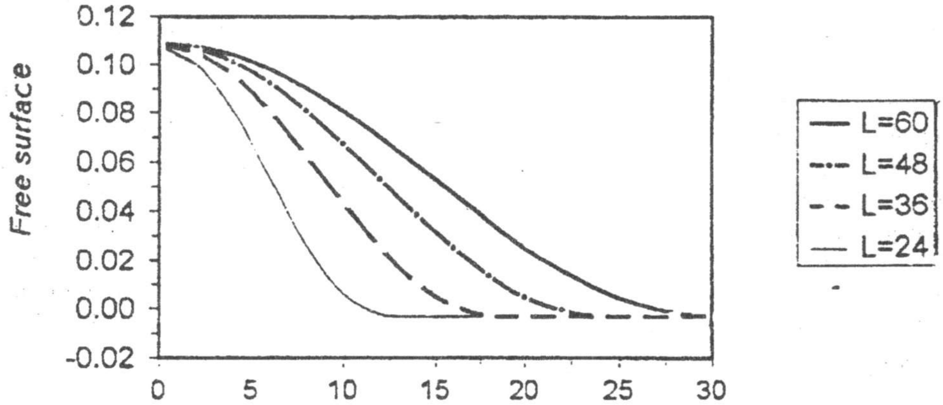

Figure 16. Effect of periodic length L on the free surface above descending part of the irregularity. Case 2: (D=L/2, T=0, F=3, ϵ=0.1).

The free surface response to this bottom topography is discussed and the solution is derived under the assumption that the height of the irregularities is small compared to the channel depth. Two different types of response are detected for subcritical flow ($F^2<1$) and supercritical flow ($F^2>1$).

Effect of Froude number F: For the subcritical flow it is found that a disturbance with depression appears immediately above the triangles in the first case and above the repeated bottom irregularity in the second case.

A small regular wave train with very small amplitude is generated in the region between two humps. Moreover, we observe that the depression of the wave increases, in magnitude, with the increase of Froude number.

Conversely, in the case of supercritical flow the free surface is in phase with the bottom configuration. The amplitude of the local disturbance above the summit of repeated triangles in the first case and above the summit of repeated irregular bottom in the second case decreases, in magnitude, with the increase of F.

The limiting form of the free surface profile for supercritical flow ($F\rightarrow\infty$) would have the same shape of the bottom which is in good agreement with the conjecture made by Forbes and Schwartz [9].

Effect of the surface tension T: It appears that the effect of the surface tension is very small and mainly affects the free surface when the radius of curvature is small.

For ($F^2<1$) the amplitude decreases with the increase of T_o, Conversely, for ($F^2>1$) the height of the free surface increases with the increase of T.

Effect of the periodic length L: As L decreases, the amplitude of the waves in the free surface profile above the region between two humps increases and consequently the wave length of these waves decreases.

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