"ON THE MOTION OF A PENDULUM WITH A MOVING SUPPORT ORBITING A CIRCLE"

F.Z. Habieb
Department of Engineering Mathematics and Physics, Faculty of Engineering,
Alexandria University, Alexandria, Egypt.

ABSTRACT

In the present study, the problem of the plane pendulum with a rapidly circular moving point of suspension has been considered. The study is made for arbitrary angular displacements of the pendulum. By applying a Kapitza type transformation, the canonical equations have been reduced to an autonomous system of differential equations. A general solution of this autonomous system has been obtained in terms of elliptic functions.

Keywords: Canonical equation, Nonlinear evolution equations, Kapitza mapping, Slowly varying procedure.

1- INTRODUCTION

Pendulums with oscillating base motions have been the subject of many investigations. Stephenson [1] considered the plane pendulum subjected to a vertical oscillation. He presented an explanation of the inverted position. Lowenstern [2] investigated the spherical pendulum with an oscillating base. Phelps and Hunter [3] presented a thorough study of the plane pendulum subjected to a vertical oscillation at an unrestricted frequency. Miles [4] considered the stability of the downward vertical position of the spherical pendulum subjected to a horizontal oscillation. Ryland and Meirovitch [5] considered stability of the vertical position of the plane flexible pendulum with a vertical harmonic oscillation at an unrestricted frequency. Schmidt [6,7,8] presented variations of pendulum with oscillating base motion. Kapitza [9] studied the plane pendulum when the support is excited vertically with high frequency and the angular displacement of the pendulum is arbitrary. Habieb [10] treated the case of a pendulum when its base is vibrating horizontally at a rapid rate. Several authors have investigated the plane pendulum subjected to non-harmonic oscillations. In this paper, we consider the motion of a pendulum with a moving support orbiting a circle. Here the pivot is describing a circle with a small radius that moves with a high angular velocity. This angular velocity is large compared to the natural frequency of the pendulum under the influence of gravity.

2- HAMILTONIAN CANONICAL EQUATIONS

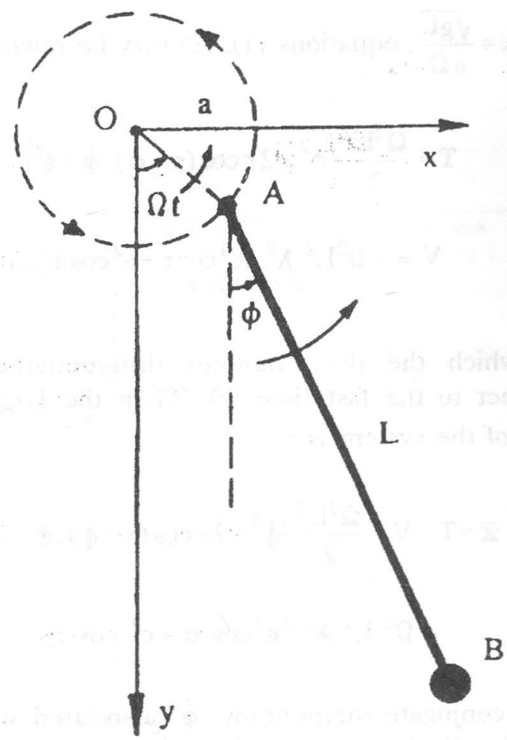

Figure 1.

Let us consider the plane pendulum shown in the annexed Figure (1) that consists of a particle (B) of unit mass connected with a massless rigid rod (AB) of length (L). The point of suspension (A) moves in a circle, of center (O) and radius (a) with a constant angular velocity (Ω). The radius (a) is small compared with the pendulum length (L) and (Ω) is very large compared with the natural frequency of the pendulum under the influence of gravity. If (ϕ) is the angular displacement of the pendulum at any instant (t), then the kinetic energy (T) takes the form:

$$T=\frac{1}{2}L^2\left(\frac{d\phi}{dt}\right)^2 + aL\Omega\cos(\Omega t-\phi).\left(\frac{d\phi}{dt}\right)+\frac{1}{2}a^2\Omega^2. \quad (1)$$

The potential energy (V)-taking as a reference level the horizontal plane passing through the centre (O) - is given by:

$$V = -g\{a\cos\Omega t + L\cos\phi\}. \quad (2)$$

Defining the fast (dimensionless) time (τ) by $\tau = \Omega t$ and introducing the following constants $\epsilon = \frac{a}{L}, \lambda = \frac{\sqrt{gL}}{a\Omega}$, equations (1), (2) may be rewritten as

$$T=\frac{\Omega^2L^2}{2}\{\dot{\phi}^2+2\epsilon\cos(\tau-\phi).\dot{\phi}+\epsilon^2\}, \quad (3)$$

$$V = -\Omega^2L^2\lambda^2\{\epsilon^3\cos\tau+\epsilon^2\cos\phi\}. \quad (4)$$

in which the dot denotes differentiation with respect to the fast time (τ). Then the Lagrangian ($\mathcal{L}$) of the system is :

$$\mathcal{L}=T-V=\frac{\Omega^2L^2}{2}\{\dot{\phi}^2+2\epsilon\cos(\tau-\phi).\dot{\phi}+\epsilon^2\}$$

$$+\Omega^2L^2\lambda^2\{\epsilon^2\cos\phi+\epsilon^3\cos\tau\}. \quad (5)$$

The conjugate momentum ($\hat{\phi}$) associated with the angular displacement (ϕ) is:

$$\hat{\phi}=\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=\Omega^2L^2\{\dot{\phi}+\epsilon\cos(\tau-\phi)\}. \quad (6)$$

Now, we construct the Hamiltonian function $H(\phi,\hat{\phi},t)$ of the system as:

$$H = \hat{\phi}\dot{\phi} - T^* + V, \quad (7)$$

where T^* is the modified kinetic energy that is defined by [11]

$$T^* = T - \frac{dF}{dt} \quad (8)$$

in which F is an arbitrary function.

Let $$F = \frac{\Omega L^2}{2}\{\epsilon^2\tau-2\epsilon\sin(\tau-\phi)\}, \quad (9)$$

then differentiating (9) with respect to (t) gives:

$$\frac{dF}{dt}=\Omega\frac{dF}{d\tau}=\frac{\Omega^2L^2}{2}\{\epsilon^2-2\epsilon\cos(\tau-\phi)+2\epsilon\dot{\phi}\cos(\tau-\phi)\}. \quad (10)$$

Substituting (3), (10) into (8) we obtain

$$T^* = \frac{\Omega^2L^2}{2}\{\dot{\phi}^2+2\epsilon\cos(\tau-\phi)\} \quad (11)$$

and substituting in (7) we obtain

$$H = H_o - \epsilon H_1 - \epsilon^2H_2 - \epsilon^3H_3, \quad (12)$$

where

$$H_o = \frac{\hat{\phi}^2}{2\Omega^2L^2}, \quad (13\text{-a})$$

$$H_1 = \Omega^2L^2\cos(\tau-\phi), \quad (13\text{-b})$$

$$H_2 = \frac{\Omega^2L^2}{2}\{\cos^2(\tau-\phi)+2\lambda^2\cos\phi\}, \quad (13\text{-c})$$

$$H_3 = \Omega^2L^2\lambda^2\cos\tau. \quad (13\text{-d})$$

Invoking Hamiltons canonical equations

$$\left\{\dot{\phi}=\frac{\partial H}{\partial \hat{\phi}},\dot{\hat{\phi}}=-\frac{\partial H}{\partial \phi}\right\}[12],$$

we get the equations of motion

$$\frac{d\phi}{d\tau}=\dot{\phi}=\frac{\hat{\phi}}{\Omega^2 L^2}, \tag{14-a}$$

$$\frac{d\hat{\phi}}{d\tau}=\dot{\hat{\phi}}=\epsilon\,\Omega^2L^2\{\sin(\tau-\phi)-\epsilon\lambda^2\sin\phi$$

$$+\epsilon\sin(\tau-\phi)\cos(\tau-\phi)\}. \tag{14-b}$$

3- THE NONLINEAR SYSTEM BEHAVIOUR

For the determination of the solution of equations (14-a,b) we introduce the new variables $\theta(\tau)$ and $\Psi(\tau)$ in the following manner:

$$\phi=\theta-\epsilon\sin(\tau-\theta), \tag{15-a}$$

$$\dot{\phi}=\epsilon\{\Psi-\cos(\tau-\theta)\}. \tag{15-b}$$

The transformation (15) is motivated by ideas due to Kapitza [9]. Substituting Equations (15) in Equations (14) and retaining terms of the lowest order of magnitude in ϵ lead to

$$\frac{d\theta}{d\tau}=\dot{\theta}=\epsilon\Psi, \tag{16-a}$$

$$\frac{d\Psi}{d\tau}=\dot{\Psi}=\epsilon\{\Psi\sin(\tau-\theta)-\lambda^2\sin\theta\}. \tag{16-b}$$

Now, applying the method of averaging [14,15] for equations (16-a,b) postulate the following autonomous system of differential equations

$$\frac{d\bar{\theta}}{d\tau}=\epsilon\bar{\Psi}, \tag{17-a}$$

$$\frac{d\bar{\Psi}}{d\tau}=-\epsilon\lambda^2\sin\bar{\theta}. \tag{17-b}$$

To integrate equations (17-a,b), we notice that

$$\frac{d\bar{\theta}}{d\bar{\Psi}}=-\frac{\bar{\Psi}}{\lambda^2\sin\bar{\theta}}. \tag{18}$$

This leads to

$$\bar{\Psi}^2-2\lambda^2\cos\bar{\theta}=\text{const.}=C \tag{19}$$

in which the constant (C) depends on the initial conditions of the motion.

Substituting equation (19) into equation (17-a) we obtain

$$\left(\frac{d\bar{\theta}}{d\tau}\right)^2=\epsilon^2(C+2\lambda^2\cos\bar{\theta})=4\epsilon^2\lambda^2\left\{\frac{C+2\lambda^2}{4\lambda^2}-\sin^2\frac{\bar{\theta}}{2}\right\}. \tag{20}$$

For the solution of equation (20) there are three cases:

(3-a) *First Case*:

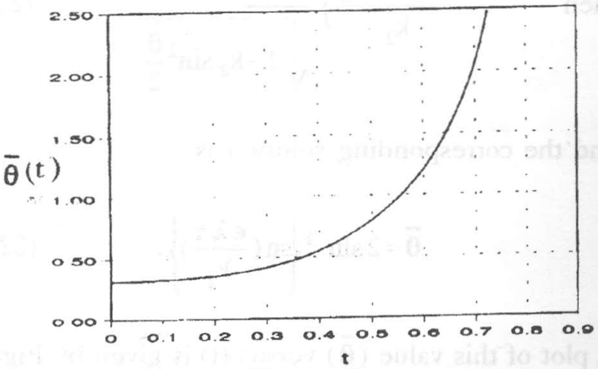

Figure 2.

$$\frac{C+2\lambda^2}{4\lambda^2}=k_1^2<1 \ \text{or}\ \Omega<\frac{\omega_o}{\epsilon}\sqrt{\frac{2}{C}};$$

then
$$2\epsilon\lambda\tau=\int\frac{d\bar{\theta}}{\sqrt{k_1^2-\sin^2\frac{\bar{\theta}}{2}}} \tag{21-a}$$

and the corresponding solution is

$$\bar{\theta}=2\sin^{-1}\{k_1\,\text{sn}(\epsilon\lambda\tau)\}. \tag{21-b}$$

A plot of this value $(\bar{\theta})$ versus (t) is given by Figure (2) at k_1=0.05 and L=100 cm.

(3-b) *Second Case:*

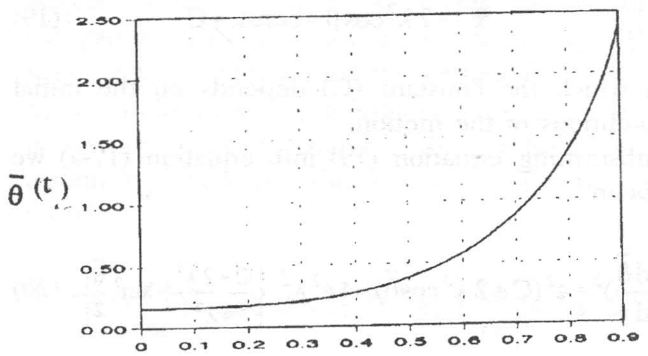

Figure 3.

$$\frac{4\lambda^2}{C+2\lambda^2}=k_2^2<1 \text{ or } \Omega > \frac{\omega_o}{\epsilon}\sqrt{\frac{2}{C}};$$

then
$$\frac{2\epsilon\lambda\tau}{k_2}=\int\frac{d\bar{\theta}}{\sqrt{1-k_2^2\sin^2\frac{\bar{\theta}}{2}}} \quad \text{(22-a)}$$

and the corresponding solution is

$$\bar{\theta}=2\sin^{-1}\left\{sn\left(\frac{\epsilon\lambda\tau}{k_2}\right)\right\}. \quad \text{(22-b)}$$

A plot of this value $(\bar{\theta})$ versus (t) is given by Figure (3) at k_2=0.1 and L=100 cm.

(3-c) *Third Case:*

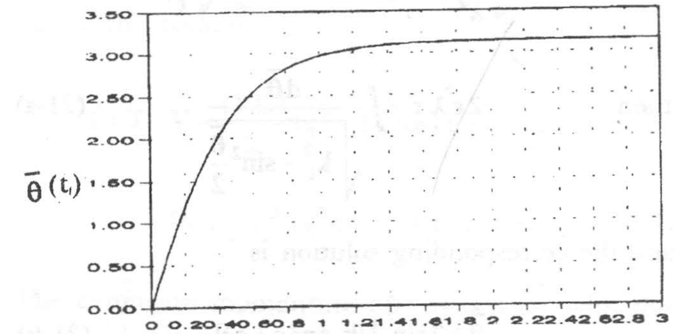

Figure 4.

$$\frac{C+2\lambda^2}{4\lambda^2}=k_3^2=1 \text{ or } \Omega=\frac{\omega_o}{\epsilon}\sqrt{\frac{2}{C}};$$

then
$$2\epsilon\lambda\tau=\int Sec\frac{\bar{\theta}}{2}d\bar{\theta} \quad \text{(23-a)}$$

and the corresponding solution is

$$\bar{\theta}=4\tan^{-1}(e^{\epsilon\lambda\tau})-\pi. \quad \text{(23-b)}$$

A plot of this value $(\bar{\theta})$ versus (t) is given by Figure (4) at L=100 cm.

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