SINGULAR VALUE OF THE NATURAL FREQUENCY FOR THE MOTION OF RIGID BODY

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ABSTRACT

The three dimensional motion of a rigid body about a fixed point in both the uniform and the Newtonian force fields is considered as one of the important problems in theoretical mechanics. The periodic solutions of this problem were obtained in the form of poincaré's expansions for different cases of the natural frequency ω as well [1-3] except some singular values of ω, namely; when $\omega=3$ and $\omega=1/3$ for examples. In this paper, the periodic solutions for the case $\omega=3$ are constructed in the form of power series expansions containing a small parameter proportional to the inverse of the initial fast spin of the rigid body. At the end, the geometric interpretation of motion is designed using Euler's angles to show the resulting motion.

Keywords: Rapid rigid body motion, Perturbation techniques, Nonlinear Oscillations, gyroscopes, Periodic solutions, Poincaré Method, Unsolved cases of periodic solutions of rigid body motion.

1. INTRODUCTION

Let us consider a rigid body of mass M rotating about a fixed point O, whose ellipsoid of inertia is arbitrary and its center of mass does not coincide with that point. Let the body be subjected to a Newtonian potential field exerted by attracting center namely O_1 locating a fixed downwards Z-axis and at distance R from the point O. Choosing the axes OX, OY and OZ to represent the fixed frame in space and the axes Ox, Oy and Oz to represent the principal axes of the ellipsoid of inertia constructed for the body at the fixed point O. Let $\underline{Z}$ be the unit vector in the direction of Z-axis, its direction cosines with respect to the moving frame (Oxyz) are γ, γ' and γ''. Assuming that at the initial instant of motion, the body spins about z-axis with a sufficiently high angular speed r_o and that this axis makes an angle $\theta_o \neq m\pi/2, (m = 0,1,2,...)$, with the Z- axis. In this case, the differential equations of motion and their first integrals are reduced to the following system with one first integral [1]:

$$\ddot{p}_2 + 9 p_2 = \mu^2 F(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu), \quad \ddot{\gamma}_2 + \gamma_2 = \mu^2 \Phi(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu); \tag{1}$$

$$\gamma_o''^{-2} - 1 = \gamma_2^2 + \dot{\gamma}_2^2 + 2\mu(\nu p_2 \gamma_2 + \nu_2 \dot{p}_2 \dot{\gamma}_2 + s_{21}) + \mu^2 [\nu_2^2 \dot{p}_2^2 - 2\dot{\gamma}_2(e_2 A_1^{-1} \dot{\gamma}_2 + A_1^{-1} \dot{p}_2 s_{21} + \frac{1}{2}\dot{\gamma}_2 s_{11} - y_o' a^{-1} A_1^{-1}) + \nu^2 p_2^2 + s_{21}^2 + 2(s_{22} - \frac{1}{2} s_{11})] + \mu^3 (\dots), \tag{2}$$

where:

$$
\begin{aligned}
F &= F_2+\mu F_3+\cdots, & \Phi &= \Phi_2+\mu\Phi_3+\cdots,\\
F_2 &= f_2+8\nu e_1 p_2, & \Phi_2 &= \phi_2-8\nu(e+e_1\gamma_2),\\
F_3 &= f_3-e_1\phi_2+8\nu e_1(e+e_1\gamma_2), & \Phi_3 &= \phi_3-\nu f_2-8\nu^2 e_1 p_2,
\end{aligned}
$$

$$
\begin{aligned}
f_2 =\; & A_1 b^{-1}x_o' s_{21}-9p_2 s_{11}+C_1A_1^{-1}p_2\dot{p}_2^2-y_o' a^{-1}p_2\dot{\gamma}_2-y_o' A_1^{-1}(A_1+a^{-1})\gamma_2\dot{p}_2\\
& +x_o'\dot{p}_2\dot{\gamma}_2-z_o' a^{-1}p_2-k([(1-C_1)\gamma_2\dot{p}_2\dot{\gamma}_2+A_1(1+B_1)\gamma_2 s_{21}-A_1p_2(1-\dot{\gamma}_2^2)],\\
\phi_2 =\; & -\gamma_2 s_{11}+(1+B_1)p_2s_{21}-(1-C_1)A_1^{-1}p_2\dot{p}_2\dot{\gamma}_2+x_o'\dot{\gamma}_2^2-y_o'\gamma_2\dot{\gamma}_2\\
& -z_o' b^{-1}\gamma_2+x_o' b^{-1}-A_1^{-2}\gamma_2\dot{p}_2^2+k(C_1\dot{\gamma}_2^2-B_1)\gamma_2,\\
f_3 =\; & C_1A_1^{-1}\dot{p}_2[e\dot{p}_2+e_1\gamma_2\dot{p}_2-2p_2(y_o' a^{-1}-e_2\dot{\gamma}_2)]-9(es_{11}+e_1\gamma_2 s_{11}+2p_2s_{12})\\
& +A_1b^{-1}x_o' s_{22}+x_o'[\nu_2\dot{p}_2^2-\dot{\gamma}_2(y_o' a^{-1}-e_2\dot{\gamma}_2)]-y_o' a^{-1}[\dot{\gamma}_2(e+e_1\gamma_2)+\nu_2p_2\dot{p}_2]\\
& +y_o'(1+A_1^{-1}a^{-1})[\gamma_2(y_o' a^{-1}-e_2\dot{\gamma}_2)-\nu p_2\dot{p}_2]+\frac{1}{2}z_o'(a^{-1}-A_1b^{-1})\gamma_2 s_{11}\\
& -z_o' a^{-1}(e+e_1\gamma_2+p_2s_{21})+k[(1-C_1)(y_o' a^{-1}-e_2\dot{\gamma}_2)\gamma_2\dot{\gamma}_2-\nu(1-C_1)p_2\dot{p}_2\dot{\gamma}_2\\
& -2\nu_2A_1p_2\dot{p}_2\dot{\gamma}_2-\nu_2(1-C_1)\gamma_2\dot{p}_2^2-\nu A_1(1+B_1)p_2s_{21}+2A_1p_2s_{21}\\
& +(9-A_1)\gamma_2s_{22}+A_1(e+e_1\gamma_2)(1-\dot{\gamma}_2^2)],\\
\phi_3 =\; & 2x_o'\nu_2\dot{p}_2\dot{\gamma}_2-2\gamma_2s_{12}-\nu p_2s_{11}+(1+B_1)[p_2s_{22}+(e+e_1\gamma_2)s_{21}]\\
& +(1-C_1)A_1^{-1}[p_2\dot{\gamma}_2(y_oa^{-1}-e_2\dot{\gamma}_2)-\nu_2p_2\dot{p}_2^2-(e+e_1\gamma_2)\dot{p}_2\dot{\gamma}_2]\\
& -z_o' b^{-1}(\nu p_2+\gamma_2s_{21})+2x_o' b^{-1}s_{21}+A_1^{-2}[2\gamma_2\dot{p}_2(y_o' a^{-1}-e_2\dot{\gamma}_2)-\nu p_2\dot{p}_2^2]\;;
\end{aligned}
$$

$$
-y_o'(\nu p_2\dot{\gamma}_2+\nu_2\gamma_2\dot{p}_2)+k(\nu p_2(C_1\dot{\gamma}_2^2-B_1)+2\gamma_2(\nu_2C_1\dot{p}_2\dot{\gamma}_2-B_1s_{21})]; \tag{3}
$$

$$
\begin{aligned}
p_2 &= P_1-\mu e-\mu e_1\gamma_2, \qquad \gamma_2 = \gamma_1-\mu\nu p_2,\\
q_1 &= -A_1^{-1}\dot{p}_2+\mu A_1^{-1}(y_o' a^{-1}-e_2\dot{\gamma}_2)+\mu^2[(aA_1)^{-1}y_o' s_{21}+\frac{1}{2}A_1^{-1}\dot{p}_2s_{11}+k\dot{\gamma}_2s_{21}\\
& -\nu_2\dot{p}_2(a^{-1}A_1^{-1}z_o'-k)]+\mu^3[(aA_1)^{-1}y_o' s_{22}+\frac{1}{2}A_1^{-1}e_1\dot{\gamma}_2s_{11}+A_1^{-1}\dot{p}_2s_{12}\\
& +a^{-1}A_1^{-2}e_1z_o'\dot{\gamma}_2+a^{-1}A_1^{-1}s_{11}(z_o'\dot{\gamma}_2-y_o')+(k-a^{-1}A_1^{-1}z_o')(a^{-1}A_1^{-1}y_o'\\
& -a^{-1}A_1^{-1}z_o'\dot{\gamma}_2+k\dot{\gamma}_2)+a^{-1}A_1^{-2}z_o'\dot{p}_2s_{21}+k(\nu\dot{p}_2s_{21}+\dot{\gamma}_2s_{22}\\
& -A_1^{-1}e_1\dot{\gamma}_2-\frac{3}{2}\dot{\gamma}_2s_{11}-2A_1^{-1}\dot{p}_2s_{21})]+\cdots,\\
r_1 &= 1+\frac{1}{2}\mu^2s_{11}+\mu^3s_{12}+\cdots,\\
\gamma_1' &= \dot{\gamma}_2+\mu\nu_2\dot{p}_2+\mu^2[(aA_1)^{-1}y_o'-A_1^{-1}(e_2\dot{\gamma}_2+\dot{p}_2s_{21})-\frac{1}{2}\dot{\gamma}_2s_{11}]\\
& +\mu^3[-A_1^{-1}(e_1\dot{\gamma}_2s_{21}+\dot{p}_2s_{22})+\frac{1}{2}(3A_1^{-1}-\nu)\dot{p}_2s_{11}-\dot{\gamma}_2s_{12}+\nu_2(k-a^{-1}A_1^{-1}z_o')\dot{p}_2\\
& +2a^{-1}A_1^{-1}y_o' s_{21}+(2k-a^{-1}A_1^{-1}z_o')\dot{\gamma}_2s_{21}]+\cdots,
\end{aligned}
$$

$$
\gamma_1'' = 1+\mu S_{21}+\mu^2(S_{22}-\frac{1}{2}S_{11})-\mu^3(S_{12}+\frac{1}{2}S_{11}S_{21})+\cdots; \tag{4}
$$

$$p_1 = p/c\sqrt{\gamma_o''}\,, \quad q_1 = q/c\sqrt{\gamma_o''}\,, \quad r_1 = r/r_o, \quad \gamma_1 = \gamma/\gamma_o''\,,$$

$$\gamma_1' = \gamma'/\gamma_o''\,, \quad \gamma_1'' = \gamma''/\gamma_o''\,, \quad \tau = r_o t, \quad (. \equiv d/d\tau); \tag{5}$$

$$\begin{aligned}
s_{11} &= a(p_{2o}^2 - p_2^2) + b(\dot{p}_{2o}^2 - \dot{p}_2^2)/A_1^2 - 2\,[x_o'(\gamma_{2o} - \gamma_2) + y_o'(\dot{\gamma}_{2o} - \dot{\gamma}_2)] \\
&\quad + k\,[a(\gamma_{2o}^2 - \gamma_2^2) + b(\dot{\gamma}_{2o}^2 - \dot{\gamma}_2^2)]\,, \\
s_{12} &= a[e(p_{2o} - p_2) + e_1(p_{2o}\gamma_{2o} - p_2\gamma_2)] - bA_1^{-2}\,[y_o'\,a^{-1}(\dot{p}_{2o} - \dot{p}_2) - e_2(\dot{p}_{2o}\dot{\gamma}_{2o} - \dot{p}_2\dot{\gamma}_2)] \\
&\quad - \nu x_o'(p_{2o} - p_2) - \nu_2 y_o'(\dot{p}_{2o} - \dot{p}_2) + (z_o' - k)s_{21} \\
&\quad + k[\nu a(p_{2o}\gamma_{2o} - p_2\gamma_2) + \nu_2 b(\dot{p}_{2o}\dot{\gamma}_{2o} - \dot{p}_2\dot{\gamma}_2)], \\
s_{21} &= a(p_{2o}\gamma_{2o} - p_2\gamma_2) - b\,A_1^{-1}(\dot{p}_{2o}\dot{\gamma}_{2o} - \dot{p}_2\dot{\gamma}_2), \\
s_{22} &= a[\nu(p_{2o}^2 - p_2^2) + e(\gamma_{2o} - \gamma_2) + e_1(\gamma_{2o}^2 - \gamma_2^2)] + b\,A_1^{-1}\,[-\nu_2(\dot{P}_{2o}^2 - \dot{p}_2^2) \\
&\quad + a^{-1}y_o'(\dot{\gamma}_{2o} - \dot{\gamma}_2) - e_2(\dot{\gamma}_{2o}^2 - \dot{\gamma}_2^2)];
\end{aligned} \tag{6}$$

$$A_1 = \frac{C-B}{A}, \quad B_1 = \frac{A-C}{B}, \quad C_1 = \frac{B-A}{C}, \quad \gamma_o \geq 0, \quad 0 < \gamma_o'' < 1,$$

$$a = \frac{A}{C}, \quad b = \frac{B}{C}, \quad c^2 = \frac{Mg\ell}{C}, \quad \mu = \frac{c\sqrt{\gamma_o''}}{r_o}, \quad x_o = \ell x_o', \quad y_o = \ell y_o',$$

$$z_o = \ell z_o', \quad \ell^2 = x_o^2 + y_o^2 + z_o^2, \quad A_1 B_1 = -9, \quad , \quad e = \frac{1}{9}x_o' A_1 b^{-1},$$

$$e_1 = \frac{1}{8}[k(9 - A_1) + z_o'(a^{-1} - A_1 b^{-1})], \nu = -\frac{1}{8}(1 + B_1), \quad e_2 = e_1 + a^{-1}z_o' - kA_1,$$

$$\nu_2 = \nu - A_1^{-1}, \quad k = N\gamma_o''/c^2, \quad N = 3g/R, \quad g = \lambda/R^2, \tag{7}$$

where A, B and C are the principal moments of inertia; x_o, y_o and z_o are the coordinates of the center of mass in the moving coordinate system: p,q and r are the projections of the angular velocity vector of the body; λ is the coefficient of attraction of the attracting center; p_o, q_o, r_o, γ_o, γ_o' and γ_o'' are the initial values of the corresponding variables. Since r_o is large, then μ is small.

2. CONSTRUCTION OF THE PERIODIC SOLUTIONS

In this section, Poincaré's small parameter method is applied to construct the periodic solutions of system (1). For this system, the following generating system ($\mu = 0$) is attained:

$$\ddot{p}_2^{(o)} + 9p_2^{(o)} = 0, \qquad \ddot{\gamma}_2^{(o)} + \gamma_2^{(o)} = 0, \tag{8}$$

which gives periodic solutions in the forms:

$$p_2^{(o)} = M_1 cos3\tau + M_2 sin3\tau, \quad \gamma_2^{(o)} = M_3 cos\tau, \tag{9}$$

with period $T_o = 2\pi$, and M_1, M_2 and M_3 are constants. The following initial condition:

$$p_2(0,0) = \dot{p}_2(0,0) = \dot{\gamma}_2(0,\mu) = 0, \tag{10}$$

does not affect the generality of the required solutions [4].

Making use of Poincaré's small parameter method, the periodic solutions for system (1) are expressed by the following forms [5]:

$$p_2(\tau,\mu) = \tilde{M}_1 cos3\tau + \tilde{M}_2 sin3\tau + \sum_{k=2}^{\infty} \mu^k G_k(\tau),$$

$$\gamma_2(\tau,\mu) = \tilde{M}_3 \cos\tau + \sum_{k=2}^{\infty} \mu^k G_k(\tau), \quad (11)$$

where

$$\tilde{M}_i = M_i + \beta_i, (i = 1,2,3), \quad (12)$$

$$U = u + \frac{\partial u}{\partial M_1}\beta_1 + \frac{\partial u}{\partial M_2}\beta_2 + \frac{\partial u}{\partial M_3}\beta_3 + \frac{1}{2}\frac{\partial^2 u}{\partial M_1^2}\beta_1^2 + \cdots,$$

$$\left\{ \begin{matrix} U = G_k, & H_k \\ u = g_k, & h_k \end{matrix} \right\}. \quad (13)$$

The quantities β_1, $3\beta_2$ and β_3 are the deviations of the initial values of p_2, $\dot{p}_2$ and γ_2 of system (1) from their initial values ($p_2^{(o)}$, $\dot{p}_2^{(o)}$ and $\gamma_2^{(o)}$) of system (8), these deviations are functions of μ and vanish when $\mu = 0$. The functions $g_k(\tau)$ and $h_k(\tau)$ take the forms [2]:

$$g_k(\tau) = \frac{1}{3}\int_0^{\tau} F_k^{(o)}(t_1) sin3(\tau - t_1) dt_1,$$

$$h_k(\tau) = \int_0^{\tau} \Phi_k^{(o)}(t_1) \sin(\tau - t_1) dt_1, (k=2,3). \quad (14)$$

The solutions (11) have the period $T = 2\pi + \alpha(\mu)$ which reduces to 2π at $\mu = 0$, that is; $\alpha(0) = 0$. The initial condition (10) can be rewritten through the following relations:

$$p_2(0,\mu) = \tilde{M}_1, \dot{p}_2(0,\mu) = 3\tilde{M}_2, \gamma_2(0,\mu) = \tilde{M}_3, \dot{\gamma}_2(0,\mu) = 0. \quad (15)$$

The solutions (9) are rewritten in the following forms:

$$p_2^{(o)} = Ecos(3\tau - \epsilon), \quad \gamma_2^{(o)} = M_3 \cos\tau, \quad (16)$$

where $E = \sqrt{M_1^2 + M_2^2}$, $\epsilon = \tan^{-1} M_2/M_1$. Making use of (16) and (6) to get:

$$s_{ij}^{(o)} = s_{ij}^{(o)}(p_2^{(o)}, \dot{p}_2^{(o)}, \gamma_2^{(o)}, \dot{\gamma}_2^{(o)}), (i,j = 1,2). \quad (17)$$

The functions $F_k^{(o)}$ and $\Phi_k^{(o)}$ are obtained from (16), (17) and (3), then using (14) to determine $g_k(2\pi)$, $\dot{g}_k(2\pi)$, $h_k(2\pi)$ and $\dot{h}_k(2\pi)$. The quantity $\tilde{M}_3$ is determined by using (15) into the first integral (2), with $\tau = 0$, and can be written:

$$\tilde{M}_3 = \sqrt{1 - \gamma_o''} / \gamma_o'' - \mu\nu\tilde{M}_1 - 9\mu^2 \nu_2^2 \tilde{M}_2^2 / 2M_3 - 3\mu^3 y_o' \nu_2 \tilde{M}_2 / aA_1 M_3 + \cdots. \quad (18)$$

The independent periodicity conditions [1] of the solutions $p_2(\tau,\mu)$, $\dot{p}_2(\tau,\mu)$, $\gamma_2(\tau,\mu)$ and $\dot{\gamma}_2(\tau,\mu)$ take the following forms:

$$(\tilde{L}_{21} - 9\tilde{N}_{21})\tilde{M}_2 = -\mu\tilde{M}_2[(\tilde{L}_{31} - 9\tilde{N}_{31}) - 9\tilde{M}_1\tilde{M}_3 N_{38}] + \cdots,$$

$$(\tilde{L}_{21} - 9\tilde{N}_{21})\tilde{M}_1 = -\mu[\tilde{M}_1[(\tilde{L}_{31} - 9\tilde{N}_{31}) + \tilde{M}_3(L_{31o}\tilde{M}_3^2 - 9N_{38}\tilde{M}_1^2)] + \cdots; \quad (19)$$

$$\alpha(\mu) = \mu^2 \tilde{M}_3^{-1}[\dot{H}_2(2\pi) + \mu\dot{H}_3(2\pi) + \cdots], \quad (20)$$

where

$$(\tilde{L}_{21} - 9\tilde{N}_{21}) = a_1(\tilde{M}_1^2 + \tilde{M}_2^2) - [a_2 + 9kb(2M_3\beta_3 + \beta_3^2)],$$

$$(\tilde{L}_{31} - 9\tilde{N}_{31}) = a_3\tilde{M}_1\tilde{M}_3, \quad a_1 = (a-1)(a+b-2)/2b,$$

$$a_2 = z_o^{\prime} (ab)^{-1}[3(a+b)-2(2ab+1)] + 18k[1-(a+b)+\frac{1}{2}bM_3^2],$$

$$a_3 = z_o^{\prime} b^{-1}(20ab-1) + ka(b-1)(31-32a),$$

$$N_{38} = \frac{1}{4}(b\omega A_1^{-1}-a)[-2(z_o^{\prime}-k)+e_1(1+B_1)-z_o^{\prime}b^{-1}-2kB_1$$

$$+kvA_1(1+B_1)]+\frac{1}{2}a(e_1+kv)+\frac{1}{4}(1-C_1)A_1^{-1}(e_2-\omega e_1)$$

$$+\frac{1}{4}kv(a-b)(1-\omega^2)+\frac{1}{2}\omega[-e_2A_1^{-2}+kv_2C_1]+\frac{1}{4}kv[-(A_1+C_1)$$

$$+\omega(1-C_1)]-\frac{1}{4}(1+B_1)(ae_1+bA_1^{-1}e_2)+\frac{1}{2}\omega\, b(e_2A_1^{-2}+kv_2),$$

$$L_{31o} = \frac{1}{4}k[8e_1(a-b)-\frac{1}{2}z_o^{\prime}(a^{-1}-A_1b^{-1})(a-b)+e_2(1-C_1)+e_1A_1$$

$$+(A_1-9)(ae_1+be_2A_1^{-1})]. \quad (21)$$

Equating to zero the terms of zero power of μ from (19), to obtain the following two equations for determining M_1 and M_2:

$$[a_1(M_1^2+M_2^2)-a_2]M_2=0, [a_1(M_1^2+M_2^2)-a_2]M_1=0, \quad (22)$$

which have the solutions:

$$(i) M_1 = M_2 = 0,$$

$$(ii) M_1 = 0, \quad M_2 = \pm[a_2a_1^{-1}]^{\frac{1}{2}},$$

$$(iii) M_1 = \pm[a_2a_1-1]^{\frac{1}{2}}, M_2 = 0,$$

$$(iv) M_i = \pm[a_2a_1^{-1}-M_j^2]^{\frac{1}{2}}, (i=1,2; j=2,1), \quad (23)$$

here M_1 and M_2 are real under the condition:

$$a_2 > 0. \quad (24)$$

This condition can be satisfied either by the choice of z'_o or a and b. The condition $a_1>0$ is satisfied at all, since the initial fast spin r_o is assumed to be given about the major or the minor axis of the ellipsoid of inertia (a > 1, b > 1 or a < 1, b <1). The solution (iv) for the constants M_1 and M_2 represents a family of arbitrary solutions in which the solutions (ii) and (iii) are included.

We consider here, for example, the periodic solutions which arise in the case of zero basic amplitudes ($M_1 = M_2 = 0$). In this case, the expansions of β_1 and β_2 must be expressed in form of power series containing only integer powers of μ, which is a vital difference between the classical problem (k = 0) and the present problem ($k \neq 0$). Assuming β_1 and β_2 take the forms:

$$\beta_1 = \sum_{k=1}^{3} \mu^k \ell_k + O(\mu^4), \quad \beta_2 = \sum_{k=1}^{3} \mu^k m_k + O(\mu^4). \quad (25)$$

Using(25), (21), (12) and (19) to yield:

$$\ell_1 = M_3^3 L_{31o} a_2^{-1}, \qquad \ell_2 = m_i = 0, (i=1,2,3),$$

$$\ell_3 = a_2^{-1}\ell_1[\ell_1(a_1\ell_1+18bvkM_3+M_3a_3-9N_{38}M_3) - 3vM_3^2L_{31o}]. \quad (26)$$

Now, using the equations (13) and (14), the functions $G_k(\tau)$ and $H_k(\tau)$ are obtained and the periodic solutions (11) are determined up to the third approximation. Returning to the formulas (4), (5) and (6), the following solutions are obtained:

$$P = c\sqrt{\gamma_o''}\ [\mu(e+e_1M_3\cos\tau+\ell_1\cos 3\tau)+\mu^3\sum_{i=0}^{3}(I_i cos i\tau+J_i sin i\tau)]+\cdots,$$

$$q = c\sqrt{\gamma_o''}\ \{\mu A_1^{-1}(y_o' a^{-1}+e_2M_3\sin\tau+3\ell_1 sin3\tau)$$

$$+\mu^3[\sum_{i=0}^{4}(I_i' \, cos i\tau+J_i' \, sin i\tau)+J_5' \, sin5\tau]\}+\cdots,$$

$$r = r_o\{1-\mu^2M_3[x_o'(1-\cos\tau)+y_o'\sin\tau+\frac{1}{4}kC_1M_3(1-cos2\tau)]+\mu^3[0]\}+\cdots,$$

$$\gamma = \gamma_o'\{M_3\cos\tau+\mu^2[\sum_{i=0}^{2}(V_i cos i\tau+W_i sin i\tau)+V_3 cos3\tau]+\mu^3[0]\}+\cdots,$$

$$\gamma' = \gamma_o''\{-M_3\sin\tau+\mu^2[\sum_{i=0}^{2}(V_i' \, cos i\tau+W_i' \, sin i\tau)+W_3' \, sin3\tau]+\mu^3[0]\}+\cdots,$$

$$\gamma'' = \gamma_o''\{1+\mu^2M_3[\frac{1}{2}M_3(e_1a+be_2A_1^{-1})+ae+x_o'+\frac{1}{4}kC_1M_3+a\ell_1$$

$$-(x_o'+ae)\cos\tau+y_o'(1+ba^{-1}A_1^{-1})\sin\tau+\frac{1}{2}(3bA_1^{-1}\ell_1-a\ell_1-\frac{1}{2}kC_1M_3$$

$$-e_1aM_3-e_2bA_1^{-1}M_3)cos2\tau-\frac{1}{2}\ell_1(a+3bA_1^{-1})cos4\tau]+\mu^3[0]\}+\cdots, \tag{27}$$

where the constants I's, J's, I''s, J''s, V's, V''s, W's and W''s are determined in terms of the rigid body parameters and are written in about ten pages. The symbol(...) means terms of order higher than $O(\mu^3)$. Using (20), the correction of the period $\alpha(\mu)$ becomes:

$$\alpha(\mu) = \pi\mu^2\{2M_3x_o'-8\nu e_1-z_o'b^{-1}+k[M_3^2(b-a)-B_1]\}+\mu^3[0]+\cdots \tag{28}$$

3. THE GEOMETRIC INTERPRETATION OF MOTION

In this section, the motion of the body is investigated by introducing Euler's angles θ,ψ and ϕ which can be determined through the obtained periodic solutions. Since the initial system is autonomous, the periodic solutions will remain periodic if (t) is replaced by ($t + t_o$) where t_o is an arbitrary interval of time. Euler's angles in terms of time take the forms [6]:

$$\cos\theta = \gamma'', \quad \frac{d\psi}{dt} = \frac{p\gamma+q\gamma'}{1-\gamma''^2},$$

$$\frac{d\phi}{dt} = r-\frac{d\psi}{dt}\cos\theta, \quad \tan\phi_o = \frac{\gamma_o}{\gamma_o'}. \tag{29}$$

Now, we investigate the geometric interpretation of motion for this body as following:

Making use of (27) and (29) and assuming that the initial instant of time corresponds to the instant $t = t_o$, one obtains:

$$\phi_o = \frac{\pi}{2}+r_ot_o+\cdots, \tag{30}$$

from the formulas (18) and (29), one obtains:

$$\theta_o = \tan^{-1}M_3, \tag{31}$$

introducing, in this way, arbitrary initial angles of spin and nutation (ϕ_o and θ_o). Making use of (27), (29), (30) and (31), with the substitution $\tau = r_o t$, the following expressions of Euler's angles can be written in the form of small parameter expansions:

$$\theta = \theta_o - \mu^2 [\theta_2(t+t_o) - \theta_2(t_o)] + \mu^3 [0] + \cdots,$$

$$\psi = \psi_o + \frac{1}{2} Mg\ell C^{-1} r_o^{-1} (e_1 - e_2 A_1^{-1}) t + \mu^2 \mathrm{cosec}\theta_o [\psi_2(t+t_o) - \psi_2(t_o)] + \mu^3 [0] + \cdots,$$

$$\phi = \phi_o + \{r_o - Mg\ell C^{-1} r_o^{-1} [\frac{1}{2}\cos\theta_o (e_1 - e_2 A_1^{-1}) + \sin\theta_o (x_o' + \frac{1}{4} k C_1 \tan\theta_o)]\} t$$

$$+ \mu^2 [\phi_2(t+t_o) - \phi_2(t_o)] + \mu^3 [0] + \cdots, \quad (32)$$

where:

$$\theta_2(t) = -(x_o' + ae)\cos r_o t + y_o' (1 + ba^{-1} A_1^{-1}) \sin r_o t + \frac{1}{2} [\ell_1 (3bA_1^{-1} - a)$$

$$- \tan\theta_o (\frac{1}{2} k C_1 + e_1 a + b e_2 A_1^{-1})] \cos 2 r_o t - \frac{1}{2} \ell_1 (a + 3bA_1^{-1}) \cos 4 r_o t,$$

$$\psi_2(t) = e \sin r_o t + y_o' a^{-1} A_1^{-1} \cos r_o t + \frac{1}{4} [(e_2 A_1^{-1} + e_1) \tan\theta_o$$

$$+ \ell_1 (1 - 3A_1^{-1})] \sin 2 r_o t + \frac{1}{8} \ell_1 (1 + 3A_1^{-1}) \sin 4 r_o t,$$

$$\phi_2(t) = (x_o' \tan\theta_o - e \cot\theta_o) \sin r_o t + y_o' (\tan\theta_o - a^{-1} A_1^{-1} \cot\theta_o) \cos r_o t$$

$$+ \frac{1}{4} [r_o^{-1} k C_1 \tan^2\theta_o - e_2 A_1^{-1} - e_1 - \ell_1 \cot\theta_o + 3 \ell_1 A_1^{-1} \cot\theta_o] \sin 2 r_o t$$

$$- \frac{1}{4} (1 + 3A_1^{-1}) [r_o^{-1} \cos 4 r_o t + \frac{1}{2} \ell \cot\theta_o \sin 4 r_o t]. \quad (33)$$

4. CONCLUSIONS

For the motion of a rigid body about a fixed point in a Newtonian force field, the following results are concluded.

- Using Poincaré's small parameter method, the periodic solutions for the equations of motion of the body are investigated in the form of power series expansions containing the assumed small parameter.

- This problem deals with the following bodies (which are classifyed according to the moments of inertia):

1. C > A > B, B < ¼C, A > ¼C,
2. C > B > A, A < ¼C, B > ¼C,

- The obtained solutions are the generalized ones of the corresponding problem in the uniform gravity field (k=0), that is; the latter solutions can be deduced from our solutions if the terms characterizing the Newtonian field are neglected.
- Using Euler's angles, expressed in the form of power series expansions containing the assumed small parameter, the geometric interpretation of motion is investigated to show that the resulting motion is of pseudo-regular precession type which depends on four arbitrary constants; namely: the initial angels of precession, nutation, pure rotation and sufficiently high spin r_o.

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