LARGE DEFLECTION BENDING OF FLEXIBLE BEAMS USING ROBOT MANIPULATORS

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ABSTRACT

The problem of large deflection bending of a straight beam with linear material properties using two manipulator arms is considered. The nonlinear governing equation is solved using a successive approximation procedure. The required trajectories of the axes of the end-effectors are calculated and the problems related to the contact nature between the end-effectors and the beam are discussed.

Keywords: elastic beam, large deflection, nonlinear bending, robot manipulators

NOMENCLATURE

E	modulus of elasticity of beam material
$f'(x)$	$df(x)/dx$
$f'(\xi)$	$df(\xi)/d\xi$
i	Subscript: i = 0: "zero" (linear) approximation; i = 1: 1st approximation; i = 2: 2nd approximation
I	2nd moment of area of beam cross-section
k	(subscript) position index
L	length of beam
M	moment acting at both beam ends
m	dimensionless moment = ML/EI
R	see Figure (1)
S^R	center of rotation of right end-effector
S^L	center of rotation of left end-effector
$w(x)$	deflection of beam
$\bar{w}(x)$	dimensionless deflection of beam = $w(x)/L$
x,y,z	reference coordinate system for beam
$\bar{x},\bar{y},\bar{z}$	reference coordinate system for end-effector
γ	increment of beam end-slope angle
δ	reduction in projected beam length
$\bar{\delta}$	dimensionless reduction in projected beam length = δ/L
ξ	x/L
$\rho(\xi)$	radius of beam curvature
$\bar{\rho}(\xi)$	dimensionless radius of beam curvature = ρ/L
$\chi(\xi)$	curvature of beam elastic line
$\bar{\chi}(\xi)$	dimensionless curvature of beam elastic line = $L\chi$
$\Psi(\xi)$	slope angle of beam elastic line
Ψ_o	slope angle of beam elastic line at $\xi = 0$

INTRODUCTION

Manipulators are increasingly used in many branches of industrial activities, where one or more of the factors: precision, reliability, speed, hazardous environment or economy might justify their use. One of the early application fields of robots was the handling of rigid bodies (lifting, transporting, etc.). Handling deformable objects by manipulators is a field that has received attention only recently. [1] Manipulating flexible objects poses more problems than those faced with rigid ones. This is due to the fact that flexible bodies may suffer uncontrolled deformation due to their own weight or the forces applied to them through the manipulator arms which might lead to unwanted change of shape, geometry, high internal stresses or even damage of the object. To avoid these effects pads are used to support flexible objects during handling, increasing weight and volume of the handled object.

Another type of application is to use manipulators to achieve certain prescribed deformation of a flexible body. This would require planning the motion trajectories of the end-effectors so that the required deformation is achieved in a controlled way. Deforming flexible bodies, such as plates and beams, using manipulators has a very high application potential in various areas, among which are: shipbuilding, aerospace, automobile and electronic industries. Transporting large plates, in shipbuilding, is a typical application in this area. Aligning beams or plates, especially when high accuracy is required, is another example. One recent application is encountered in the production of printed wiring boards where every sheet is first bent to facilitate its alignment with two guide pins, with very strict tolerance requirements, and then unfold again for final alignment with the remaining guides. This problem was addressed by Zheng and Chen [1] who used a controlled buckling approach to achieve the required deformation. The plate was approximated by a beam. They calculated "ideal" trajectories of the manipulators which would minimize forces and moments on the end-effectors and hence on the manipulator arms. Arriving at the required deformation of the beam through buckling subjects it to unnecessarily high stresses, which can cause permanent deformation, or even damage in some cases.

In this work an alternative approach is adopted in which the beam is subjected to bending moments through moving the end-effectors, just as a human would use hands to achieve the same result. It is aimed to keep the beam free from axial forces, contrary to the buckling case, with subsequent reduction in stress levels.

In most cases the required deformation, or more precisely the slope of the deformation line, will be large, with slopes approaching 40°. Therefore nonlinear analysis is required. A method is used based on successive approximation of the solution. [2] It was found to give very good results when applied to the problem of beam post-buckling behavior. [2,3]

The problem of finding analytical solution to the nonlinear bending problem has been tackled before in e.g. [4,5]. In [4] a solution was found using elliptical integrals and Maclaurin's series. The method is however lengthy and very complicated to apply. Moreover, and more importantly, it would not be of much help for the present problem which is different due to the fact that, unlike in classical structural analysis, the question posed here is not: which deformation, internal stresses etc. result from a certain known loading, but rather: what are the required trajectories of the end points of a beam that would produce a certain required deformation.

In [5] a solution was found using an integral power series method. The results obtained there are formally equivalent to those found here. But the expression used for the nonlinear curvature relation used there is different from the expression used here, as will be explained shortly.

One final main consideration is that it is important to arrive at straightforward expressions for the required quantities, since this would allow operating the manipulator arms on-line.

THE LINEAR SOLUTION

The problem considered here is that of a beam supported by two end-effectors as shown in Figure (1). The boundary conditions are such that no vertical deflection, relative to the end-effectors, is allowed at either beam end and no axial force should be imposed on the beam. The end-effectors can perform plane motion, including translation in $\bar{x}$ and $\bar{y}$- and -directions as well as rotation about the axes S^R and S^L. The end-effectors are mounted on the ends of the robot arms, not shown in the figure. The end-effectors should not allow any motion of the beam in the transverse (y-) direction. This is to avoid the beam slipping sideways from the end-effectors. The beam is assumed to have constant, warping-free cross-section. The loading on the beam consists of two end moments initiated through rotation of the end-effectors as will be discussed later.

For this configuration the elastic line of the beam is governed by the equation:

$$EI\chi(x) = -M, \tag{1}$$

where E is the modulus of elasticity of the beam material, I is the 2nd moment of area of its cross-section about the y-axis and χ is the curvature of the deflection curve. In the linear (small slope) beam theory the curvature is approximated by:

$$\chi(x)=\psi'(x)=(\tan^{-1} w'(x))' \approx (w'(x))'=w''(x), \quad (2)$$

where ψ is the angle of the tangent to the elastic line, leading to the well known beam equation:

$$w''(x) = -M/EI \quad (3)$$

Introducing the following dimensionless quantities:

$$\xi = x/L; \ \bar{w}(x) = w(x)/L; \ m = ML/EI$$

eqn. (3) can be written as

$$\bar{w}_o''(\xi) = -m \quad (4)$$

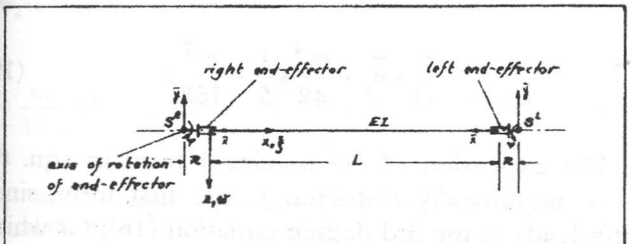

Fig. (1) Problem and axes definition for beam, right and left end-effectors

Figure 1. Problem and axes definition for beam, right and left end-effectors.

where $\bar{w}_o(\xi)$ denotes the "zero" (linear) approximation of the elastic line as opposed to the 1st and 2nd approximations which will follow.

Integrating eqn.(4) twice and satisfying the boundary conditions

$$\bar{w}(0) = 0 \text{ and } \bar{w}(1) = 0 \quad (5)$$

(where the subscript "o" is omitted here since these boundary conditions need always, i.e. for all approximations, to be satisfied), the deflection of the beam is given by

$$\bar{w}_o(\xi) = \frac{m}{2}(\xi - \xi^2) \quad (6)$$

The beam in this case will deflect in an arc of a circle with the constant, dimensionless radius

$$\bar{\rho} = \rho/L = -1/\bar{\chi}(\xi) = m \quad (7)$$

The beam will be subject to uniformly distributed bending moment along its length with a maximum deflection at the middle of m/8.

The dimensionless reduction in the projected length of the beam can be calculated from the formula:

$$\bar{\delta}_o = \int_0^1 \bar{w}'^2 d\xi$$

yielding $\bar{\delta}_o = \dfrac{m^2}{24}$

THE FIRST APPROXIMATION

The previous solution is only valid for small slopes (which in most cases are associated with small deflections). For larger slopes, the curvature - in the absence of axial forces [3] - is given by:

$$\bar{\chi}(\xi) = (\tan^{-1}\bar{w}'(\xi))' = \bar{w}''/(1+\bar{w}'^2) \quad (8)$$

leading to the nonlinear equation

$$\bar{w}''(1+w'^2)^{-1} = -m \quad (9)$$

It is worth mentioning here that the form of eqn. (9) differs from the form normally encountered in the literature (e.g. [5-8]), namely

$$\bar{\chi}(\xi) = \bar{w}''(\xi)/(1+\bar{w}'^2(\xi))^{2/3} \quad (10)$$

The difference is attributed to the coordinate system used in each case. The system used in eqn. (8) is the so-called material or lagrangian system , while that in eqn. (10) is the space or eulerian system, both systems being attributed to Novozhilov [9]. The expression in eqn. (8) takes into account that the beam was initially straight and then bent in the given shape, while eqn. (10) does not. This point is discussed in more detail in [10]. We will adopt eqn. (8) since this system of coordinates is more convenient for the problem at hand.

In order to find a better approximation for the deflection of the beam for large slopes, we start from equation (9) and expand the binomial term in parentheses up to first term, yielding

$$\bar{w}''(1+\bar{w}'^2)=-m \quad (11)$$

or, rearranging,

$$\bar{w}''_1=-m+\bar{w}'^2_o\,\bar{w}''_o \quad (12)$$

which is valid as long as $-1<\bar{w}'<1$, i.e. for slopes less than 45°. The subscripts "o" and "1" in eqn. (12) denote the zero (linear) and first approximation, respectively. A scheme which is basically a successive substitution iteration [11], is used to find higher approximations. Inserting the linear solution from eqn. (6) into the r.h.s. of eqn. (12), integrating twice and satisfying the boundary conditions (5), we arrive at an expression for the 1st approximation of the deflection, namely

$$\bar{w}_1(\xi)=\bar{w}_o(\xi)+\frac{m^3}{24}(\xi-3\xi^2+4\xi^3-2\xi^4) \quad (13)$$

with

$$\bar{w}_1(\xi)=\bar{w}'_o(\xi)+\frac{m^3}{24}(1-6\xi+12\xi^2-8\xi^3) \quad (14)$$

and

$$\bar{w}''_1(\xi)=\bar{w}''_o(\xi)+\frac{m^3}{24}(-6+24\xi-24\xi^2) \quad (15)$$

From (14) the slope at $\xi=0$ is

$$\bar{w}'_1(0)=\frac{m}{2}(1+\frac{m^2}{12}) \quad (16)$$

Hence for a given slope ψ_o at $\xi=0$, i.e. tan $\Psi_o=\bar{w}'_1(0)$, the required (dimensionless) moment can be calculated from equation (16) which is third degree in m, with a positive discriminant, i.e. it has one real and two complex conjugate roots. The only real root of (16) with the l.h.s. equals Ψ_o can be explicitly given by [12]:

$$m=\{12\psi_o+8\sqrt{1+(\frac{3\psi_o}{2})^2}\}^{\frac{1}{3}}+\{12\psi_o-8\sqrt{1+(\frac{3\psi_o}{2})^2}\}^{\frac{1}{3}} \quad (17)$$

This equation determines the end moment required to deflect the beam end by an angle Ψ_o. For the problem under consideration one is mainly concerned with the kinematics of the manipulator arms, while, on the force side, it should only be ensured that the end-effectors are capable of exserting the required forces. Having determined m, from eqn. (17), other relevant quantities can be directly calculated. The dimensionless reduction in length is calculated using the expression [10]:

$$\bar{\delta}_1=\int_1^0\bar{w}_1'^2d\xi \quad (18)$$

yielding

$$\bar{\delta}_1=\bar{\delta}_o+\frac{m^4}{48}(\frac{1}{5}+\frac{m^2}{168}) \quad (19)$$

The expansion of the nonlinear term in eqn. (9) was intentionally restricted to the first term, since this leads to the 3rd degree equation (16), for which an explicit solution can be given. With only one more term in the binomial expansion we would have arrived at a 5th degree equation with no possibility of giving explicit expressions to any of its roots. Indeed, the given real root of the 3rd degree equation can be used a starting value for roots of higher order equations as will be done with the following 2nd approximation.

Before turning our attention to the second approximation, we examine eqn. (14) more closely. It can be put in the form

$$\bar{w}_1'(\xi)=\bar{w}_o'(\xi)+\frac{1}{3}\bar{w}_o'^3(\xi)$$

This equation is in formal agreement with the result arrived at in [5], according to which the slope in the case of large deflections is a power series in the linear solution, thus validating the procedure used here.

THE SECOND APPROXIMATION

In order to further improve the accuracy of the solution for the deflection, a second approximation is sought. It is obtained by using eqn. (12) again, however with the first approximation inserted on the r.h.s., namely

$$\bar{w}''_2 = -m + \bar{w}_1'^2 \bar{w}_1'' \tag{20}$$

This method of successive approximation was applied to the problem of post-buckling behavior of bars and found to give very good results, as mentioned earlier.

Integrating eqn. (20) twice and satisfying the boundary conditions yields

$$\begin{aligned}\bar{w}_2(\xi) = \bar{w}_1(\xi) &+ \frac{m^5}{96}\{\xi - 5\xi^2 + \frac{40}{3}\xi^3 - 20\xi^4 + 16\xi^5 - \frac{16}{3}\xi^6\} + \\ &+ \frac{m^7}{1552}\{\xi - 7\xi^2 + 28\xi^3 - 70\xi^4 + 112\xi^5 - 112\xi^6 + 64\xi^7 - 16\xi^8\} + \\ &+ \frac{m^9}{41472}\{\xi - 9\xi^2 + 48\xi^3 - 168\xi^4 + 403.2\,\xi^5 - 672\xi^6 + 768\xi^7 \\ &- 576\xi^8 + 256\xi^9 - 51.2\,\xi^{10}\}\end{aligned} \tag{21}$$

The expressions for $\bar{w}_2{}'(\xi), \bar{w}_2{}''(\xi)$ which may be obtained through straightforward, though lengthy manipulation are omitted here for brevity. Here again the slope for the large deflection case can be expressed as a power series, namely

$$\bar{w}_2{}'(\xi) = \bar{w}_1{}'(\xi) + \frac{1}{3}\bar{w}_o{}'^5(\xi) + \frac{1}{9}\bar{w}'^7(\xi) + \frac{1}{81}\bar{w}_o{}'^9(\xi)$$

The dimensionless reduction in length is given by

$$\begin{aligned}\bar{\delta}_2 = \frac{m^2}{24}(1 &+ \frac{m^2}{10} + \frac{m^4}{48} + \frac{m^6}{432} + \frac{m^8}{4472.4} \\ &+ \frac{m^{10}}{53914} + \frac{m^{12}}{995327} + \frac{m^{14}}{33841100} + \frac{m^{16}}{2723210000})\end{aligned} \tag{22}$$

The slope $\bar{w}'_2(0)$ at the beam's left end is given by the expression

$$\bar{w}_2{}'(0) = \bar{w}_1{}'(0) + \frac{m^5}{96}(1 + \frac{m^2}{12} + \frac{m^4}{432}) \tag{23}$$

For a given slope at $\xi = 0$, it would in this case be necessary to solve eqn. (23) to obtain the corresponding required moment m. This can only be done numerically, since the equation is of the 9th degree. The obvious choice to converge directly onto the required root is to use the root as given by eqn. (17), corresponding to the first approximation, as a starting value and improve it through iteration using eqn. (23) without a need to solve for all its roots. In fact, for an end-slope angle of up to 30° the error in the value of m as given by eqn. (17) and that obtained by solving eqn. (23) iteratively is in the order of 2%; the corresponding error in $\bar{\delta}$ is also of the same order of magnitude. Beyond this angle, very few iterations would be required to obtain m with a high degree of accuracy.

KINEMATICS OF THE END-EFFECTORS

The required deflection of the beam will be introduced through moving the manipulator arms which carry the end-effectors. The total motion is to be divided into several steps to ensure stepwise synchronization between the performed transnational and rotational movements so that they are always consistent. The motion should be slow enough to avoid dynamic effects. The manipulator arms will step through from the original to the final position in steps of, say, 10°. This will be accomplished through successive MOVE instructions to the manipulator arms. The coordinates for the (k+1)-position will be given relative to the previous k-position.

Figure (2) shows two consecutive positions of the right manipulator axis S^R corresponding to the end-slope angle of the beam right end of Ψ_k and Ψ_{k+1}, respectively. The coordinates of S^R_{k+1} relative to $S^R{}_k$ can, after elementary geometrical manipulation, be given by the values triple:

$\bar{x}$*-travel:*

$$\bar{x}_{k+1} - \bar{x}_k = L(\bar{\delta}_{k+1} - \bar{\delta}_k)/2 + R(\cos\psi_k - \cos\psi_{k+1}) \tag{24}$$

$\bar{y}$*-travel:*

$$\bar{y}_{k+1} - \bar{y}_k = R(\sin\psi_{k+1} - \sin\psi_k) \quad (25)$$

rotation:

$$\psi_{k+1} - \psi_k = \gamma_k \quad (26)$$

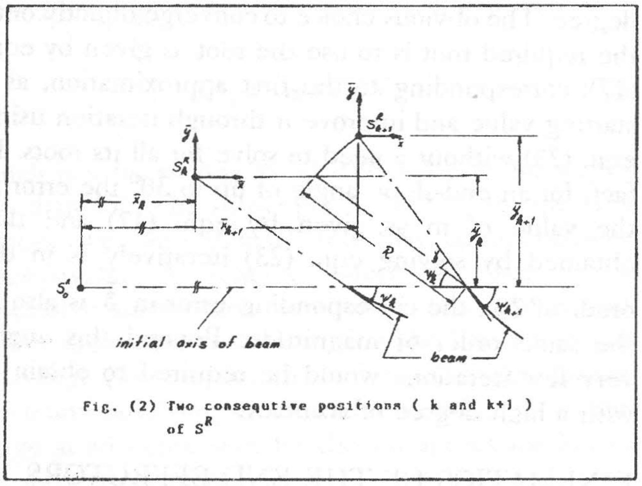

Figure 2. Two consecutive positions (k and k+1) of S^R.

where, γ_k is the increment of the slope angle at the beam right end.

These coordinates are given in a system of coordinates fixed to, but not rotating with, the manipulator rotation axis. These coordinates produce the mode of bending shown in Figure (3-a), where both ends of the beam are kept on the original axis, i.e. without change in the vertical level of its axis. Another mode of bending may be thought of in which the tangent to the beam at the point of maximum deflection (at midspan) does not change its level, i.e. the lowest point of the beam is always on the original axis of the beam, as shown in Figure (3-b). This can be achieved by increasing the $\bar{y}$-travel by an amount corresponding to the middle point deflection of the beam given by:

for the linear solution:

$$\bar{w}_o(\xi = 0.5) = \frac{m}{8} \quad (27)$$

for the first approximation

$$\bar{w}_1(\xi = 0.5) = \bar{w}_o(\xi = 0.5) + \frac{m^3}{192} \quad (28)$$

and finally for the second approximation

$$\bar{w}_2(\xi = 0.5) = \bar{w}_1(\xi = 0.5) + \frac{m^5}{1152}\left(1 + \frac{m^2}{16} + \frac{m^4}{720}\right) \quad (29)$$

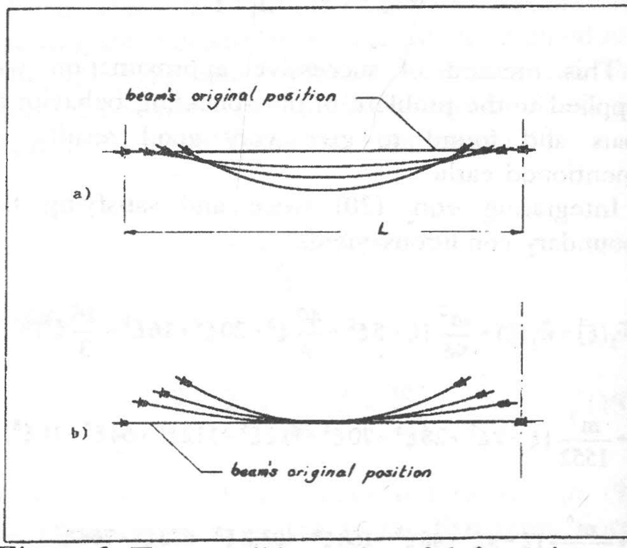

Figure 3. Two possible modes of deformation.

THE FORCE REGIME AT THE BEAM ENDS

The contact between the manipulator end-effectors and the beam needs some consideration. The loading at the beam ends is assumed to be consisting only of the bending moment M. It is virtually impossible to achieve a pure moment loading exactly at the very end of the beam, without introducing other force side-effects. Assuming a tight fit between the beam and clamp, the pressure distribution between the two bodies in the rotated position will roughly be as shown in Figure (4-a), with the result that the beam is actually loaded by two resultant forces, a small distance apart. The portion of the beam subjected to pure bending moment being the unsupported span of the beam. Since the analysis assumes that no axial forces are to be present, a tight fit would violate this assumption, rendering the beam statically indeterminate, and possibly resulting in buckling of the beam.

To avoid this, very small clearance should be allowed for between the end-effectors and beam, allowing for sliding motion between both. In this case the contact zone will be as shown in Figure (4-b). This has two effects. The first is that a difference in inclination between the beam and end-effectors will occur, as denoted by the clearance angle in the figure. However for small clearances this angle can be kept very small, in fact not exceeding few tenths of a degree. Even with large clearances, this effect can be accounted for by adding the clearance angle to the rotation angle of the end-effector as calculated by eqn. (26). The second effect is that, again, the beam is practically loaded by concentrated forces, with some small friction forces at the contact edges.

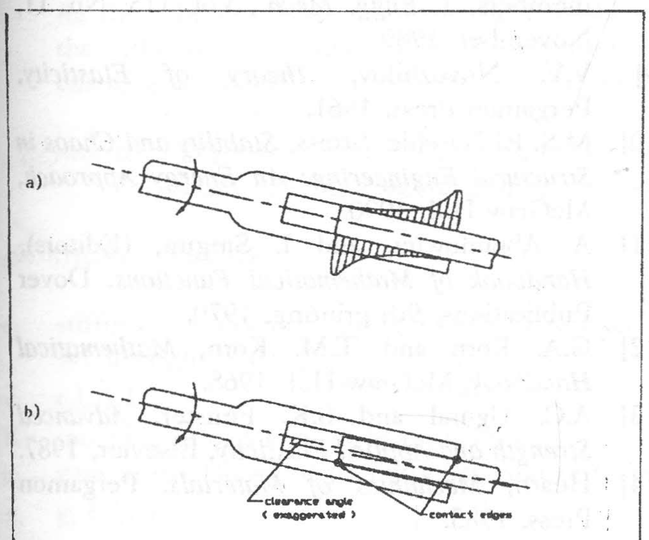

Figure 4. The contact between end-effector and beam.

All these effects causing deviation from the ideal assumed loading case will have no detrimental effect on the results. In fact there are no supports that qualify for being able to satisfy any ideal support requirements. According to St. Venants' Principle [13], however, the difference in effect of two different, yet equivalent, groups of forces is confined to a "small" region in their vicinity; far away from the point of application the effects of both force groups become equal. Since we are interested in the global behavior of the beam, the effect of this non-ideal support condition can be safely neglected.

EXAMPLE

The example given here is to present a typical case where the deflection of an aluminum bar is treated. We consider a beam having a rectangular cross section of 2x0.4 cm and a length of 50 cm, with the following material properties [14]:

Modulus of elasticity E = 69 GN/m^2
Elastic limit = 230 MN/m^2
Density = 2770 kg/m^3

Table (1) shows the $\bar{y}$ -,$\bar{y}$- travel and the rotation of the right end-effector for the different approximations for the mode of bending shown in Figure (3-a), using eqns. (24-26). The table shows, as expected, that for the larger angles the second approximation should be used as otherwise large deviation between the actual and required position of the end-effector would occur leading to increased forces on the end-effector and/or beam. For example, the error in the $\bar{x}$-travel between the linear and 2nd approximation is 7% and 16.85% for the rotation angles 30°° and 40°, respectively. On the other hand it is not necessary to introduce higher degree approximations since the differences would, in this example, be a fraction of a millimeter, and will not have any appreciable effect on the overall behavior of the beam. It is also noted that the $\bar{y}$-travel decreases as the degree of approximation becomes higher. This is to be expected, since ignoring higher terms usually adds stiffening to a structure.

Table 1.

degree of approx.	$\bar{x}$-travel (cm)	$\bar{y}$-travel (cm)	rotation (degree)
"0" approx.	1.003	1.684	20°
1st approx.	0.986		
2nd approx.	0.983		
"0" approx.	1.574	1.580	30°
1st approx.	1.499		
2nd approx.	1.471		
"0" approx.	2.545	1.428	40°
1st approx.	2.308		
2nd approx.	2.178		

This phenomenon is also noted in dynamic problems. For this example the bending stresses in the beam are approximately one order of magnitude less than the elastic modulus, so that the beam will not suffer any permanent deformation. The required moment M that should be applied by the end-effectors has in this case a maximum value of 25 N.m.

CONCLUSION

The process of bending a beam up to large deflections using a pair of manipulator arms is presented. Since large deflections were allowed, solution of the nonlinear deflection equation is found using successive approximation. It was found that this method agrees formally with solutions obtained before using integral power series. The end slopes, end moments and the reduction in length were all calculated. It was further found that the linear solution would lead to relatively high errors in calculating these quantities. More accurate expressions based on a second approximation were derived. The kinematics of the manipulator arms' movement are investigated together with aspects of the end fixation of the beam.

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