EXACT FORMULATION FOR DISCRETE FEEDBACK CONTROL OF AN ELASTICALLY SUPPORTED VIBRATING BEAM CARRYING CONCENTRATED MASSES AND SUBJECTED TO AXIAL FORCE

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ABSTRACT

An exact formulation for the problem of discrete displacement and velocity feedback for structural control of an elastically supported vibrating beam carrying concentrated masses and subjected to axial force is given. Using Laplace transform method (on the spatial variable), a closed form expression for the solution of the boundary value problem is obtained. The analysis is completed by deriving the orthogonality condition for the eigenforms. The formulation allows for studying the effect of changing actuators' positions and/or gains on the response of the system.

Keywords: Discrete feedback, Direct output feedback control, Structural control, Elastically supported beam, Beam carrying concentrated masses, Boundary value problem.

NOMENCLATURE

c_i	stiffness of torsional spring i (i=0 for left end, i=1 for right end)
$\bar{c}_i$	$c_i L/EI$
E	modulus of elasticity of beam material
g_i	gain of displacement-proportional control force
$\bar{g}_i$	$g_i L^3 / EI$
h_i	gain of velocity-proportional control force
$\bar{h}_i$	$h_i L/\sqrt{EI\mu}$
i	$\sqrt{-1}$
I	2nd moment of area of beam cross-section about y-axis
$\bar{J}_k$	$J_k / \mu L^3$
J_k	mass moment of inertia of k-th mass about y-axis
k_i	stiffness of translational spring i (i=0 for left end, i=1 for right end)
$\bar{k}_i$	$k_i L^3/EI$
K	total number of concentrated masses
L	beam length
M	total number of actuators
M_k	k-th concentrated mass
m_k	$M_k/\mu L$
P	axial force, positive when compressive
$q(\tau)$	dimensionless time function
t	time
T	$\sqrt{\mu L^4/EI}$
$x_k^{(m)}$	x-position of k-th mass
$x_i^{(c)}$	x-position of i-th actuator
u(x-a)	unit step function at x = a
w(x,t)	beam deflection
x, y, z	right-handed coordinate system
α	$\sqrt{\lambda/2}$
β	PL^2/EI
χ_i	$x_i^{(c)}/L$
δ(x-a)	Dirac delta function at x = a
$\bar{\delta}$(x-a)	$L\delta$(x-a), dimensionless Dirac delta function at x = a
$\epsilon_i(\xi)$	see appendix I
Γ, Γ_{ij}	overall system matrix and its submatrices, see eq. (16)
γ_{ij}	elements of submatrices Γ_{ij} (see eq. (16) and appendix II)
η	see eqs. (8,9)
$\phi(\xi)$	separated spatial dimensionless deflection function

parameter systems, two types of control are possible: discrete or continuous control. In discrete control the control forces are applied at specific finite number of points on the structure, whereas in the continuous type they are distributed all over the structure. It is obvious that the use of discrete control is much easier to achieve than continuous control, which needs special materials (e.g., viscoelastic polymers [8,9]) and is more difficult to design and manufacture for full-scale, even only moderately-sized structures.

For the mathematical treatment of discrete control of distributed-parameter systems it is customary to employ some sort of eigenfunction expansion, normally with a small number of eigenfunctions participating in the calculated overall response of the structure. In such so-called projection methods, where the infinite-dimensioned original problem is "projected" on a finite-dimensioned space, truncation error is inherently present. Moreover, in the special case of structural control, the phenomenon of spill-over [3] may occur, whereby the uncontrolled modes might be adversely affected by the control of other modes. Hence, it would be advantageous to be able to treat discretely controlled distributed-parameter systems without reverting to the truncated eigenfunction expansion approach. It is the aim of this paper to introduce such an exact formulation of this problem.

Vibration of distributed-parameter (often, though less appropriately, also called continuous) systems, taking concentrated effects due to a variety of origins into account (sometimes referred to as "combined" systems [4]), has been a subject of research for a very long time. More than 60 years ago Gassmann [19] has studied the problem of a beam carrying a concentrated mass. Earlier work has however been hampered by the limitation on possible calculation complexity required to solve these problems. Different techniques has since been employed to solve the problem, eigenfunction expansion being used in the majority of cases. So did, for example, Buttler [20], in 1967, who treated the problem of a beam carrying a spring-mass-damper system using eigenfunction expansion. Akin and Mofid [21], in 1989, applied the same method while considering the problem of a beam with a moving mass. (They refer in their paper to several earlier treatments of this problem, some dating back as early as 1929). The other approach which proved very efficient is the energy approach (which is also in essence a projection method). Warburton, who has for long time studied extensively the application and optimization of vibration absorbers for lumped-parameter systems, has, together with Ayorinde, employed the energy method to treat the problem of reducing the vibration of distributed-parameter systems using a spring-mass-damper system [22,23]. The discrete vibration absorber used, being a passive one, was tuned only to the fundamental vibration mode of the considered distributed-parameter structures, namely plates and shells. Laura et al. [24] has extensively studied the vibration of beams carrying concentrated masses also using the energy approach. In recent papers, Low et al. [25] and Boay [26] studied the vibrations of plates carrying concentrated masses using energy method. Meanwhile, Leckie and Lindberg [27] used the finite difference method to study the effect of lumping beam parameters on its frequencies. Transfer matrix methods has also been extensively developed, e.g., in the classical work of Pestel and Leckie [28]. Plunkett [29] used influence coefficients to calculate optimum concentrated damping for distributed-parameter systems. Still other solution approaches exist for combined systems, e. g. [30,31], where another exact method, the transfer function synthesis, is introduced. This method is suitable for complex, multi-connected, multi-branched combined systems. A short survey of other methods is also given in [30].

Contrary to the field of classical control, using Laplace transform to solve vibration roblems has not been very widely adopted; it was mostly reserved for special problems. Yen [32] treated the problem of a vibrating beam with time-dependent boundary conditions, while Florence [33] treated the problem of a traveling force on a Timoshenko beam, both using Laplace transform on the time variable. However, it is usually more productive to transform the spatial variable because the time dependence is often a simple initial value problem [4]. Pan [34] considered the problem of a beam carrying concentrated masses and used the Laplace transform on the spatial variable. Grant [35] used the same approach to solve the problem of vibration of a

Timoshenko beam carrying a single mass in the middle. Chonan [36] treated the problem of a moving harmonic load using Fourier transform on the spatial variable. The use of transform methods has the advantage of easily dealing with discontinuities in the system. Also, the exponential form of the time and spatial functions lends itself readily to Laplace and Fourier types of transforms. Moreover, transform methods can yield exact closed form solutions, though at the cost of moderately increased amount of analytical and numerical burden. Hence, we adopt this approach here.

The configuration of the problem considered here is aimed at representing as near as possible actual unidimensional structures. The choice of elastic restraint at beam ends as the support condition is motivated by the fact that it represents a realistic simulation of actual supports, since structural members are usually supported by other members that always show some degree of flexibility. On the other hand, concentrated masses resemble real structures, where equipment and other types of concentrated loads are present.

PROBLEM DEFINITION

Consider a uniform slender beam restrained at each end by one translational and one rotational spring, Figure (1).

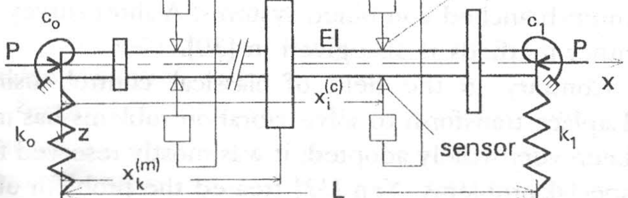

Fig. (1)

The beam carries a number of arbitrarily-located concentrated masses M_k with their centers of gravity laying on the beam longitudinal axis. Each mass has a rotary mass moment of inertia J_k about the beam y-axis. An axial force P, considered positive when compressive, acts on the beam. This axial force should be less than the buckling load of the beam for the given boundary conditions. Further, sensors are used to measure the displacement and velocity of the beam at a finite number of arbitrarily-located points. Based on these measurements, control forces are generated, with the proportionality constants g_i and h_i as the control law, and applied at the respective measuring points through actuators. This type of control is called direct output feedback control and is considered particularly useful for distributed-parameter systems. [3,37-40] Physical external or internal damping is not considered. However, its addition to the problem poses no difficulty. It would simply add to the corresponding term resulting from the velocity-proportional control force.

The differential equation of motion governing the transverse vibration of the beam is [34,35,41,42]:

$$\begin{aligned} &EIw''''(x,t)+Pw''(x,t)+\mu\,\ddot{w}(x,t)+ \\ &+\sum_{k=1}^{K}\{M_k\ddot{w}(x_k^{(m)},t)\delta(x-x_k^{(m)})\}- \\ &-\sum_{k=1}^{K}\{J_k\ddot{w}'(x_k^{(m)},t)\delta(x-x_k^{(m)})\}'+ \\ &+\sum_{i=1}^{M}\{[g_i w(x_i^{(c)},t)+h_i\dot{w}(x_i^{(c)},t)]\delta(x-x_i^{(c)})\}=0 \end{aligned} \tag{1}$$

The concentrated effects are taken into account by making use of the Dirac delta function, which allows expressing concentrated effects as distributed functions, thus - formally - removing the discontinuities and hence allowing integration. It should be noted that the dimensions of Dirac delta function here is (1/length). The inclusion of a velocity-proportional control force leads to the appearance of a "damping" term, although no physical damping devices are present. The displacement-proportional term contributes towards changing the original stiffness term. The boundary conditions for the elastically restrained ends are [43]:

at x = 0:

$$EIw'''(0,t) + k_o\, w(0,t) = 0 \tag{2a}$$

$$EIw''(0,t) - c_o\, w'(0,t) = 0 \tag{2b}$$

and at x = L :

$$EIw'''(L,t) - k_1 w(L,t) = 0 \tag{3a}$$

$$EIw''(L,t)+c_1w'(L,t)=0 \quad (3b)$$

These boundary conditions can be specialized to represent any type of boundary conditions, especially the three "standard" boundary conditions: free, simply supported and built-in end, by assigning proper limiting values (0 or ∞) to respective spring stiffnesses.

Before solving eq. (1) it will be convenient to cast it in a dimensionless form by changing to the dimensionless spatial and time variables:

$$\xi = x/L$$

$$\tau = t/T$$

and the dimensionless deflection

$$\bar{\phi}(\xi,\tau) = w(\xi,\tau)/L$$

The resulting equation for $\bar{\phi}(\zeta,\tau)$ can be solved using separation of variables by assuming a solution in the form:

$$\bar{\phi}(\xi,\tau)=\phi(\xi).q(\tau)=\phi(\xi)\,.\,e^{\lambda\tau}\quad,$$

where λ is a dimensionless parameter. The separated equation for the dimensionless spatial function $\phi(\xi)$ is then:

$$\phi''''(\xi)+\beta^2\phi''(\xi)+\lambda^2\phi(\xi)+ +\lambda^2\sum_{k=1}^{K}\{m_k\phi(\nu_k)\bar{\delta}(\xi-\nu_k)\}- -\lambda^2\sum_{k=1}^{K}\{j_k\phi'(\nu_k)\bar{\delta}(\xi-\nu_k)\}'+ +\sum_{i=1}^{M}\{(\bar{g}_i+\bar{h}_i\lambda)\bar{\delta}(\xi-\chi_i)\phi(\chi_i)\}=0 \quad (4)$$

with the boundary conditions:

at $\zeta = 0$:

$$\phi'''(0)+\bar{k}_0\phi(0)=0 \quad (5a)$$

$$\phi''(0)-\bar{c}_0\phi'(0)=0 \quad (5b)$$

at $\xi = 1$:

$$\phi'''(1)-\bar{k}_1\phi(1)=0 \quad (6a)$$

$$\phi''(1)+\bar{c}_1\phi'(1)=0 \quad (6b)$$

SOLUTION

The solution of eq. (4) will be found using Laplace transform with respect to the spatial variable ξ. Straight forward application of Laplace transform and its inverse [44] leads to the solution:

$$\phi(\xi)=\left[\frac{-\eta^2\cos\eta\xi+\zeta^2\cos\zeta\xi}{(\zeta^2-\eta^2)}+\beta^2\frac{\cos\eta\xi-\cos\zeta\xi}{(\zeta^2-\eta^2)}\right]\phi(0)+ +\left[\frac{-\eta\sin\eta\xi+\zeta\sin\zeta\xi}{(\zeta^2-\eta^2)}+\beta^2\frac{\zeta\sin\eta\xi-\eta\sin\zeta\xi}{\eta\zeta(\zeta^2-\eta^2)}\right]\varphi'(0)+ +\left[\frac{\cos\eta\xi-\cos\zeta\xi}{(\zeta^2-\eta^2)}\right]\phi''(0)+\left[\frac{\zeta\sin\eta\xi-\eta\sin\zeta\xi}{\eta\xi(\zeta^2-\eta^2)}\right]\phi'''(0)- -\lambda^2\sum_{k=1}^{K}m_k\left[\frac{\zeta\sin\eta(\xi-\nu_k)-\eta\sin\zeta(\xi-\nu_k)}{\eta\zeta(\zeta^2-\eta^2)}\right]\phi(\nu_k)u(\xi-\nu_k)+ +\lambda^2\sum_{k=1}^{K}j_k\left[\frac{\cos\eta(\xi-\nu_k)-\cos\zeta(\xi-\nu_k)}{(\zeta^2-\eta^2)}\right]\phi'(\nu_k)u(\xi-\nu_k)- -\sum_{i=1}^{M}(\bar{g}_i+\bar{h}_i\lambda)\left[\frac{\zeta\sin\eta(\xi-\chi_i)-\eta\sin\zeta(\xi-\chi_i)}{\eta\zeta(\zeta^2-\eta^2)}\right]\phi(\chi_i)u(\xi-\chi_i) \quad (7)$$

where η and ζ are given by:

case a: for $\frac{\beta^4}{4} > \lambda^2$, η and ζ are both real:

$$\eta=\pm\sqrt{(\beta^2/2)-\sqrt{(\beta^4/4)-\lambda^2}} \quad (8a)$$

$$\zeta=\pm\sqrt{(\beta^2/2)+\sqrt{(\beta^4/4)-\lambda^2}} \quad (8b)$$

case b:

for $\frac{\beta^4}{4} < \lambda^2$, η and ζ are both complex:

$$\eta=\pm\sqrt{(\beta^2/2)-i\sqrt{(\lambda^2-(\beta^2/4)}} \quad (9a)$$

$$\zeta=\pm\sqrt{(\beta^2/2)+i\sqrt{(\lambda^2-(\beta^2/4)}} \quad (9b)$$

The solution for the special case where no axial force is applied to the beam ends will also be given here:

$$\begin{aligned}
\phi(\zeta) &= [\cos\alpha\xi\cosh\alpha\xi]\phi(0) + \\
&+ [(\sin\alpha\xi\cosh\alpha\xi + \cos\alpha\xi\sinh\alpha\xi)/2\alpha]\phi'(0) + \\
&+ [(\sin\alpha\xi\sinh\alpha\xi)/2\alpha^2]\phi''(0) + \\
&[(\sin\alpha\xi\cosh\alpha\xi - \cos\alpha\xi\sinh\alpha\xi)/4\alpha^3]\phi'''(0) - \\
&-(\frac{\lambda^2}{4\alpha^3})\sum_{k=1}^{K} m_k[\sin\alpha(\xi-\nu_k)\cosh\alpha(\xi-\nu_k) - \\
&-\cos\alpha(\xi-\nu_k)\sinh\alpha(\xi-\nu_k)]\phi(\nu_k)u(\xi-\nu_k) + \\
&+(\frac{\lambda^2}{2\alpha^2})\sum_{k=1}^{K} j_k[\sin\alpha(\xi-\nu_k)\sinh\alpha(\xi-\nu_k)]\phi'(\nu_k)u(\xi-\nu_k) - \\
&-(\frac{1}{4\alpha^3})\sum_{i=1}^{M} \{(\bar{g}_i+\bar{h}_i\lambda)[\sin\alpha(\xi-\chi_i)\cosh\alpha(\xi-\chi_i) - \\
&-\cos\alpha(\xi-\chi_i)\sinh\alpha(\xi-\chi_i)]\phi(\chi_i)u(\xi-\chi_i),
\end{aligned}$$

with $\alpha = \sqrt{\lambda/2}$

However, the analysis will be further developed below for the more general case including the axial force. The special case (with P = 0) can be treated following completely analogous path.

Using the abbreviations given in appendix I, the deflection $\varphi(\xi)$ and the boundary conditions can be expressed in a more concise form, namely

$$\begin{aligned}
\phi(\xi) &= \bar{\epsilon}_o(\xi)\phi(0) + \bar{\epsilon}_1(\xi)\phi'(0) + \\
&\epsilon_2(\xi)\phi''(0) + \epsilon_3(\xi)\phi'''(0) - \\
&-\lambda^2\sum_{k=1}^{K}\{m_k\epsilon_3(\xi-\nu_k)\phi(\nu_k)\}u(\xi-\nu_k) + \\
&+\lambda^2\sum_{k=1}^{K}\{j_k\epsilon_2(\xi-\nu_k)\phi'(\nu_k)\}u(\xi-\nu_k) - \\
&-\lambda^2\sum_{i=1}^{M}\{(\bar{g}_i+\bar{h}_i\lambda)\epsilon_3(\xi-\chi_i)\phi(\chi_i)\}u(\xi-\chi_i)
\end{aligned} \tag{10}$$

with boundary conditions:

at $\xi = 0$:

$$\phi'''(0) + \bar{k}_o\phi(0) = 0 \tag{11a}$$

$$\phi''(0) - \bar{c}_o\phi'(0) = 0 \tag{11b}$$

and at $\zeta = 1$:

$$\begin{aligned}
&[\bar{\psi}_o(1) - \bar{k}_1\bar{\epsilon}_o(1)]\phi(0) + [\bar{\psi}_1(1) - \bar{k}_1\bar{\epsilon}_1(1)]\phi'(0) + \\
&+[\psi_2(1) - \bar{k}_1\epsilon_2(1)]\phi''(0) + [\psi_3(1) - \bar{k}_1\epsilon_3(1)]\phi'''(0) + \\
&+\lambda^2\sum_{k=1}^{K}\{m_k[\psi_3(1-\nu_k) - \bar{k}_1\epsilon_3(1-\nu_k)]u(1-\nu_k)\}\phi(\nu_k) - \\
&-\lambda^2\sum_{k=1}^{K}\{j_k[\psi_2(1-\nu_k) - \bar{k}_1\epsilon_2(1-\nu_k)]u(1-\nu_k)\}\phi'(\nu_k) + \\
&+\sum_{i=1}^{M}\{(\bar{g}_i+\bar{h}_i\lambda)[\psi_3(1-\chi_i) - \bar{k}_1\epsilon_3(1-\chi_i)]u(1-\chi_i)\}\phi(\chi_i) = 0
\end{aligned} \tag{12a}$$

$$\begin{aligned}
&[\bar{k}_o(1) + \bar{c}_1\bar{\theta}_o(1)\phi(0) + [\bar{k}_1(1) + \bar{c}_1\bar{\theta}_1(1)]\phi'(0) + \\
&+[k_2(1) + \bar{c}_1\theta_2(1)]\phi''(0) + [k_3(1) + \bar{c}_1\theta_3(1)]\phi'''(0) + \\
&+\lambda^2\sum_{k=1}^{K}\{m_k[k_3(1-\nu_k) + \bar{c}_1\theta_3(1-\nu_k)]u(1-\nu_k)\}\phi(\nu_k) - \\
&-\lambda^2\sum_{k=1}^{K}\{j_k[k_2(1-\nu_k) + \bar{c}_1\theta_2(1-\nu_k)]u(1-\nu_k)\}\phi'(\nu_k) + \\
&+\sum_{i=1}^{M}\{(\bar{g}_i+\bar{h}_i\lambda)[k_3(1-\chi_i) + \bar{c}_1\theta_3(1-\chi_i)]u(1-\chi_i)\}\phi(\chi_i) = 0
\end{aligned} \tag{12b}$$

The four above boundary conditions produce four equations for the first four unknowns: $\phi(0)$, $\phi'(0), \phi''(0)$, $\phi\prime(0)$. The rest of the equations for the rest of the unknowns is obtained through constructing the so-called consistency equations [34], found by consecutively substituting $\zeta = \upsilon_k$-a in eq. (10) then in its first derivative (with k = 1,2,...,K), and again in eq. (10) (with i = 1,2,...,M), where 'a' is a small quantity, and in each case letting a→0. [34,35] This is, in effect, equivalent to successively solving for the rest of the unknowns, namely the K deflections and K slopes at ν_k and the M deflections at χ_i. The consistency equations thus obtained are:

$$\begin{aligned}
&\bar{\epsilon}_o(\nu_r)\phi(0) + \bar{\epsilon}_1(\nu_r)\phi'(0) + \\
&+\epsilon_2(\nu_r)\phi''(0) + \epsilon_3(\nu_r)\phi'''(0) + \\
&+\lambda^2\sum_{k=1}^{r-1}\{m_k\epsilon_3(\nu_r-\nu_k)u(\nu_r-\nu_k)\}\phi(\nu_k) - \\
&-\lambda^2\sum_{k=1}^{r-1}\{j_k\epsilon_2(\nu_r-\nu_k)u(\nu_r-\nu_k)\}\phi'(\nu_k) + \\
&+\sum_{i=1}^{M}\{(\bar{g}_i+\bar{h}_i\lambda)\epsilon_3(\nu_r-\chi_i)u(\nu_r-\chi_i)\}\phi(\chi_i) - \\
&-\phi(\nu_r) = 0, \qquad r = 1,\ldots,K
\end{aligned} \tag{13}$$

$$\bar{\theta}_o(\nu_r)\phi(0)+\bar{\theta}_1(\nu_r)\phi'(0)+\theta_2(\nu_r)\phi''(0)+\theta_3(\nu_r)\phi'''(0)+\lambda^2\sum_{k=1}^{r-1}\{m_k\theta_3(\nu_r-\nu_k)u(\nu_r-\nu_k)\}\phi(\nu_k)-\lambda^2\sum_{k=1}^{r-1}\{j_k\theta_2(\nu_r-\nu_k)u(\nu_r-\nu_k)\}\phi'(\nu_k)+\sum_{i=1}^{M}\{(\bar{g}_i+\bar{h}_i\lambda)\theta_3(\nu_r-\chi_i)u(\nu_r-\chi_i)\}\phi(\chi_i)-\phi'(\nu_r)=0,\qquad r=1,\ldots,K \tag{14}$$

$$\bar{\epsilon}_o(\chi_q)\phi(0)+\bar{\epsilon}_1(\chi_q)\phi'(0)+\epsilon_2(\chi_q)\phi''(0)+\epsilon_3(\chi_q)\phi'''(0)+\lambda^2\sum_{k=1}^{K}\{m_k\epsilon_3(\chi_q-\nu_k)u(\chi_q-\nu_k)\}\phi(\nu_k)-\lambda^2\sum_{k=1}^{K}\{j_k\epsilon_2(\chi_q-\nu_k)u(\chi_q-\nu_k)\}\phi'(\nu_k)+\sum_{i=1}^{q-1}\{(\bar{g}_i+\bar{h}_i\lambda)\epsilon_3(\chi_q-\chi_i)u(\chi_q-\chi_i)\}\phi(\chi_i)-\phi(\chi_q)=0,\qquad q=1,\ldots,M \tag{15}$$

Equations (11-15) form a homogeneous system of equations for the (4+2K+M) unknowns: $\phi(0), \phi'(0), \phi''(0), \phi'''(0), \phi(\upsilon_k), \phi'(\upsilon_k)$ and $\phi(\chi_i)$. This can be put in the matrix form:

$$\Gamma.\,\Phi = 0, \tag{16a}$$

or

$$\begin{bmatrix}\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14}\\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24}\\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{34}\\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44}\end{bmatrix}\begin{bmatrix}\bar{\Phi}_1\\ \bar{\Phi}_2\\ \bar{\Phi}_3\\ \bar{\Phi}_4\end{bmatrix}=0, \tag{16b}$$

where

$$\begin{aligned}\bar{\Phi}_1&=[\phi(0)\quad \phi'(0)\quad \phi''(0)\quad \phi'''(0)]^t\\ \bar{\Phi}_2&=[\phi(\nu_1)\quad \phi(\nu_2)\quad \ldots\quad \phi(\nu_k)]^t\\ \bar{\Phi}_3&=[\phi'(\nu_1)\quad \phi'(\nu_2)\quad \ldots\quad \phi'(\nu_k)]^t\\ \bar{\Phi}_4&=[\phi(\chi_1)\quad \phi(\chi_2)\quad \ldots\quad \phi(\chi_M)]^t\end{aligned} \tag{17}$$

The elements of the submatrices Γ_{ij} are given i Appendix II. The overall matrix Γ is general i nature since elastic end restraints were considere Should any of the actual boundary conditions at th left beam end in a considered structure be pu geometric, i.e. calling for a vanishing deflection slope, the corresponding row and column in Γ shoul be deleted.

It is to be noted here that the square matrix Γ not a stiffness matrix, but rather a form of an overa transfer matrix. It is neither symmetric nor has an special structural characteristics. Its elements a transcendental functions of the eigenvalue λ Equating the determinant of the overall matrix Γ zero, for a nontrivial solution, yields the frequenc equation, which will have, in general, complex root λ_n. It is then possible to solve for the correspondin eigenforms Φ_n, within one degree of indeterminac The imaginary part of any root gives the (dampe frequency, while the real part gives the decay rat The roots can be found using any of the existin efficient numerical libraries (e.g., NAG, IMSI EISPACK, etc. [45]). The system matrix wi normally be of relatively low order since the numbe of masses and actuators in practical cases will b small. In the case of a low order matrix it woul even be possible to use a symbolic mathematic software package to obtain a closed form expressio for the characteristic equation of the system.

The steady state general solution for th dimensionless deflection of the beam can finally b expressed as the infinite sum:

$$\bar{\phi}(\xi,\tau) = \sum_{n=1}^{\infty} \phi_n(\xi).q_n(\tau) \tag{18}$$

where the shape of the time function $q_n(\tau)$ woul depend on the nature of the eigenvalues and th initial conditions.

ORTHOGONALITY CONDITION

Having obtained the eigenforms, it is essential t establish the corresponding orthogonality conditions in order to complete the analysis of the problem since this would be needed for solving th corresponding forced vibration problem, wher eigenfunction expansion is used for arbitrar (continuously differentiable) forcing functions in the closed -interval [0,1]. This is done by multiplying eq. (10), written for the n-th eigenform by the m-th eigenform ϕ_m, and integrating over the domain from $\zeta=0$ to $\zeta=1$, as follows

$$\int_0^1 \phi_m\phi_n^{''''}d\xi+\beta^2\int_0^1\phi_m\phi_n^{''}d\xi+\lambda_n^2\int_0^1\phi_m\phi_n d\xi+$$
$$+\lambda_n^2\sum_{k=1}^{K}m_k\int_0^1\phi_m(\xi)\bar{\delta}(\xi-\nu_k)\phi_n(\nu_k)d\xi-$$
$$-\lambda_n^2\sum_{k=1}^{K}j_k\int_0^1\{\phi_m(\xi)\bar{\delta}(\xi-\nu_k)\phi_n^{\prime}(\nu_k)\}^{\prime}d\xi+ \qquad (19)$$
$$+\sum_{i=1}^{M}(\bar{g}_i+\bar{h}_i\lambda_n)\int_0^1\phi_m(\xi)\bar{\delta}(\xi-\chi_i)\phi_n(\chi_i)d\xi=0$$

It should be noted here that the differential operator $\partial^4/\partial\zeta^4$ is self-adjoint with respect to the "standard" boundary conditions mentioned above, and that the differential operator $\partial^2/\partial\zeta^2$ is self-adjoint with respect to the built-in and simply supported boundary conditions.[4,46] Hence, for the two latter boundary conditions, the first and second integrations in eq. (19) vanish, and the orthogonality condition can be given by:

$$\int_0^1\phi_m\phi_n d\xi+\sum_{k=1}^{K}m_k\phi_m(\nu_k)\phi_n(\nu_k)+$$
$$+\sum_{k=1}^{K}j_k\phi_m^{\prime}(\nu_k)\phi_n^{\prime}(\nu_k)+ \qquad (20)$$
$$+\frac{1}{\lambda_n^2}\sum_{i=1}^{M}(\bar{g}_i+\bar{h}_i\lambda_n)\phi_m(\chi_i)\phi_n(\chi_i)=0,\quad m\neq n$$

where the following properties of Dirac delta function were utilized [34]:

$$\int_{-\infty}^{+\infty}\phi(\xi)\delta(\xi-a)d\xi=\phi(a),\ \text{and}$$
$$\int_{-\infty}^{+\infty}\phi(\xi)\delta^{\prime}(\xi-a)d\xi=-\phi^{\prime}(a)$$

For other boundary conditions the full form of the orthogonality relationship, eq.(19), should be retained. For the special case of an uncontrolled beam, with simply supported or built-in ends, carrying a single mass m_1 (ignoring its rotary inertia), eq. (20) reduces to the known form [34]:

$$\int_0^1\phi_m\phi_n d\xi+m_1\phi_m(\nu_1)\phi_n(\nu_1)=0,\qquad m\neq n.$$

It is seen from eq. (20) that the presence of the concentrated masses and control forces, complicates the orthogonality relationship. For the case where no concentrated masses are present, Weaver and Silverberg [41] observed that locating the actuators at specific points on the beam restores the orthogonality of the eigenforms under certain conditions. Their solution was based on eigenfunction expansion with a small number of eigenforms participating in the overall response.

CONCLUSION

An exact formulation for the problem of discrete displacement and velocity feedback control for an elastically supported vibrating beam carrying concentrated masses and subjected to axial force is given. The positions of the masses as well as the control forces are arbitrary. The solution is based on the use of Laplace transform with respect to the spatial variable and arriving at an exact form of the eigenvalue problem of a general matrix through satisfying boundary and intermediate conditions regarding the deflections and slopes at the positions of the concentrated masses and actuators. The eigenforms are given in terms of the boundary and intermediate conditions. The resulting matrix has complex roots due to the presence of terms arising from the velocity feedback which are - mathematically - equivalent to damping terms. The complex roots govern the frequency and the decay rate of vibration. The given exact formulation allows for studying the effect of changing the number and positions of the actuators as well as the gains of the feedback loops for displacement and velocity on the frequency and decay rate. Optimum placement of actuators satisfying certain criterion such as minimum control energy might be sought for. The exact formulation does not suffer from the shortcomings of series solutions where spill-over and truncation errors are bound to occur. Other problems can be treated using the same approach, such as tubes conveying fluids, as in the case of marine risers, where the mathematical formulation is completely analogous to the effect of the axial force considered here.

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Appendix I

$$\varepsilon_0(\xi) = \frac{-\eta^2\cos\eta\xi + \zeta^2\cos\zeta\xi}{(\zeta^2 - \eta^2)}$$

$$\varepsilon_1(\xi) = \frac{-\eta\sin\eta\xi + \zeta\sin\zeta\xi}{(\zeta^2 - \eta^2)}$$

$$\varepsilon_2(\xi) = \frac{\cos\eta\xi - \cos\zeta\xi}{(\zeta^2 - \eta^2)}$$

$$\varepsilon_3(\xi) = \frac{\zeta\sin\eta\xi - \eta\sin\zeta\xi}{\eta\zeta(\zeta^2 - \eta^2)}$$

$$\varepsilon_4(\xi) = \frac{\eta^3\sin\eta\xi - \zeta^3\sin\zeta\xi}{(\zeta^2 - \eta^2)}$$

$$\varepsilon_5(\xi) = \frac{\eta^4\cos\eta\xi - \zeta^4\cos\zeta\xi}{(\zeta^2 - \eta^2)}$$

$$\varepsilon_6(\xi) = \frac{-\eta^5\sin\eta\xi + \zeta^5\sin\zeta\xi}{(\zeta^2 - \eta^2)}$$

$\bar{\varepsilon}_0(\xi) = \varepsilon_0(\xi) + \beta^2 \varepsilon_0(\xi)$

$\bar{\varepsilon}_1(\xi) = \varepsilon_1(\xi) + \beta^2 \varepsilon_3(\xi)$

Although the rest of the abbreviations, given below, can be completely expressed in terms of the $\varepsilon_i(\xi)$ and its derivatives, it is much more convenient to give them different symbols to ease the construction of the equations (11-15).

$\bar{\theta}_0(\xi) = \varepsilon_4(\xi) + \beta^2\varepsilon_1(\xi)$

$\bar{\theta}_1(\xi) = \bar{\varepsilon}_0(\xi)$

$\theta_2(\xi) = \varepsilon_1(\xi)$

$\theta_3(\xi) = \varepsilon_2(\xi)$

$\bar{\kappa}_0(\xi) = \varepsilon_5(\xi) + \beta^2\varepsilon_6(\xi)$

$\bar{\kappa}_1(\xi) = \bar{\theta}_0(\xi)$

$\kappa_2(\xi) = \varepsilon_0(\xi)$

$\kappa_3(\xi) = \varepsilon_1(\xi)$

$\bar{\psi}_0(\xi) = \varepsilon_6(\xi) + \beta^2\varepsilon_4(\xi)$

$\bar{\psi}_1(\xi) = \bar{\kappa}_0(\xi)$

$\psi_2(\xi) = \varepsilon_4(\xi)$

$\psi_3(\xi) = \varepsilon_0(\xi)$

Appendix II

The elements of the submatrices Γ_{ij} in eq. (16) are given by:

Γ_{11} (4x4):

$\gamma_{11} = \bar{k}_0$, $\gamma_{12} = \gamma_{13} = 0$, $\gamma_{14} = 1$

$\gamma_{21} = 0$, $\gamma_{22} = -\bar{c}_0$, $\gamma_{23} = 1$, $\gamma_{24} = 0$

$\gamma_{31} = \bar{\psi}_0(1) - \bar{k}_1\bar{\varepsilon}_0(1)$

$\gamma_{32} = \bar{\psi}_1(1) - \bar{k}_1\bar{\varepsilon}_1(1)$

$\gamma_{33} = \psi_2(1) - \bar{k}_1\varepsilon_2(1)$

$\gamma_{34} = \psi_3(1) - \bar{k}_1\varepsilon_3(1)$

$\gamma_{41} = \bar{\kappa}_0(1) + \bar{c}_1\bar{\theta}_0(1)$

$\gamma_{42} = \bar{\kappa}_1(1) + \bar{c}_1\bar{\theta}_1(1)$

$\gamma_{43} = \kappa_2(1) + \bar{c}_1\theta_2(1)$

$\gamma_{44} = \kappa_3(1) + \bar{c}_1\theta_3(1)$

Γ_{12}(4xK):

$\gamma_{ij}=0$, $i=1,2$; $j=1, \dots ,K$

$\gamma_{3j}=\lambda^2 m_j[\psi_3(1-\nu_j)-\bar{k}_1\varepsilon_3(1-\nu_j)]$, $j=1, \dots ,K$

$\gamma_{4j}=\lambda^2 m_j[\kappa_3(1-\nu_j)+\bar{c}_1\theta_3(1-\nu_j)]$, $j=1, \dots ,K$

Γ_{13} (4xK):

$\gamma_{ij}=0$, $i=1,2$; $j=1, \dots ,K$

$\gamma_{3j}=-\lambda^2 j_j[\psi_2(1-\nu_j)-\bar{k}_1\varepsilon_2(1-\nu_j)]$, $j=1, \dots ,K$

$\gamma_{4j}=-\lambda^2 j_j[\kappa_2(1-\nu_j)+\bar{c}_1\theta_2(1-\nu_j)]$, $j=1, \dots ,K$

Γ_{14} (4xM):

$\gamma_{ij}=0$, $i=1,2$; $j=1, \dots ,M$

$\gamma_{3j}=(\bar{g}_j + \bar{h}_j\lambda)[\psi_3(1-\chi_j)-\bar{k}_1\varepsilon_3(1-\chi_j)]$, $j=1, \dots ,M$

$\gamma_{4j}=(\bar{g}_j + \bar{h}_j\lambda)[\kappa_3(1-\chi_i)+\bar{c}_1\theta_3(1-\chi_i)]$, $j=1, \dots ,M$

Γ_{21} (Kx4):

$\gamma_{i1} = \bar{\varepsilon}_0(\nu_i)$, $i=1, \dots ,K$

$\gamma_{i2} = \bar{\varepsilon}_1(\nu_i)$, $i=1, \dots ,K$

$\gamma_{i3} = \varepsilon_2(\nu_i)$, $i=1, \dots ,K$

$\gamma_{i4} = \varepsilon_3(\nu_i)$, $i=1, \dots ,K$

Γ_{22} (KxK):

$\gamma_{ii}=-1$, $i=1, \dots ,K$

$\gamma_{i,i+1}=0$, $i=1, \dots ,(K-1)$

$\gamma_{ij}=\lambda^2 m_j\varepsilon_3(\nu_i-\nu_j)$, $i=2, \dots ,K$; $j=1, \dots ,(i-1)$

Γ_{23} (KxK):

$\gamma_{i,j}=0$, $i=1, \dots ,K$; $j=i, \dots ,K$

$\gamma_{ij}=-\lambda^2 j_j\varepsilon_2(\nu_i-\nu_j)$, $i=2, \dots ,K$; $j=1, \dots ,(i-1)$

Γ_{24} (KxM):
$\gamma_{ij}=[(\bar{g}_j+\bar{h}_j\lambda)\varepsilon_3(\nu_i-\chi_j)]u(\nu_i-\chi_i)$,
$i=1,\ldots,K;\ j=1,\ldots,M$

Γ_{31} (Kx4):
$\gamma_{i1}=\bar{\theta}_0(\nu_i)$, $i=1,\ldots,K$
$\gamma_{i2}=\bar{\theta}_1(\nu_i)$, $i=1,\ldots,K$
$\gamma_{i3}=\theta_2(\nu_i)$, $i=1,\ldots,K$
$\gamma_{i4}=\theta_3(\nu_i)$, $i=1,\ldots,K$
Γ_{32} (KxK):
$\gamma_{i,j}=0$, $i=1,\ldots,K;\ j=i,\ldots,K$
$\gamma_{ij}=\lambda^2 m_j\theta_3(\nu_i-\nu_j)$, $i=2,\ldots,K;\ j=1,\ldots,(i-1)$
Γ_{33} (KxM):
$\gamma_{ii}=-1$, $i=1,\ldots,K$
$\gamma_{i,i+1}=0$, $i=1,\ldots,(K-1)$
$\gamma_{ij}=-\lambda^2 j_j\theta_2(\nu_i-\nu_j)$, $i=2,\ldots,K;\ j=1,\ldots,(i-1)$
Γ_{34} (KxM):
$\gamma_{ij}=[(\bar{g}_j+\bar{h}_j\lambda)\theta_3(\nu_i-\chi_j)]u(\nu_i-\chi_j)$,
$i=1,\ldots,K;\ j=1,\ldots,M$

Γ_{41} (Mx4):
$\gamma_{i1}=\bar{\varepsilon}_0(\chi_i)$, $i=1,\ldots,M$
$\gamma_{i2}=\bar{\varepsilon}_1(\chi_i)$, $i=1,\ldots,M$
$\gamma_{i3}=\varepsilon_2(\chi_i)$, $i=1,\ldots,M$
$\gamma_{i4}=\varepsilon_3(\chi_i)$, $i=1,\ldots,M$
Γ_{42} (MxK):
$\gamma_{ij}=\lambda^2[m_j\varepsilon_3(\chi_i-\nu_j)]u(\chi_i-\nu_j)$,
$i=1,\ldots,M;\ j=1,\ldots,K$
Γ_{43} (MxK):
$\gamma_{ij}=-\lambda^2[j_j\varepsilon_2(\chi_i-\nu_j)]u(\chi_i-\nu_j)$, $i=1,\ldots,M;\ j=1,\ldots,K$
Γ_{44} (MxM):
$\gamma_{ii}=-1$, $i=1,\ldots,M$
$\gamma_{i,i+1}=0$, $i=1,\ldots,(M-1)$
$\gamma_{ij}=(\bar{g}_j+\bar{h}_j\lambda)\varepsilon_3(\chi_i-\chi_j)$,
$i=1,\ldots,M;\ j=1,\ldots,(i-1)$