ON FUZZY LINEAR REPRESENTATIONS OF FINITE GROUPS

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ABSTRACT

A linear representation ρ of a finite group G in afinite dimensional vector space V induces, through Zadeh's extension principle, a function, $\tilde{\rho}$, from I^G into $I^{GL(V)}$, where GL(V) is the group of all linear automorphisms of V. If *W* is a fuzzy subspace of V, the group of all fuzzy linear automorphisms of *W*, GL(*W*), is a subgroup of GL(V). *W* is said to be stable under the action of a fuzzy subgroup *A* of G if $\tilde{\rho}$ (*A*) is a fuzzy subset of GL(*W*) i.e. $\tilde{\rho}$ (*A*) is zero at every f inside GL(V) and outside GL(*W*). If *W* is stable under the action of *A*, then its support subspace is stable under the action of the support subgroup of *A* in the crisp sense . Finally , we show that if there are two stable fuzzy subspaces one of them is contained in the other, then the support subspace of the smaller one is a direct summand of the support subspace of the larger one.

Keywords: Linear representation of finite groups, Fuzzy subgroup, Fuzzy subspace, Stable fuzzy subspace, Fuzzy direct summand.

1. PREREQUISITES

Throughout, G will be a finite group with identity e, and V will be a vector space over a field F. 0 will denote both the zeros of F and V. GL(V) will be the group of all linear automorphisms of V. A group homomorphism ρ: G→ GL(V) is called a *linear representation* of G in V. See [5] .

A fuzzy subset *A* of G is called a *fuzzy subgroup* of of G if it satisfies for all x , y ε G:

(G 1) $A(xy) \geq A(x) \wedge A(y)$,

(G 2) $A(x^{-1}) = A(x)$,

(G 3) $A(e) = 1$.

A fuzzy subset *W* of V is called a *fuzzy subspace* of of V if it satisfies for all u , v ε V and α, β ε F:

(S 1) $W(\alpha u + \beta v) \geq W(u) \wedge W(v)$,

(S 2) $W(0) = 1$.

Let X , Y be any two sets, a function f: X → Y is extended by means of Zadeh's extension principle to a function $\tilde{f}: I^X \to I^Y$ defined by ; for all A ε I^X and

$$y \in Y \text{ let } \tilde{f}(A)(y) = \sup\{A(x) : f(x) = y\}. \quad (1)$$

See [2], [9].

Remark 1.1

If f: X → Y is a function and A is a crisp subset of X, then $\tilde{f}$(A) is also a crisp subset of Y and $\tilde{f}$(A) = f(A).

If X ⇔ Y and A ε I^Y, considering *A* as a function from Y into I, the restriction $A|_X$ is a fuzzy subset of X.

If *A* is a fuzzy subset of a set X, the *support subset* of *A* in X, denoted by $\bar{A}$, is;

$$\bar{A} = \{x \; \varepsilon \; X : A(x) > 0\}. \qquad (2)$$

One easily prove that:

$$\tilde{f}(A) = f(A). \qquad (3)$$

Lemma 1.2 [see 2]

1) If W is a fuzzy subspace of V, then $\bar{W}$ is a subspace of V.
2) If A is a fuzzy subgroup of G, then $\bar{A}$ is a subgroup of G.

If W is a fuzzy subspace of V, GL(W) will denote the set of all linear automorphisms of W, see Definition 3.3.1 of [6] So;

$$GL(W) = \{f: f \; \varepsilon \; GL(V), \; \tilde{f}(W) = W\}. \qquad (4)$$

Lemma 1.3 Let W be a fuzzy subspace of V, then GL(W)={f ε GL(V): W(v)= W(f(v)) for all v ε V}.

Proof Straight forward from (1) and f being an invertible homomorphism.

Lemma 1.4 GL(W) is a subgroup of GL(V).

Lemma 1.5 If f ε GL(W), the restriction of f to $\bar{W}$, $f|_{\bar{W}}$, is an automorphism of $\bar{W}$.

Proof Let v ε $\bar{W}$, then $W(v) = W(f(v)) > 0$, so f(v) ε $\bar{W}$.

From the above lemma we can see that ; GL (W) is a subgroup of GL ($\bar{W}$).

An operation o on a set X is extended, by Zadeh's extension principle [9], to an operation on I^X denoted by the same o as follows ; for all A, B ε I^X and each x ε X let

$$(A \circ B)(x) = \sup \{A(a) \wedge B(b): a \circ b = x\}. \qquad (5)$$

See [2], [4] and [7] for all the details.

The operation of G and the two operations of both V and F are extended to operations on the appropriate fuzzy subsets. These extensions are denoted by the same notations for the original operations.

If W_1, W_2 are fuzzy subspaces of V such that $W_1 \cap W_2 = 1_o$, we say that their sum $W_1 + W_2$ is *direct* and it is denoted by: $W_1 \oplus W_2$. Note that; a_x is the fuzzy subset taking value a ($0 \leq a \leq 1$) at x and 0 elsewhere.

Lemma 1.6 Let W_1, W_2 be two fuzzy subspaces of V such that $W_1 \cap W_2 = 1_o$. Then $\bar{W}_1 \oplus \bar{W}_2 = \overline{(W_1 \oplus W_2)}$. Moreover for all v ε V, $(W_1 \oplus W_2)(v) = W_1(c) \wedge W_2(d)$, where c and d are the unique vectors, c ε $\bar{W}_1$, d ε $\bar{W}_2$, such that v = c + d.

Proof It is straight forward to show that $\bar{W}_1 \cap \bar{W}_2 = \{0\}$. Let v ε $\bar{W}_1 \oplus \bar{W}_2$, so v = a + b with $W_1(a)>0$, $W_2(b) > 0$. Then $(\bar{W}_1 \oplus \bar{W}_2)(v) =$
$= \sup\{W_1(c) \wedge W_2(d): c+d=v\} \geq W_1(a) \wedge W_2(b) > 0$.
Let v ε $(\overline{W_1 \oplus W_2})$, then
$(W_1 \oplus W_2)(v) = \sup\{W_1(c) \wedge W_1(d): c+d=v\} > 0$.
Then, there are a , b ε V and a + b = v such that $W_1(a) > 0$ and $W_2(b) > 0$. The second part comes from; if v = c + d with c ε $\bar{W}_1$ and d ε $\bar{W}_2$, then c and d are the only elements such that: $\bar{W}_1(c) \wedge \bar{W}_2(d) > 0$.

2. STABLE FUZZY SUBSPACE

Definition 2.1

A fuzzy subspace W of V is said to be *stable under the action of a fuzzy subgroup A* of G with respect to the linear representation ρ: G → GL (V) if it satisfies for all f ε GL (V) and f ∉ GL (W) ;

$$\tilde{\rho}(A)(f) = 0. \qquad (6)$$

Equation (6) means; for every f ε GL (V)/GL (W) we have:

$$\tilde{\rho}(A)(f) = \sup\{A(a): a \; \varepsilon \; G, \; \rho(a) = f\} = 0.$$

Lemma 2.2

1) W is stable under the action of A if and only if it is stable under the action of $\bar{A}$.
2) If W is stable under the action of A then $\bar{W}$ is also stable under the action of A.

Proof

1) Assume that W is stable under the action of A. To prove that W is stable under the action of $\bar{A}$; let $f \varepsilon GL(W)$, then it is enough to prove $f \notin \rho(\bar{A})$ because $\rho(\bar{A}) = \tilde{\rho}(\bar{A})$. This is straight forward from (3) and (6).
On the other hand ; assume that W is stable under the action of $\bar{A}$. Let $f \notin GL(W)$, then (3) and (6) imply also that $\rho(A)(f) = 0$.
2) If $f \notin GL(\bar{W})$ then $f \notin GL(W)$. This proves the second part of the lemma.

Proposition 2.3

If a fuzzy subspace W is stable under the action of a fuzzy subgroup A. Then $\bar{W}$ is stable under the action of A in the crisp sense i.e. $\rho(a)(w) \varepsilon \bar{W}$ for all $a \varepsilon \bar{A}$ and $w \varepsilon \bar{W}$.

Proof Suppose not, then
$W(\rho(a)(w)) = 0$ for some $a \varepsilon \bar{A}$, $w \varepsilon \bar{W}$. So
$0 = W(\rho(a)(w)) \neq W(w) > 0$. Then, by Lemma 1.3: $\rho(a) \notin GL(W)$. So, since W is stable;
$\tilde{\rho}(A)(\rho(a)) = 0$, but we have:
$\tilde{\rho}(A)(\rho(a)) = \sup\{A(x): \rho(x) = \rho(a)\} \geq A(a) > 0$, which is a contradiction.

For the linear representation $\rho : G \to GL(V)$, if $a \varepsilon G$ the linear automorphism $\rho(a)$ will be denoted by ρ_a.

Proposition 2.4

Let W be a stable fuzzy subspace under the action of a fuzzy subgroup A. Then $\rho_a \varepsilon GL(W)$ for all $a \varepsilon A$.

Proof Suppose not, then there is some $a \varepsilon A$ with $\rho_a \notin GL(W)$. So, $\tilde{\rho}(A)(\rho_a) = 0$. But we have:
$\tilde{\rho}(A)(\rho_a) = \sup\{A(x): \rho(x) = \rho_a\} \geq A(a) > 0$, which is a contradiction.

Since the set of linear transformations $\{\rho_a: a \varepsilon G\}$ is finite ; the sum $\Sigma \rho_a$ is defined and is also an automorphism of V. So we get the following lemma which is important in the proof of the next theorem.

Lemma 2.5 Let $\rho_G = \Sigma \rho_a$ then $\rho_x \rho_G = \rho_G \rho_x = \rho_G \varepsilon GL(W)$ for all $x \varepsilon G$ and any stable fuzzy subspace W.

Theorem 2.6 Let $U \subseteq W$ be two fuzzy subspaces of V both are stable under the action of a fuzzy subgroup A. Then there is a fuzzy subspace V of V satisfying:
1) V is stable under the action of A.
2) $V \subseteq W$.
3) The sum $U \oplus V$ is direct i.e. $U \cap V = 1_0$.
4) $\bar{U} \oplus \bar{V} = \bar{W}$.

Proof Let W be a subspace of V stable under the action of $\bar{A}$ such that $\bar{W} = \bar{U} \oplus$ W, see Theorem 1 of 1.3 of [5]. Define a fuzzy subset V of V as follows: for all $v \varepsilon V$;

$$V(v) = \begin{cases} W(\rho_g(V)) & \text{if } v \varepsilon w, \\ 0 & \text{if } v \notin W \end{cases}.$$

We claim that V is a fuzzy subspace of V. Let u, v ε V and $\alpha, \beta \varepsilon F$. Consider $V(\alpha u + \beta v)$, we may assume both u and v are in W. Then

$$V(\alpha u + \beta v) = W(\rho_G(\alpha u + \beta v)) =$$

$$= W(\alpha\rho_G(u) + \beta\rho_G(v)) \geq W(\rho_G(u)) \wedge W(\rho_G(v))$$

$$= V(u) \wedge V(v).$$ This proves our claim.

We claim again that $\bar{V}$ = W. Let $w \varepsilon W \subseteq \bar{W}$. Then $\rho_G(w) \varepsilon \bar{W}$, because $\bar{W}$ is stable under the action of $\bar{A}$ by Proposition 2.3. Then $V(w) = W(\rho_G(w)) > 0$, which means $w \varepsilon \bar{V}$. On the other hand, let $w \varepsilon \bar{V}$, so $V(w) > 0$ which means $w \varepsilon$ W. This

proves the second claim. Our third claim is that, $U \cap V = 1_o$. It is straight forward to prove that $U \cap V = 1_o$ if and only if $\bar{U} \cap \bar{V} = \{0\}$. Finally, it remains to prove that V is stable under the action of A. Suppose not. Then there is $f \notin GL(V)$ such that
$\tilde{\rho}(A)(f) > 0$. Then
$\tilde{\rho}(A)(f) = \sup\{A(a): a \in G, \rho_a = f\} > 0$.

So there is $x \in \bar{A}$ such that $\rho_x = f \notin GL(V)$. Then there is $v \in V$ such that
$\tilde{\rho}_x(V)(v) \neq V(v)$ because $\tilde{\rho}_x(V) \neq V$. Then;
$\tilde{\rho}_x(V)(v) = \sup\{V(u): \rho_x(u) = v\} = V(\rho_x^{-1}(v))$. So, $V(\rho_y(v)) \neq V(v)$, where y is the inverse of x in $\bar{A}$. We have two cases for v.

Case 1: $v \notin W$ then $\rho_y(v) \notin W$ because W is stable under the action of $\bar{A}$ and we get
$V(\rho_y(v)) = V(v) = 0$.

Case 2: $v \in W$ then $\rho_y(v) \in W$, so we get
$V(\rho_y(v)) = W(\rho_G \rho_y(v)) = W(\rho_G(v)) = V(v)$.

In both cases, we get a contradiction and V is stable under the action of A.

3. AN EXAMPLE

let G be a finite group of order n which is equal to the dimension of V. Let $\{v_x\}_{x \in G}$ be a basis for V indexed by the elements of G. Define the linear representation
$\rho: G \to GL(V)$ by ; $\rho(x)(v_y) = v_{xy}$ for all x, y $\in$ G and extend it by linearity. This linear representation is called the *regular representation* of G. Let W be the subspace of V generated by $w = \Sigma v_x$. Then $\rho(a)(v) = v$ for all $v \in W$ and $a \in G$. Let W be the fuzzy subset of V defined by

$$W(v) = \begin{cases} 1/2 & \text{if } v \in W, v \neq 0, \\ 1 & \text{if } v = 0 \\ 0 & \text{otherwise}. \end{cases}$$

Then W is a fuzzy subspaces of V with $\bar{W} = W$. $GL(W) = \{f: f \to GL(V),\ W(v) = W(f(v)) \text{ for all } v \in V\}$. So $f \in GL(W)$ if and only if $f(w) = \alpha w$ for some $\alpha \in F$. In particular $\{\rho(x): x \in G\} \subseteq GL(W)$. Now, let A be any fuzzy subgroup of G. Let $f \in GL(V)$, then;
$\tilde{\rho}(A)(f) = \sup\{A(x): \rho(x) = f\}$.
Then, $\tilde{\rho}(A)(f)$ may be different from zero only at f for which f (w) = w. This means W is stable under any fuzzy subgroup of G.

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