ANALYTICAL STUDY OF SOLUTIONS OF A GENERALIZED VAN DER POL MULTI-DEGREE SYSTEM

F.Z. Habieb
Department of Engineering Mathematics and Physics, Faculty of Engineering,
Alexandria University, Alexandria, Egypt.

ABSTRACT

An analytical study has been carried out for investigating the solutions of 4-degrees of freedom highly dissipative dynamical system. The system studied is a generalization of the well-known van der pol equation. Results reveal that there exist three regimes of the system behaviour. The first regime is characterized by the occurrence of the summed and difference harmonic oscillations. The second regime indicates a quasi periodic behaviour, Finally, the third regime is purely chaotic.

Keywords: Nonlinear differential equations, Rayleigh dissipation function, Variation of parameters, Chaos.

1. INTRODUCTION

In this paper, we are concerned with an analytical study of the solutions of the generalized van der pol's multi-degrees of freedom system viz. :

$$\frac{d^2x}{dt^2} - \lambda\left[(1-x^2)+(x+y)y\right]\frac{dx}{dt} + x = 0 \qquad (1\text{-a})$$

$$\frac{d^2y}{dt^2} - \lambda\left[(1-y^2)-(x+y)x\right]\frac{dy}{dt} + p^2y = 0 \qquad (1\text{-b})$$

in which "λ" is a positive real parameter.
System (1) can be written in an equivalent form as a system of first order differential equations as follows:

$$\frac{dx}{dt} = u \qquad (2\text{-a})$$

$$\frac{du}{dt} = -x + \lambda\left[(1-x^2)+(x+y)y\right]u \qquad (2\text{-b})$$

$$\frac{dy}{dt} = v \qquad (2\text{-c})$$

$$\frac{dv}{dt} = -p^2y + \lambda\left[(1-y^2)-(x+y)x\right]v \qquad (2\text{-d})$$

Subject to the initial conditions:

$$x(o) = y(o) = 0 \text{ and } u(o) = v(o) = 1 \qquad (3)$$

Here, the variables (x,y:u,v) lie inside a closed bounded domain D inside an open sphere of radius R having its center at the origin and satisfy:

$$(x^2 + y^2 + u^2 + v^2) < R^2 \qquad (4)$$

It can be easily seen that for the vanishing of y in Eq. (1.a) (or x in Eq. (1.b)) one can obtain.

$$\frac{d^2x}{dt^2} - \lambda(1-x^2)\frac{dx}{dt} + x = 0 \qquad (5)$$

which is the equation known as "Van der pol's Equation" originally discovered by the dutch scientist Bertold Van der pol in 1926 [Ref. (1)]. The dissipative parameter "λ" characterizes the coupling between the oscillatory electric circuit and the vacuum tube in which "x" denotes the current. Since 1926 interest in Eq. (5) never ceases and up till now. Thus Mandelstam and Papaleksi [Ref. (2)] had given a new mathematical approach for tackling Eq. (5). Several investigations were later carried out on Van der pol Eq. (5). [Refs. (3) & (4]) being the most significant of these. In this paper, we will now look into the system (1) which as we shall see later that the translation from a single equation to a system is accompanied by the appearance of new effects which can not take place in a single freedom case.

2. MECHANICAL GENERATION OF THE SYSTEM (1)

Let us consider two equal masses (mass is taken as unity) that are connected separately by two linear springs of different elasticity. Let the two masses undergo a motion in a dissipative medium whose properties are dependent on the position of every mass that occupies during its motion.

Thus, this dissipative medium will act as a coupling between the two masses during their motion and naturally, it is expected to exert its influence on their motion characteristics.

To obtain the equations of motion of the masses, one may get them by invoking Lagrange's equations with the help of "Rayleigh dissipation function "F" which in this context may take the form

$$F = \frac{\lambda}{2}\{(x^2-xy-y^2-1)\ u^2+(x^2+xy+y^2-1)\ v^2\} \qquad (6)$$

and, naturally, the kinetic and potential energies T and V are given by

$$2T = u^2 + v^2,\ 2V = x^2 + p^2\ y^2 \qquad (7)$$

Substituting in Lagrange's equations [Ref. (5)].

$$\frac{d}{dt}\left(\frac{\partial T}{\partial u}\right) + \frac{\partial V}{\partial x} + \frac{\partial F}{\partial u} = 0 \qquad (8\text{-a})$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v}\right) + \frac{\partial V}{\partial y} + \frac{\partial F}{\partial v} = 0 \qquad (8\text{-b})$$

yields the system (1).

3. ANALYTICAL SOLUTIONS OF THE SYSTEM (1)

If we ignore the dissipative effect in system (1), one is lead to the motion of every mass follows the path.

$$x = \sin t \qquad (9\text{-}$$

$$y = \frac{1}{p}\sin pt \qquad (9$$

Thus, it is plausible to suggest

$$x = \sin t + \lambda\xi \qquad (10$$

$$y = \frac{1}{p}\sin pt + \lambda\zeta \qquad (10$$

as a solution of system (1) which bears t
dissipation. The form (10.a,b) is one of the varia
of the technique "variation of parameters" due
Lagrange [Ref. (8)]. The validity of this form
solution will be shown in the last section remar
Substituting in Eqs. (1) and retaining terms up
the order of λ, one can get

$$\frac{d^2\xi}{dt^2} + \xi = \phi(t) \qquad (11$$

$$\frac{d^2\zeta}{dt^2} + p^2\zeta = \psi(t) \qquad (11\text{-}$$

in which

$$\Phi(t)=(\cos^2 t+p^{-1}\ \sin t \sin pt+p^{-2}\ \sin^2 pt)\ \cos t \qquad (12$$

$$\Psi(t)=(\cos^2 t-p^{-1}\ \sin t \sin pt-p^{-2}\ \sin^2 pt)\ \cos pt \qquad (12$$

solving Eqs. (11), one can get

$$\xi = \left\{\frac{1}{32} - \frac{1}{4p[1-(p-2)^2]} + \frac{1}{4p[1-(p+2)^2]} + \frac{1}{4p^2[1-(2p+1)^2]} + \frac{1}{4p^2[1-(2p-1)^2]}\right\}\cos t - \frac{1}{32}\cos 3t + \frac{1}{4p[1-(p-2)^2]}\cos(p$$

$$- \frac{1}{4p[1-(p+2)^2]}\cos(p+2)t - \frac{1}{4p^2[1-(2p+1)^2]}\cos(2p+1)t - \frac{1}{4p^2[1-(2p-1)^2]}\cos(2p-1)t \qquad ($$

and

$$\zeta = \left\{\frac{1}{32p^4} - \frac{1}{4[p^2-(p+2)^2]} - \frac{1}{4[p^2-(p-2)^2]} + \frac{1}{4p[p^2-(2p-1)^2]}\right.$$

$$-\frac{1}{4p[p^2-(2p+1)^2]}\Bigg\}\cos pt-\frac{1}{32p^4}\cos 3pt$$
$$+\frac{1}{4[p^2-(p+2)^2]}\cos(p+2)t+\frac{1}{4[p^2-(p-2)^2]}\cos(p-2)t$$
$$-\frac{1}{4p[p^2-(2p-1)^2]}\cos(2p-1)t+\frac{1}{4p[p^2-(2p+1)^2]}\cos(2p+1)t \qquad (14)$$

4. REMARKS

In section (3) we have obtained an analytic solution for system (1) as given by Eqs. (10-14). It can be clearly seen that the solution is affected by two parameters namely: λ and p.

It is instructive to note that the natural frequency "p" may take the integral, rational and irrational values. Let us take every case separately:

(a) p is an integer: In this case the solution is purely periodic one accompanied by the existence of the summed and difference harmonic oscillations of the type ($k\,p \pm l$) in which k and l are positive integers.

(b) p is rational: In this case the solution is characterized by being quasi periodic.

(c) p is irrational: In this final case, the solution is identified by observing on a time history the absence of periodicity or quasi-periodicity. Thus, the solution is called in this case as chaotic [Ref. (6), (7)]. The reason of this chaos is due to strange attractors in the system (1) that causes this phenomenon to occur. It is hoped that this case should be studied in a future work.

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