FINITENESS CONDITIONS ON FUZZY MODULES

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ABSTRACT

Chain conditions play an important role in the ring theory ever since Emmy Noether and Emil Artin introduced their modules and rings. When we switch to fuzzy set theory; the natural question is: Need we have similar finiteness conditions? The main result of this paper is an answer " yes " to this question . When we work with fuzzy modules and rings the word " finiteness " appears strongly. In this paper, the fuzzy chain conditions are introduced (both a.c.c. and d.c.c.) and it shown that many of the theorems in the crisp case work in the fuzzy case. For example ; a module is fuzzy Noetherian if and only if its submodules and quotient-modules are fuzzy Noetherian . Also , for fuzzy Artinian rings fuzzy prime ideals are fuzzy maximal .

Keywords: Fuzzy submodule, Fuzzy ideal, Fuzzy homomorphism, Fuzzy finitely generated fuzzy submodule, Finite fuzzy submodule, Finite fuzzy ascending and descending chain conditions, Finite fuzzy maximal and minimal conditions .

1 . INTRODUCTION

In what follows, R will denote a ring with identity 1. Modules over R will be denoted by capital letters A, B, C, M. Their fuzzy submodules will be denoted by capital italic letters *A, B, C, M* respectively. Recall, a fuzzy submodule A of A is a fuzzy subset of A which satisfies for all a , b ε A and $\alpha, \beta \;\varepsilon$ R:

(S 1) $A(\alpha a + \beta b) \geq A(a) \wedge B(b)$,

(S 2) $A(0) = 1$.

Let X be any set and let $\chi \;\varepsilon\; I^X$. The support subset of χ, denoted by $\bar{X}$ is defined to be:

$$\bar{X} = \{x : x \;\varepsilon\; A,\ \bar{X}(x) > 0\} \qquad (1)$$

It is well known that if A is a fuzzy submodule of A, then $\bar{A}$ is a submodule of A.

Zadeh's extension principle, [7], is applied to the two operations of A to get another two operations on the set of all fuzzy subsets of A (I^A). These new two operations are denoted by + and juxtaposition as well and defined as follows: for all $A, B \;\varepsilon\; I^A$ and for every $\alpha \;\varepsilon$ R and a ε A let

$$(A + B)(a) = \sup\{A(x) \wedge B(y) : x + y = a\}, \qquad (2)$$

$$(\alpha A)(a) = \begin{cases} \sup\{A(x) : x \varepsilon A,\ \alpha x = a\} & \text{if } \alpha \neq 0, \\ 1 & \text{if } \alpha = 0, a = 0, \\ 0 & \text{if } \alpha = 0, a \neq 0. \end{cases} \qquad (3)$$

Then, if $A_1, A_2, \ldots, A_n \;\varepsilon\; I^A$ we get the sum:

$$(A_1 + A_2 + \ldots + A_n)(a) = \sup\{A(x_1) \wedge A(x_1) \wedge \ldots \wedge A(x_n) : \textstyle\sum x_i = a\}.$$

If all A_i 's are fuzzy submodules of A then so is their sum, see [2].

A function f: A $\rightarrow$ B is extended to $\tilde{f}: I^A \rightarrow I^B$, where $\tilde{f}$ is defined as follows: for all $A \;\varepsilon\; I^A$ and b ε B let

$$\tilde{f}(A)(b) = \sup\{A(x) : x \in A,\ f(x) = b\} \qquad (4)$$

If $B \in I^B$; define $\tilde{f}^{-1}(B)$ to be the fuzzy subset of A defined by; for all a ε A let

$$\tilde{f}^{-1}(B)(a) = B(f(a)) \qquad (5)$$

2. FUZZY HOMOMORPHISMS

An R-module homomorphism f from A into B is called a *fuzzy homomorphism* from *A* into *B* if $\tilde{f}(A) \subseteq B$. It is denoted by f: $A \to B$. See Definitions 3.3.1 and 7.2.1 of [4]. In these definitions, our fuzzy homomorphisms are called weak fuzzy homomorphisms. If f: $A \to B$ is a fuzzy homom-orphism, we call $\check{f}(A)$ a fuzzy homomorphic image of *A* or simply a fuzzy homomorphic image. This $\check{f}(A)$ is a fuzzy submodule of B, [4]. Hom (A,B) will denote the abelian group of all R-module homomorphisms from A into B and Hom (*A*, *B*) will denote the subset of all fuzzy homomorphisms *A* into *B*.

Lemma 2.1 Let f: $A \to B$ be a fuzzy homomorphism. Then f $(\bar{A}) \subseteq \bar{B}$

Proof Let a $\in \bar{A}$ then;
$B(f(a)) \geq \check{f}(A)(f(a)) = \sup\{A(x) : x \in A,\ f(x) = f(a)\}$
$\geq A(a) > 0$.

Proposition 2.2 Hom (*A*, *B*) *is a subgroup of* Hom (A,B).

Proof Let f, g ε Hom (*A*,*B*). Then it is enough to prove f - g ε Hom (*A*, *B*). For all x ε A we have:
$B(f(x)) \geq \check{f}(A)(f(x)) = \sup\{A(a) : a \in A, f(a) = f(x)\} \geq A(x)$.
Similarly $B(g(x)) \geq A(x)$. Let h = f - g, then for all b ε B we get:
$\check{h}(A)(b) = \sup\{A(a) : a \in A,\ h(a) = b\}$
$\leq \sup\{B(f(a)) \wedge B(g(a)) : a \in A,\ h(a) = b\}$
$\leq \sup\{B(f(a) - g(a)) : a \in A,\ h(a) = b\}$
$= \sup\{B(h(a)) : a \in A,\ h(a) = b\} = B(b)$.
Hence h = f - g ε Hom (*A*, *B*).

By Lemma 2.1 , the restriction of f (as an R-module homomorphism) to $\bar{A}$ is a homomorphism from $\bar{A}$ into $\bar{B}$ and is denoted by f as well .

Definition 2.3 Let X be a subset of A and let $0 \leq a \leq 1$ define $a_X \in I^A$ as follows ; for all x ε A let: a

$$a_X(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{if } x \notin X. \end{cases}$$

If X is a singleton {x} then a will be denoted by a_x, in this case a is called a fuzzy point . Define <a> ε I by; for all y ε A let

$$\langle a_x \rangle(y) = \begin{cases} a & \text{if } y = \alpha x \neq 0, \alpha \in R, \\ 1 & \text{if } y = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to prove (see [3] and [4]):

Lemma 2.4 $\langle a_x \rangle$ is the smallest fuzzy submodule of A containing a_x. It is called the fuzzy submodule generated by a_x.

Lemma 2.5 Let f : A→ B be an R-module homomorphism, and let x ε A. Then $\check{f}(\langle a_x \rangle) = \langle a_y \rangle$, where f(x) = y.

Proof Let z ε B and z = αy for some α ε R, then
$\check{f}(\langle a_x \rangle)(z) = \sup\{\langle a_x \rangle(u) : u \in A,\ f(u) = \alpha y\} = a$.

Also we have:
$\check{f}(\langle a_x \rangle)(0) = \sup\{\langle a_x \rangle(u) : u \in A,\ f(u) = 0\} = 1$.
Suppose z ε B such that $z \neq \alpha y$ for all α ε R, so
$\check{f}(\langle a_x \rangle)(z) = \sup\{\langle a \rangle(u) : u \in A,\ f(u) = z\} = 0$,
otherwise we can find β ε R such that f(βx) = z which contradicts the assumption of z.

A fuzzy submodule *A* is said to be fuzzy finitely generated if there are $0 \leq a^1, a^2, .., a^n \leq 1$ and $x_1, x_2, .., x_n \in A$ such that:

$$A = \langle a^1{}_{x_1} \rangle + \langle a^2{}_{x_2} \rangle + \ldots + \langle a^n{}_{x_n} \rangle, \qquad (6)$$

see [4] , [5]. Then for all x ε A we get:
$A(x) = \sup\{\min(a^j) : 1 \leq j \leq n,\ x = \Sigma\alpha_j,\ 0 \neq \alpha_j \in R\}$.
Hence $A(x) > 0$ if and only if x is a linear

combination of the x_i 's. So, $\bar{A}$ is finitely generated by $\{x_i\}$ as an R-module. The image of A in I is the finite set $\{a^1 ,..., a^n\}$ and $A(x_i) \geq a^i$ for all $1 \leq i \leq n$. Also we have for each x_i

$$(<a^1{}_{x_1}> + ... + <a^n{}_{x_n}>) (x_i) \geq a^i \qquad (7)$$

Lemma 2.6 Fuzzy homomorphic images of fuzzy finitely generated fuzzy submodules are also fuzzy finitely generated .

Proof Let $f : A \rightarrow B$ be a fuzzy homomorphism and let;

$A = <a^1{}_{x_1}> + ... + <a^n{}_{x_n}>$. Then by Theorem 7.1.2 of [4] we get:

$\check{f}(<a^1{}_{x_1}>+...+ <a^n{}_{x_n}>) = \check{f}(<a^1{}_{x_1}>) + ... + \check{f}(<a^n{}_{x_n}>)$

$= <a^1{}_{x_1}> + ... + <a^n{}_{x_n}>$, where $f(x_i) = y_i$, $i = 1 ,..., n$.

Remark 2.7 If μ is a fuzzy subset of A, the intersection of all fuzzy submodules of A that contain μ is also a fuzzy submodule of A. It is the smallest fuzzy submodule of A containing μ; called the fuzzy submodule of A generated by μ and denoted by $<\mu>$. Theorem 7.3.2 of [4] reads ; for all $x \epsilon A$ we get:

$$<\mu>(x) = \begin{cases} 1 & \text{if } x = 0, \\ \sup \{\min(\mu(x_i)) : x = \sum \alpha_i x_i, \alpha_i \varepsilon R\} & \text{if } x \neq 0. \end{cases}$$

If $\bar{\mu}$ is a finite subset of A, $<\mu>$ is said to be a finitely generated fuzzy submodule of A, see Definition 7.3.1 of [4]. If we put $\bar{\mu} = \{x_1, x_2, ...\}$ and $\mu(x_i) = a^i$, this definition of finitely generated fuzzy submodule agrees with our previous definition of fuzzy finitely generated fuzzy submodule .

3. FUZZY CHAIN CONDITIONS

A fuzzy submodule A of an R-module A is said to be a finite fuzzy submodule if the image of the membership function of A in I is finite i.e. the cardinality of $A(A)$, $|A(A)|$, is finite. Then any submodule of A is a finite fuzzy submodule because the image of its membership function has cardinality 2 or 1 .

A set of increasing finite fuzzy submodules of an R-module A;

$$A_1 \subseteq A_2 \subseteq A_3 .... \qquad (8)$$

is said to satisfy the finite fuzzy ascending chain condition (f.f.a.c.c. for short) if there is an integer n such that $\bar{A}_n = \bar{A}_m$ for all $m \geq n$.

A set of decreasing finite fuzzy submodules of an R-module A;

$$A_1 \supseteq A_2 \supseteq A_3 .... \qquad (9)$$

is said to satisfy the finite fuzzy descending chain condition (f.f.d.c.c. for short) if there is an integer n such that $\bar{A}_n = \bar{A}_m$ for all $m \geq n$.

A set of finite fuzzy submodules $\{A_1, A_2, ... \}$ of an R-module A is said to satisfy the finite fuzzy maximal (minimal) condition if it has a fuzzy submodule A_o such that: if A_n is another fuzzy submodule that satisfies $A_o \subseteq A_n$ ($A_n \subseteq A_o$) then we must have $\bar{A}_o = \bar{A}_n$.

Example 3.1 Let A be the module of even integers over the ring of integers. For any $0 \neq x \epsilon A$, let k_x be the natural number such that 2^{k_x} is greatest even divisor of x i.e. $x = 2^{k_x} (q)$, where q is an odd integer. For any natural number n , define a fuzzy subset A_v of A as follows; for all $x \epsilon A$ let

$$A_n (x) = \begin{cases} 1 - \dfrac{1}{2^{k_x + n}} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then we get an ascending sequence of fuzzy submodules

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq .....$$

The image of the membership function of each A_i (i= 1, 2,..) is infinite, and $\bar{A}_i$=A. Although $\bar{A}_i$ are finitely generated as R-modules and A is a Noetherian module, no one of the A_i's is a fuzzy finitely generated fuzzy submodule of A .

Example 3.2 Let k be a field , the polynomial ring

$k[x_1, x_2, ..]$ in a countably infinite number of indeterminates satisfies neither chain conditions on ideals . Let A_i be the ideal generated by $x_1, x_2, ..., x_i$ and for each i define a fuzzy submodule A_i of A_i that takes 1 at 0 and 1 / 2 otherwise. Then each A_i is a finite fuzzy submodule which is finitely generated but does not satisfy the f.f.a.c.c. .

Theorem 3.3 For any R-module A, the following conditions are equivalent

1) Every finite fuzzy submodule is finitely generated.
2) Every set of increasing finite fuzzy submodules satisfies the finite fuzzy ascending chain condition.
3) Every non-empty set of finite fuzzy submodules satisfies the finite fuzzy maximal condition .

Proof See Theorem 3.5 of [5] , see also [4].

Theorem 3.4 For any R-module A, the following conditions are equivalent :

1) Every set of decreasing finite fuzzy submodules satisfies the finite fuzzy descending chain condition.
2) Every non-empty set of finite fuzzy submodules satisfies the finite fuzzy minimal condition .

Proof

(1) → (2)

Let S be a non-empty set of finite fuzzy submodules and let $A \in S$.If A does not satisfy the finite fuzzy minimal condition, then there is A_1 such that $A \supseteq A_1$ with $\bar{A} \neq \bar{A}_1$ we continue with this procedure till it stops by the f.f.d.c.c.

(2) → (1)

For the decreasing set of finite fuzzy submodules $A_1 \supseteq A_2 \supseteq$........, the set $S = \{A_1, A_2, ... \}$ must satisfy the finite fuzzy minimal condition.

Definition 3.5 We shall say that an R-module A is fuzzy Noetherian (respectively fuzzy Artinian) if it satisfies any one of the conditions of Theorem 3.3 (respectively Theorem 3.4).

Proposition 3.6 Let A be a fuzzy Noetherian module . Then every submodule and every factor module of A is fuzzy Noetherian.

Proof Let B be a submodule of A, then every fuzzy submodule of B is also a fuzzy submodule of A which is fuzzy finitely generated. Now, consider the factor module A / B , let f: A→A/B be the canonical homomorphism. Let B be a finite fuzzy submodule of A/B. We claim that $\check{f}^{-1}(B)$ is also a finite fuzzy submodule of A. Let a, b $\in$ A and α, b $\beta \in$ R. Then $\check{f}^{-1}(B)(\alpha a + \beta b) = B(f(\alpha a + \beta b)) = B(\alpha f(a) + \beta f(b)) \geq B(f(a)) \wedge B(f(b)) = \check{f}^{-1}(B)(a) \wedge \check{f}^{-1}(B)(b)$. This shows that $\check{f}^{-1}(B)$ is a fuzzy submodule of A. From (5) we see that $\check{f}^{-1}(B)(A) \subseteq B(A/B)$; this completes the proof of the claim. Our second claim is that; let $A = \check{f}^{-1}(B)$ then $f(\bar{A}) = \bar{B}$. Let a $\in \bar{A}$ then $B(f(a)) = \check{f}^{-1}(B)(a) > 0$, so $f(a) \in \bar{B}$.

On the other hand let $x + B \in \bar{B}$ for some $x \in A$ then $0 < B(x + B) = B(f(x)) = \check{f}^{-1}(B)(x)$ hence $x \in \bar{A}$. Let $C_1 \subseteq C_2 \subseteq ...$ be an increasing finite fuzzy submodules of A/B. For each i let $A_i = \check{f}^{-1}(C_i)$, then we get an increasing finite fuzzy submodules of A $A_1 \subseteq A_2 \subseteq ...$ then there is n such that $\bar{A}_n = \bar{A}_m$ for all m $\geq$ n. Therefore by the second claim; $\bar{C}_n = \bar{C}_m$ for all m $\geq$ n.

Proposition 3.7 Let B be a submodule of A. Assume that B and A / B are fuzzy Noetherian. Then A is fuzzy Noetherian.

Proof Let $A_1 \subseteq A_2$ be fuzzy submodules of A, For i=1, 2 define $C_i = A_i \cap B$. Let f: A → A/B be the canonical homomorphism, for i = 1, 2 define $D_i = \check{f}(A_i + B)$. Then we get $C_1 \subseteq C_2$ fuzzy submodules of B and $D_1 \subseteq D_2$ fuzzy submodules of A/B. We claim that if $\bar{C}_1 = \bar{C}_2$ and $\bar{D}_1 = \bar{D}_2$ then $\bar{A}_1 = \bar{A}_2$.

Let $x \in \bar{A}_2$ then

$D_2(x + B) = \check{f}(A_2 + B)(x + B)$
$= \sup \{(A_2 + B)(a): f(a) = x + B\}$
$= \sup \{(A_2 + B)(a) : a + B = x + B\}$
$= \sup \{\sup \{A_2(u) \wedge B(v): u + v = a\}: a+B=x+B\}$
$= \sup \{\sup \{A_2(u) : v \in B, u+v=a\}: a + B = x + B\} =$
$\sup \{A_2(u) : v \in B, u + v + B = x + B\}$
$= \sup \{A_2(u) : u + B = x + B\}$
$= \sup \{A_2(u) : x - u \in B\} \geq A_2(x) > 0.$

Then $x + B \in \bar{D}_2 = \bar{D}_1$. So $D_1(x + B) > 0$. Then as before; $0 < D_1(x + B) = \sup \{A_1(u) : x - u \in B)$. This means; there is $z \in A$ such that $A_1(z) > 0$ and $x - z \in B$. Then $z \in \bar{A}_1 \subseteq \bar{A}_2$, so $z \in \bar{A}_2$ also $x \in \bar{A}_2$ then $x - z \in \bar{A}_2$. So we have: $C_2(x - z) = A_2(x - z) \wedge B(x - z) = A_2(x - z) > 0$. So; $x - z \in \bar{C}_2 = \bar{C}_1$, so $C_1(x - z) > 0$.
Then $A_1(x - z) \wedge B(x - z) > 0$. So $A_1(x - z) > 0$ and $x - z \in \bar{A}_1$, also $z \in \bar{A}_1$ so $x \in \bar{A}_1$ and the claim is proved. Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq$ ....be an increasing finite fuzzy submodules of A. Then we get the two sequences of increasing finite fuzzy submodules; $C_1 \subseteq C_2 \subseteq C_3 \subseteq$ .... and $D_1 \subseteq D_2 \subseteq D_3 \subseteq$....., of A and A/B respectively, then there is n big enough such that $\bar{C}_n = \bar{C}_m$ and $\bar{D}_n = \bar{D}_m$ for all $m \geq n$. Then by the claim; $\bar{A}_n = \bar{A}_m$ for all $m \geq n$.
Now, we arrive at our main result in this section.

Theorem 3.8 Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules . Then A is fuzzy Noetherian if and only if both B and A / B are fuzzy Noetherian.

Corollary 3.9 Let A_1, ... A_n be fuzzy Noetherian R-modules, then $A_1 \oplus ... \oplus A_n$ is also fuzzy Noetherian.

Proof We Apply induction to the exact sequence:

$0 \to A_n \to A_1 \oplus ... \oplus A_n \to A_1 \oplus ... \oplus A_{n-1} \to 0.$

Corollary 3.10 If one of two isomorphic R-modules is fuzzy Noetherian (respectively fuzzy Artinian), then the other is also fuzzy Noetherian (respectively Artinian).

Corollary 3.11 Let $f : A \to B$ be an R-module epimorphism. If A is fuzzy Noetherian (respectively fuzzy Artinian) then so is B.

Proof If we factor A by the kernel of f; we get a quotient- module of A which is isomorphic to B.

4. FUZZY NOETHERIAN AND ARTINIAN RINGS

A fuzzy subset I of a ring R is said to be a fuzzy ideal if it satisfies for all $a, b \in R$;
(I 1) $I(a + b) \geq I(a) \wedge I(b)$,
(I 2) $I(ab) \geq I(a) \vee I(b)$,
(I 3) $I(0) = 1$.

So if we consider R as a module over itself, then any fuzzy ideal of R may be considered as a fuzzy submodule of R.
A ring is said to be a fuzzy Noetherian ring (respectively a fuzzy Artinian ring) if finite fuzzy ideals satisfy one of the equivalent conditions of Theorem 3.3 (respectively Theorem 3.4).

Proposition 4.1 If R is a fuzzy Noetherian ring and f is a ring epimorphism of R onto R' then R' is also Noetherian.
For every $0 \leq t \leq 1$, the t-level subset of a fuzzy ideal I, denoted by I_t is defined to be;

$$I_t = \{x : x \in R, I(x) \geq t\} \qquad (10)$$

Then I is a fuzzy ideal in R if and only if I_t is an ideal in R for every $0 \leq t \leq 1$. See [4]. A fuzzy ideal I is said to be a fuzzy prime ideal (respectively fuzzy maximal ideal) if and only if I_1 is a prime (respectively maximal) ideal in R and there is s ($0 \leq s < 1$) such that

$$I(x) = \begin{cases} 1 & \text{if } x \in I_1 \\ s & \text{otherwise} \end{cases}$$

See Theorem 5.2 of [6] and sections 6.5 and 6.7 of [4].

Proposition 4.2 Let R be a fuzzy Artinian ring, then every fuzzy prime ideal is a fuzzy maximal

Proof Let P be a fuzzy prime ideal, then P_1 is a prime ideal in R . Then it is a maximal ideal in R because R is an Artinian ring (see Proposition 8.1 of [1]) . Then P is a fuzzy maximal ideal in R . p

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