INELASTIC SPATIAL STABILITY OF RESTRAINED IMPERFECT BEAM-COLUMNS

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ABSTRACT

A mathematical model governing the inelastic behaviour of imperfect, restrained beam-columns is developed. The model incorporates also residual strains. The members are composed of arbitrary thin-walled cross-sections. Geometrical joint and member imperfections are included in the analysis. Spatial loadings, displacement and rotation springs are employed. Minimum total potential energy principle is used to develop the equilibrium equations. Material and geometrical non-linearities are considered iteratively.Comparison with the European Buckling Curves for struts is given. The deviations due to imperfections, residual strains and inelastic behaviour from the theoretical elastic stability theory for elastically connected columns are demonstrated.

Keywords: Beam-Columns, Spatial stability, Plastic analysis, Elastic springs.

INTRODUCTION

The stability of framed steel structures depends primarily on the stability their compression members.

In fact, a perfect elastic strut does not exist. The developed European Strut Curves, which are supported by enormous theoretical and experimental works, take into account the imperfections, residual strains and the inelastic behaviour of compression members [1,2]. Shear effects were proved to be small in compression members [5]. On the other hand the end conditions affect their behaviour significantly. The influence of member imperfections and joint properties on the elastic behaviour of space frame member is presented in [3]. Elastic stability expressions for elastically connected columns were developed in [4]. The aim of this study is to develop a numerical technique to determine the inelastic behaviour of imperfect beam-columns spatially loaded and restrained when considering residual strains. Spatial elastic displacement and rotation springs, distributed and concentrated, are employed. The equilibrium equations are generated using the minimum total potential energy principle. Geometrical and material non-linearities are considered iteratively.

Comparison between the proposed technique and the European Strut Curves is made.

The effects of imperfections, residual strains and inelastic behaviour on the elastic stability of elastically connected columns are demonstrated.

EQUILIBRIUM EQUATIONS

The following assumptions are considered in the present analysis:

1- Distortion of the cross-section during deformations is not allowed.
2- Small deformations.
3- Only shear deformations due to St. Venant torsion is considered.
4- The distribution of residual strains in the longitudinal direction is constant along the beam length.

The longitudinal normal strain ϵ and the St-Venant shear strain γ at a point (η, ξ, ω) at a distance x measured from the origin are given from differentiating the deformations of an arbitrary pole of the cross-section (u,v,w,β) due to external loadings and those due to the member and joint imperfections (v_o, w_o, β_o) as follow:

$$\epsilon = u' + v'_o v' + \frac{1}{2} v'^2 - (y - z(\beta_o + \beta)) v''$$
$$+ w'_o w' + \frac{1}{2} w'^2 - (z + y(\beta_o + \beta)) w''$$
$$+ z\beta v''_o - y \beta w''_o + (y^2 + w^2)(\beta'\beta'_o + \frac{1}{2}\beta'^2) - w\beta'' \quad (1\text{-a})$$
$$\gamma = 2 r \beta' \quad (1\text{-b})$$

where, r is the distance to the middle line of the thin wall and (y,z) are the local coordinate system before rotation.

The equilibrium equations are derived using the minimum total potential energy principle. For a beam segment cut from a deformed system, the edge forces must be in equilibrium with the internal stress resultants and the external loadings. The internal potential Π_i is given as:

$$\Pi_i = \int_{x_1}^{x_2} \left\{ \int_A \left(\int \sigma . d_\epsilon + \int \tau d\gamma \right) dA + \frac{1}{2}(c_y v_{cy}^2 + c_z w_{cz}^2 + c_\beta . \beta^2) \right\} dx \quad (2\text{-a})$$

where (c_y, c_z) are the distributed displacement spring constants and c_β is the distributed rotation spring constant.

The external potential Π_e is:

$$\Pi_e = - \int_{x_1}^{x_2} (p_y . v_{p_y} + p_z . w_{p_z} + p_x . u_{p_x}) dx$$
$$- (A_y . v + A_z . w - A_x . u - M_y . w' + M_z v' + M_x . \beta - M_\omega . \beta')_{x_1}^{x_2} \quad (2\text{-b})$$

where p,A,M are the applied loads, edge forces and edge flexural and warping moments in the three directions.

The total potential energy π is:

$$\Pi = \Pi_i + \Pi_e \quad (2\text{-c})$$

It could be written in the form:

$$\Pi = \int_{x_1}^{x_2} (Q_{\sigma,\tau} + Q_c + Q_p) dx + R_{(A,M)} \Big|_{x_1}^{x_2}$$
$$= \int_{x_1}^{x_2} Q . dx + R \Big|_{x_1}^{x_2} \quad (2\text{-d})$$

where Q and R are functions of deformations as follows:

$$Q = Q\,(x, \Psi, \Psi', \Psi'')$$
$$R = R\,(x, \Psi, \Psi')$$

Ψ presents global deformations (u,v,w,β). The system state of equilibrium requires that the total potential energy is to be minimum, i.e. $\delta\Pi = 0$. According to [6], and using Eq. (2) yields

$$\delta\Pi = \int_{x_1}^{x_2} \delta Q . dx + \delta R \Big|_{x_1}^{x_2}$$
$$= \int_{x_1}^{x_2} \left[\frac{\partial Q}{\partial \psi} - \left(\frac{\partial Q}{\partial \psi'}\right)' + \left(\frac{\partial Q}{\partial \psi''}\right)'' \delta \Psi . dx \right.$$
$$+ \left[\frac{\partial Q}{\partial \psi'} - \left(\frac{\partial Q}{\partial \psi''}\right)' + \frac{\partial R}{\partial \psi} \right]_{x_1}^{x_2} \delta \psi$$
$$+ \left[\frac{\partial Q}{\partial \psi''} + \frac{\partial R}{\partial \psi'} \right]_{x_1}^{x_2} \delta \psi' = 0 \quad (3)$$

This expression is zero if, and only if each expression between brackets is zero, hence

$$\frac{\partial Q}{\partial \psi} - \left(\frac{\partial Q}{\partial \psi'}\right)' + \left(\frac{\partial Q}{\partial \psi''}\right)'' = 0 \quad \text{Euler differential equation}$$
$$\left[\frac{\partial Q}{\partial \psi'} - \left(\frac{\partial Q}{\partial \psi''}\right)' + \frac{\partial R}{\partial \psi} \right]_{x_1}^{x_2} = 0$$
$$\left[\frac{\partial Q}{\partial \psi''} + \frac{\partial R}{\partial \psi'} \right]_{x_1}^{x_2} = 0 \quad \text{edge conditions} \quad (4\text{a-c})$$

Using Eq. (1) together with the expression of the first variation of the potentials for each of the deformations (u,v,w,β) and upon applying the stress resultant expressions, three second order differential equations, one first order equation and 7 edge conditions are obtained.

Seven 1^{st} order equations are formed by substituting the differentiations of the edge conditions into the 2^{nd} order differential equations. The yielding system of equations could be transformed into a linear system by replacing the stress resultants in all terms containing their product with displacements, with those determined in a previous step due to the first order theory (N_x^I, M_y^I, M_z^I and M_p^I).

The equilibrium equations due to the 2^{nd} order theory considering imperfections and distributed displacement and rotation elastic springs are given as:

$$A_y' = c_y\,(v - z_{cy}\beta) - p_y$$

$$M_z' = -A_y + N_x^{\,I}\,(v_o+v)' + p_x\,(y_{p_x} - z_{p_x}\,(\beta_o + \beta))$$

$$A_z' = c_z\,(w + y_{c_z}\,\beta) - p_z$$

$$M_y' = A_z - N_x^{\,I}\,(w_o+w)' - p_x\,(z_{p_x} + y_{p_x}\,(\beta_o + \beta))$$

$$M_x' = -c_y\,(v - Z_{c_y}\,\beta)\,\bar{z}_{cy}^{\,I} + C_z(w + y_{cz}\,\beta)\,\bar{y}_{c_z}{}^{I} + c_\beta \cdot \beta$$

$$- M_y^{\,I\prime}\,(v_o+v)' - M_z^{\,I\prime}\,(w_o+w)' + p_y\,\bar{z}_{p_x} - p_z\bar{y}_{p_z}$$

$$+ p_x\,(y_{p_x}(w_o+w)' \quad {}_{p_x}(v_o+v))$$

$$M_\omega' = M_x - M_T - M_p^{\,I}\,(\beta_o+\beta)' + M_y^{\,I}\,(v_o+v)'$$

$$+ M_z^{\,I}\,(w_o+w)' - p_x\omega$$

$$A_x' = -p_x \tag{5a-g}$$

where, the following abbreviated terms are introduced:

$$\bar{y} = y - z(\beta_o + \beta)\ ,\ \bar{z} = z + y(\beta_o + \beta) \tag{6}$$

STRESS RESULTANT DISPLACEMENT RELATIONSHIPS

The beam cross-section is discretized into a fine mesh. For each element i (x,y,z,ω) the stress σ_i is related to the strain ϵ_i through linear secant modulus S_i

$$\sigma_i = S_i\,\epsilon_i \tag{7}$$

By using the linear terms of Eq. (1-a), and considering the residual strains ϵ_r, the total strain is given as:

$$\epsilon_i = u' - y_i\,v'' - z_i\,w'' - w_i\,\beta'' - \epsilon_{ri} \tag{8}$$

S_i is determined corresponding to ϵ_i from the column material relationship.

Integrating the stresses Eq. (7) along the cross-sectional area according to the stress resultant definitions gives:

$$\{K\} = [D]\,\{V\} + \{D_{rs}\} \tag{9}$$

where,

{K} is the stress resultant vector, [D] is the rigidity matrix of the cross-section properties, {V} is the displacement vector and $\{D_{rs}\}$ is the vector determined from the residual stresses integrations along the cross-sectional area.

On the other hand the St-Venant torsion moment M_T is calculated directly according to the elastic shear modulus G:

$$M_T = 2\int 2\,G\,r^2\,dA\,\beta' = G\,I_t\,\beta' \tag{10}$$

From Eqs. (9) the stress resultant-displacement relations are expressed as:

$$\{V\}_{4x1} = [D]^{-1}\,\{K\} - [D]^{-1}\,\{D_{rs}\} \tag{11}$$

It is completed with the following new definitions:

$$\{V\}_{3x1} = \{\phi_z, -\phi_y, \rho\}^T \tag{12}$$

CONCENTRATED LOADS AND SPRINGS

The stress resultants and displacement vectors before and after the point of application of the concentrated loads and springs are related to each other through point matrix. This point matrix is developed by applying direct equilibrium conditions at the cross-section faces J and J+1 of the deformed beam-column. Hereby, concentrated displacement springs C_Ψ, rotation springs CC_Ψ and warping spring C_ω are considered. The point matrix is expressed as:
$\{\delta_{J+1}\}_{7x1} = \{\delta_J\}_{7x1}$

And the stress resultants:

$$A_{y_{J+1}} = A_{y_J} + C_y(v - z_{cy} - \beta) - P_y$$

$$A_{z_{J+1}} = A_{z_J} + C_z(w + y_{c_z}\cdot\beta) - P_z$$

$$A_{x_{J+1}} = A_{x_J} - P_x$$

$$M_{y_{J+1}} = M_{y_J} + CC_y\phi_y - P_x(\bar{Z}_{p_x})$$

$$M_{z_{J+1}} = M_{z_J} + CC_z\phi_z + P_x(\bar{Y}_{p_x})$$

$$M_{x_{J+1}} = M_{x_J} - C_y(v - z_{cy}.\beta)(\bar{Z}_{cy})$$

$$+ C_z(w + y_{cz}.\beta)(\bar{Y}_{cz}) + CC_x.\beta$$

$$+ P_y(\bar{Z}_{py}) - P_z(\bar{y}_{p_z})$$

$$M_{\omega_{J+1}} = M_{\omega J} - C_\omega \rho \qquad (13a\text{-}m)$$

SOLUTION TECHNIQUE

The beam-column is divided into parts; for each part the field transport matrix is developed through numerical integrations of Eqs. (5,11,12). AT the locations of concentrated loads or elastic springs, the point matrix, Eq. (13), is applied. The resulting system of linearized equations becomes solvable when it is separated into known and unknown variables which are either displacements or stress resultants according to the supporting conditions.

Inelastic behaviour of the material is considered in an internal iteration loop The secant modulus of each element of the cr[illegible]-sections along the beam length is modified till [illegible]vergence. Convergence is achieved, when the difference between the internal stress resultants (resulting from the integration of stresses along cross-sectional area) and the external stress resultants (given from the equilibrium equations) does not exceed a predefined convergence factor. In an external loop the loads are increased incrementaly till the critical load.

NUMERICAL VERIFICATIONS

Example 1

In order to verify the accuracy and efficiency of the proposed model the buckling curve is determined for the column of I-shaped cross-section, Figure (1). The initial member imperfection is taken a 2nd parabola, with a maximum midspan displacement v_o equal to:

$$v_o = \frac{1}{1000} \text{ (where l=span)}$$

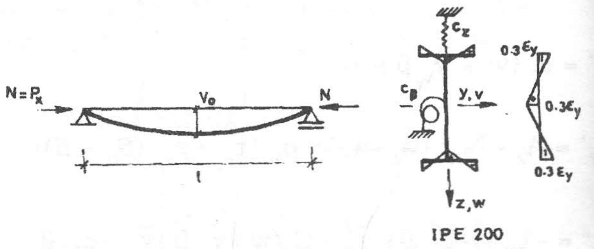

Figure 1. Example 1, Cross-section, imperfections, residual strains and distributed springs.

The residual strains are assumed as depicted in the figure with a maximum value of 0.3 of the yield strain. The material property is linear elastic -ideal plastic with a yield stress of 2.4 t/cm^2, E = 2100 t/cm^2 and μ = 0.3. Distributed displacement spring c_z and twisting spring c_β are initiated to ensure an inplane failure of the column. The buckling curve; buckling load versus slenderness ratio, is plotted in Figure (2) in a dimensionless form such that:

$$\bar{N} = \frac{N_u}{N_p}$$

$$\bar{\lambda} = \sqrt{N_p/N_e}$$

where, N_u is the ultimate load capacity, N_p is the plastic normal force, and N_e is the Euler buckling load.

The European Buckling Curve is plotted as well, in the same figure. Values of the European Strut Curve fall exactly on the present analysis model results. This is true for all slenderness ratios.

Example 2

The following example is intended to demonstrate the effect of "real world" conditions in comparison with the "idealistic" assumption.

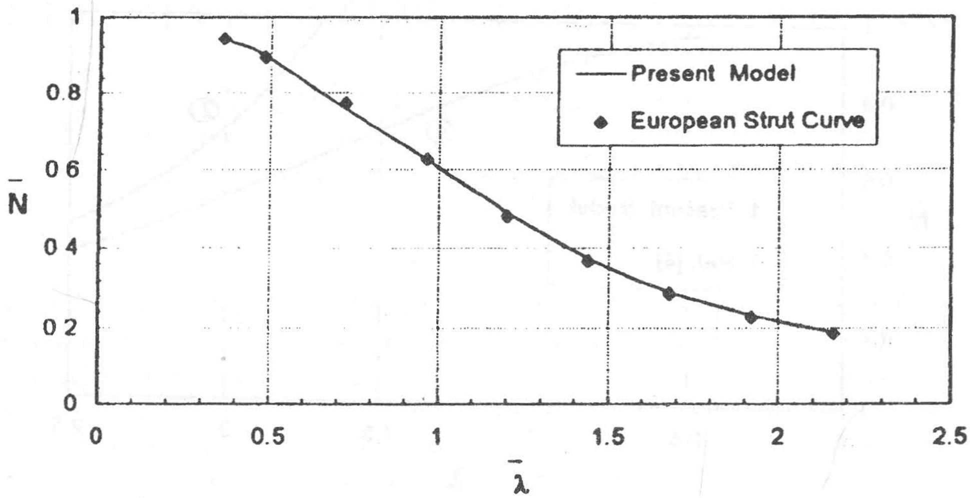

Figure 2. Bukling curves for the column of example 1.

The H-shape column cross-section of Figure (3) is considered. Imperfections, residual strains, distributed springs and material properties are similar to those of the first example. In addition, two concentrated elastic rotational springs CC_z exist at both ends of the column. Results are compared with the theoretical elastic stability expressions defined in [4], Figure (4).

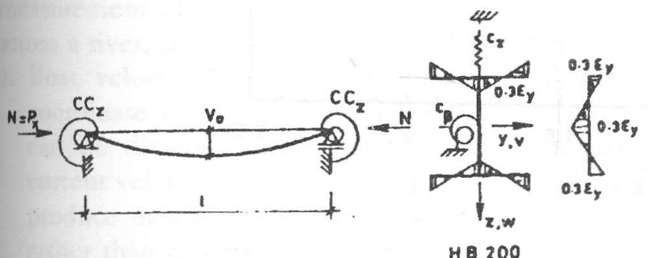

Figure 3. Elastically connected column of example 2, imperfections, residual strains and distributed springs.

The elastic rotational spring CC_z is taken similar to Ref. [4] as:

$CC_z = \xi \dfrac{EI}{l}$ where ξ was taken equals to 10.

The corresponding normal force parameter q defined in Ref. [4] was:

$$q = \frac{N_u}{N_e} = 2.825$$

Figure (5) shows the normal force parameter q as developed in [4] and in the present analysis. The ratio between the actual value of q as developed by the present model and the theoretical one given in [4] is 0.84 and 0.64 when $\bar{\lambda}$ is 2.5 and 1.68 respectively. For smaller values of $\bar{\lambda}$, q (theoretical) is not practically applicable.

CONCLUSIONS

A mathematical model representing the inelastic spatial stability of restrained imperfect beam-columns is developed. Influence of residual strains, arbitrary cross-section and arbitrary material properties are included. The comparison with the European Buckling Curve proves the accuracy of the model. Applying the present analysis on elastically connected columns shows the necessity of incorporating inelastic behaviour, imperfections and residual strains in designing such columns. With the aid of the present mathematical model and the corresponding computer program, better design formulae applicable to real column conditions cloud be established.

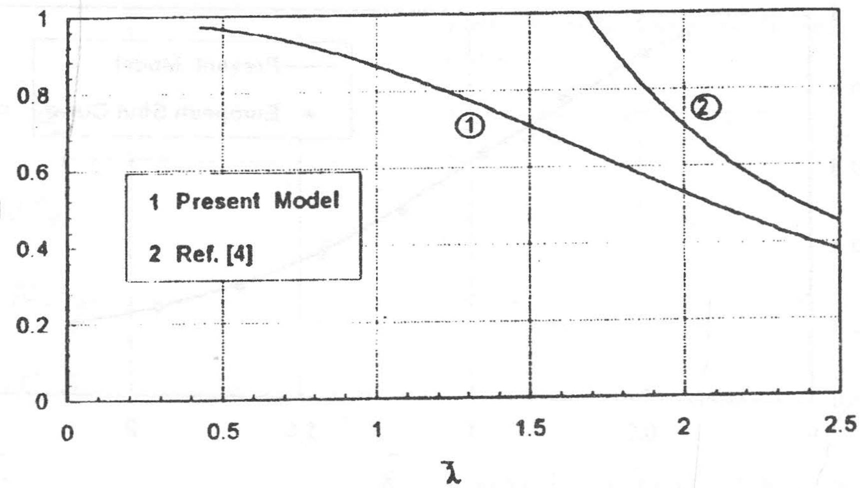

Figure 4. Buckling curves for the elastically connected column of example 2.

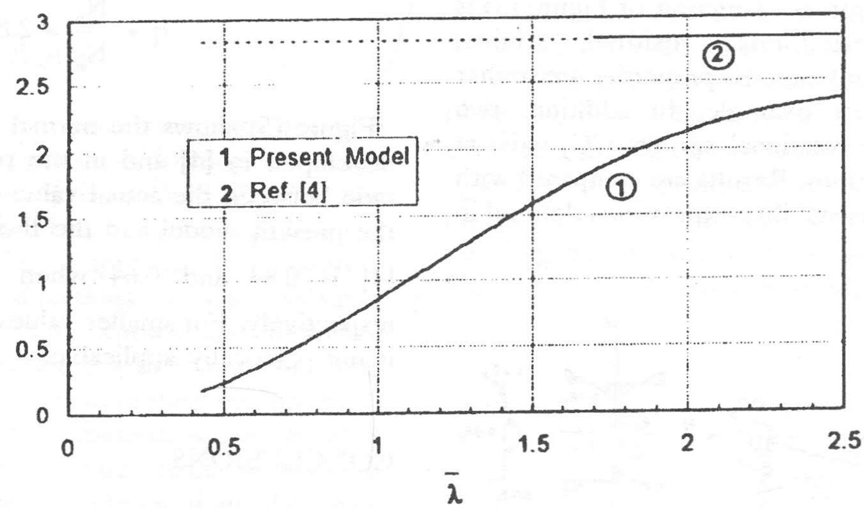

Figure 5. Normal force parameter $q = N_u/N_e$ vs. selenderness ratio $\bar{\lambda}$.

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