SERIES SOLUTION OF NON LINEAR UNDAMPED ASYMMETRIC FORCED OSCILLATIONS

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ABSTRACT

An approximate series solution of a nonlinear undamped forced oscillation is obtained. The restoring force is asymmetric and the external driving force is harmonic. The series solution is proved to be absolutely convergent and is identical with the exact solution for some particular values of the constants and initial conditions. The effects of varying the initial conditions, the frequency and amplitude of the external force are discussed. To the same order of approximation, our results proved to be more accurate than those obtained by the method of Bubnov-Galerkin.

Keywords: Differential equations, nonlinear oscillations, forced, asymmetric, series solution.

INTRODUCTION

Oscillations are the most general form of motion of dynamic systems about their equilibrium position. The method of small parameter is widely used in the theory of nonlinear oscillations [1], [2], [3]. This method is based on the fundamental researches of Poincaré [4]. Further developments were connected with Russian school of research [2]. Various phenomena are due to the nonlinearity of the oscillating system,[5],[6]. Mainly the dependence of the period of the oscillation on its amplitude [1], [7], [8], the emergence of different harmonics in the oscillation [1], [3], [9], [10] and for a given amplitude of the forcing function there may be three distinct response amplitudes with an associated jump phenomena [1], [7]. There has been different methods used to obtain approximate solution of nonlinear oscillations [11], [12]. In [13], they found an approximate series solution of two nonlinear differential equations previously studied in [14] and given by

$$\ddot{x} + \alpha^2 x^2 = \beta^2 x^4 \tag{1}$$

$$\ddot{x} + \alpha^2 x^2 = -\beta x^3 \tag{2}$$

with initial conditions

$$x(0) = A \tag{3}$$

$$\dot{x}(0) = 0 \tag{4}$$

where α, β, A are constants Convergence of their series solution had been proved. The same method of solution had been used in [15] and an approximate solution of a nonlinear undamped asymmetric free oscillation of the form

$$\ddot{x} + 3x^2 - 6x + 2 = 0 \tag{5}$$

with the same initial conditions (3), (4) was obtained. In this paper we shall get in section 2 an approximated series solution of a nonlinear undamped forced oscillation of the form

$$\ddot{x} + ax^2 + bx + c = P_o \sin\omega t \tag{6}$$

where a,b,c are constants, P_o, ω are the amplitude and frequency of the external harmonic driving force. The initial conditions are the same as given by equations (3), (4). It is of importance to note that when c, P_o are set equal to zero and a is a small negative quantity equation (6) reduces to the satellite equation [16]. So when only c is equal to zero equation (6) represents the forced oscillation of the satellite equation. Also it is obvious that the preciously solved equations (1),(2) and (5) are nonlinear autonomous differential equations because the time t does not appear explicitly in these equations. Equation (6) is nonautonomous and different kinds of mathematical difficulties are encountered when we try to get its solution. The series solution we got is periodic with period T equal to $\frac{2\pi}{\omega}$. Convergence of the solution is proved in section 3. In section 4, the solution is proved to be identical with the exact solution for particular values of the constants a, P_o, A. Results and conclusion are presented in the same section. Comparison of our results with those obtained from using the method of Bubnov-Galerkin is shown in an appendix.

2. SERIES SOLUTION

We introduce a new variable u such that

$$u = x + \frac{b}{2a} \tag{7}$$

Hence equation (6) takes the form

$$\ddot{u} + au^2 + d = P_o \sin \omega t \tag{8}$$

where

$$d = c - \frac{b^2}{4a} \tag{9}$$

$$u(0) = A + \frac{b}{2a} \tag{10}$$

and

$$\dot{u}(0) = 0 \tag{11}$$

We assume the solution of equation (8) to be a series in the form

$$u(t) = \sum_{n=0}^{\infty} C_n \sin^n \omega t \tag{12}$$

where the coefficients C_o, C_1, C_2,... are constants to be determined.

Substituting the initial conditions (10) and (11) in equation (12) we get the values of the coefficients C_o and C_1 which are

$$C_o = A + \frac{b}{2a} \tag{13}$$

$$C_1 = 0 \tag{14}$$

By forward straight differentiation of equation (12) we get

$$\ddot{u} = \omega^2 \left[\sum_{n=0}^{\infty} n(n+1) C_{n+1} \operatorname{Sin}^{n-1} \omega t - \sum_{n=0}^{\infty} (n+1)^2 C_{n+1} \sin^{n+1} \omega t \right] \tag{15}$$

Now using equation (12), u^2 can be written as

$$u^2 = \sum_{n=0}^{\infty} b_n \sin^n \omega t \tag{16}$$

$$b_n = C_o C_n + C_1 C_{n-1} + \dots + C_n C_o \tag{17}$$

Substituting from equations (15) and (16)in L.H.S of equation (8) and equating the sum of the coefficients of each $\sin^n \omega t$, n = 0,1,2,.. to zero we get for n = 0,1 the following two equations.

$$2C_2\omega^2 + aC_o^2 + d = 0 \tag{18}$$

$$\omega^2 (6\, C_3 - C_1) + 2a\, C_o C_1 = P_o \tag{19}$$

and for n = 2,3,... the recurrence relation

$$C_{n+2} = \frac{1}{\omega^2 (n+1)(n+2)} \left[\omega^2 n^2 C_n - ab_n \right] \tag{20}$$

Using equations (9), (13) and (14) in equations (18) and (19) we get for the coefficients C_2, C_3 the alues

$$C_2 = - \frac{1}{2\omega^2} \left[aA^2 + bA + c \right] \tag{21}$$

and

$$C_3 = \frac{P_o}{6\omega^2} \tag{22}$$

Using the recurrence relation (20), equations (17) and (14) we can get for n = 2,3 the values of C_4 and C_5 as

$$C_4 = \frac{C_2}{12\omega^2}(4\omega^2 - 2aC_o) \qquad (23)$$

$$C_5 = \frac{C_3}{20\omega^2}(9\omega^2 - 2aC_o) \qquad (24)$$

It is easy to prove by forward straight calculations that all other even coefficients C_6, C_8,... are multiples of C_2 while the odd coefficients C_5, C_7, ... are multiples of C_3. This is in agreement with the results of [13] and [15] where $P_o = 0$ and only terms with even coefficients were obtained in their series solutions. Hence from equations (12), (7), (13), (21), (22), (23) and (24), the solution x(t) of the differential equation (6) has the value

$$x(t) = A - \frac{1}{2\omega^2}(aA^2 + bA + c)\sin^2\omega t + \frac{P_o}{6\omega^2}\sin^3 wt$$

$$- \frac{1}{24\omega^4}(aA^2 + bA + c)(4\omega^2 - 2aA - b)\sin^4 wt + \qquad (25)$$

$$\frac{P_o}{120\omega^4}[9\omega^2 - 2aA - b]\sin^5 wt + ..$$

where coefficients of further terms can be evaluated from recurrence relation (20) and the values of the coefficients previously obtained.

3. CONVERGENCE OF THE SOLUTION

Summing the recurrence relations (20) for n = 2,3,... to n = ∞ we get

$$\sum_{n=2}^{\infty}(n+2)(n+1)C_{n+2} - \sum_{n=2}^{\infty} n^2 C_n = -\frac{a}{\omega^2}\sum_{n=2}^{\infty} b_n \qquad (26)$$

from equation (17) we have

$$\because \sum_{n=2}^{\infty} b_n = (C_oC_2 + C_1C_1 + C_2C_o) + (C_oC_3 + C_1C_2 + C_2C_1 + C_3C_o) + (C_oC_4 + C_1C_3 + C_2C_2 + C_3C_1 + C_4C_o) + \ldots\ldots$$

$$\therefore \sum_{n=2}^{\infty} b_n = C_o(C_o + C_1 + C_2 + ..) - C_o^2 - C_oC_1 + C_1(C_oC_1 + C_2 + ..) - C_oC_1 + C_2(C_o + C_1 + C_2 + \ldots) + C_3(C_o + C_1 + C_2 \ldots) + \ldots\ldots$$

$$\therefore \sum_{n=2}^{\infty} b_n = (\sum_{n=0}^{\infty} C_n)(\sum_{n=0}^{\infty} C_n) - C_o^2 - 2C_oC_1$$

$$\text{i.e} \sum_{n=2}^{\infty} b_n = (\sum_{n=0}^{\infty} C_n)^2 - C_o^2 - 2C_oC_1 \qquad (27)'$$

also

$$\because \sum_{n=2}^{\infty}(n+2)(n+1)C_{n+2} = \sum_{n=0}^{\infty}(n+1)C_{n+2} - 2C_2 - 6C_3$$

and

$$\because \sum_{n=2}^{\infty} n^2 C_n = \sum_{n=0}^{\infty} n^2 C_n - C_1$$

$$\therefore \text{R.H.S. of equation (26)} = \sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2} - \sum_{n=0}^{\infty} n^2 C_n - 2C_2 - 6C_3 - C_1 = -\sum_{n=0}^{\infty} nC_n - 2C_2 - 6C_3 - C_1 \qquad (27)''$$

Substituting from equations (27)' and (27)" into equation (26) we get

$$-\omega^2\sum_{n=0}^{\infty} nC_n + a(\sum_{n=0}^{\infty} C_n)^2 = a(C_o^2 + 2C_oC_1) + \omega^2(2C_2 + 6C_3 + C_1) \qquad (28)$$

Since R.H.S. of equation (28) has a definite value, hence each one of the two infinite series on the L.H.S. of this equation converges. As a consequence the series of coefficients $\sum_{n=0}^{\infty} C_n$ converges because each term of this series is less than the corresponding term of either the series $\sum_{n=0}^{\infty} nC_n$ or the series $(\sum_{n=0}^{\infty} C_n)^2$ except for n = 0. By the same argument, the series solution u(t) given by equation (12) converges. Hence it can represent a solution of equation (8).

4. RESULTS AND CONCLUSION

It is importance to note that, putting a = 0 and P_o = o in equation (6) we get the linear free undamped oscillations , of the form

$$\ddot{x} + bx + c = 0 \qquad (29)$$

whose exact solution subject to the same initial conditions given by equations (3) and (4) is.

$$x(t) = (A + \frac{c}{b})\cos \sqrt{b}\ t - \frac{c}{b} \tag{30}$$

Thus $x(t) = -\frac{c}{b}$ if $A = -\frac{c}{b}$

This is the same solution that we get from equation (25); if we put a =0, P_o = o and $A = \frac{-c}{b}$. Hence our solution is identical with the exact solution for particular values of the constants a,P_o and initial conditions A.

Also assuming that the period of the series solution u(t) given by equation (12) is the same as the period of the harmonic driving force P(t) was quite justified in numerous previous studies of linear and nonlinear forced oscillations [1],[3],[5],[17].

Also if we put the constants a,b,c,ω, and P_o equal to 3,-6,2, $\sqrt{3}$ and o respectively, our approximate series solution given by equation (25) becomes the same as the approximate series solution of the nonlinear undamped asymmetric free oscillation studied by Eid [15].

In our calculations, we let the constants a,b,c and A to be equal to the forementioned values used in [15]. Hence, the effect of the external driving force can be discussed by comparing our results with those of [15]. In Figures (1),(2),(3),(4) and (5) P_o = 1 and $\omega = \sqrt{3}$ while A takes the values of 1,2,3,4 and 7 respectively. In Figure (1), we notice that x(t) is $\geq$ A while in Figures (2),(3),(4),(5),x(τ) is $\leq$ A.This is identical with the results of Eid [15].

Also during the interval $0 \leq \tau \leq 2\pi$ in Figures (2),(3),(4) there are 3 maximum values of x that are equal A at τ=0, π, and 2 π. This is also the same as the results of Eid [15]. But, the two minimum value of x at $\tau = \frac{\pi}{2}$ and $\tau = \frac{3}{2}\pi$ are not equal which differ from their results. So the external driving force causes the nonlinear oscillation to be asymmetric. The difference between the minimum values decreases with the increase of A as shown in Figures [4] and [5]. From Figure (4) when A=4, x(τ) assumes - ve values this is not the same as he results of Eid [10] where x is always + ve Also in Figure (5),x(τ) are + ve again, but number of maximum and minimum values of x are increased by two during a period.

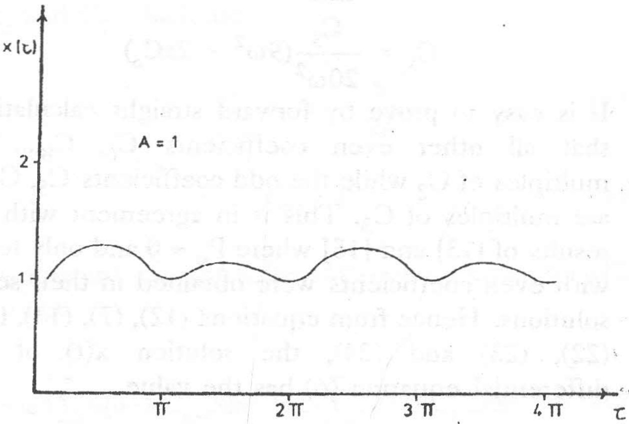

Figure 1. Variation of the amplitude of the non linear oscillation with the normalized time τ, A=1.

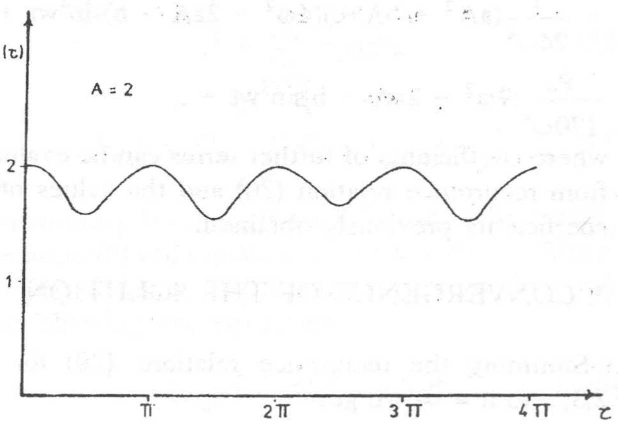

Figure 2. Same as Figure (1) but for A=2.

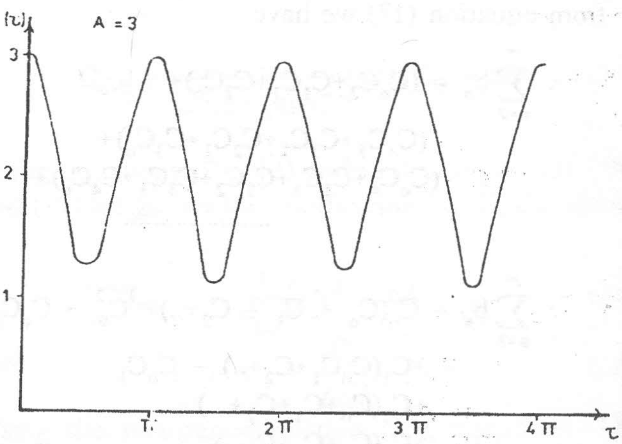

Figure 3. Same as Figure (1) but for A=3.

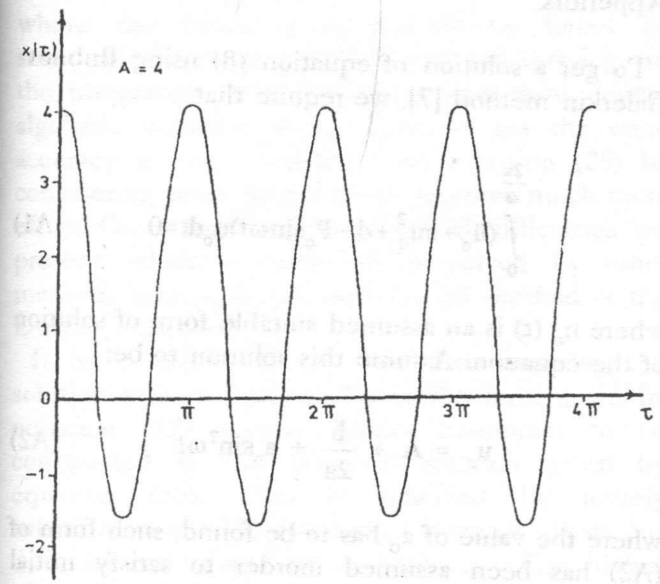

Figure 4. Same as Figure (1) but for A=4.

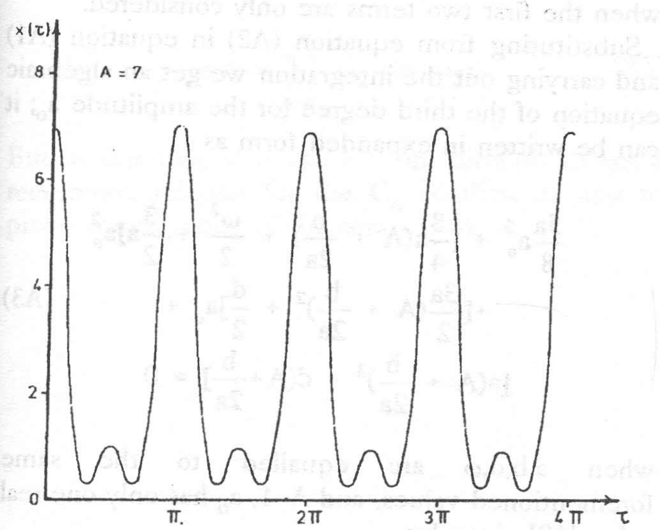

Figure 5. Same as Figure (1) but for A=7.

In Figure (6) A,P_o kept equal to 1 and ω is varied. It is obvious that x (τ) and the difference between the two maximum value of x decreases as ω is increased . In Figure (7) A and ω are equal to 1 and $\sqrt{3}$ respectively, P_o is varied. When $0 < \tau < \pi$, $x(\tau)$ increases as P_o increases and when $\pi < \tau < 2\pi$, $x(\tau)$ decreases as P_o increases Hence from the obtained results we find that the symmetry of the oscillation is reduced by the increase of P_o and the decrease of A and ω. Meanwhile the amplitude of the oscillation increases as P_o, and A increases and ω decreases.

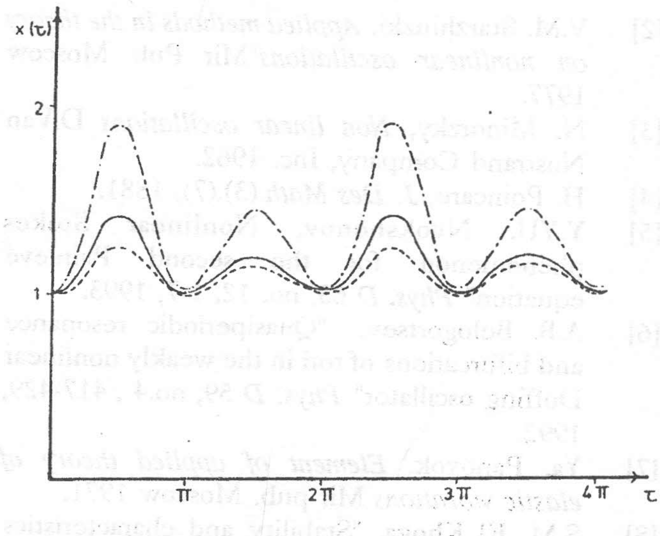

Figure 6. Variation of the amplitude of the non linear oscillation with the normalized time τ for different values of driving frequency ω. P_o=1, A=1, -.- ω=1, ------- $\omega=\frac{3}{2}$ and ----- ω=2.

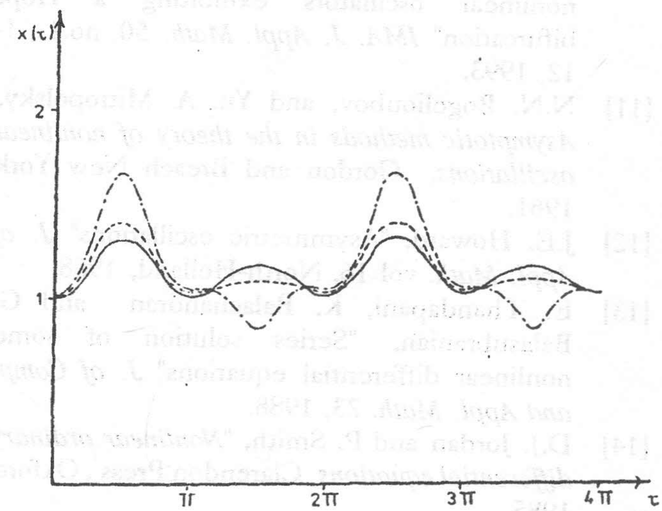

Figure 7. Variation of the amplitude of the non linear oscillation with the normalized time τ for different values of the amplitude of the driving force P_o. A=1,$\omega = \sqrt{3}$,-.- P_o= 5,-------- P_o=2, —P_o=1.

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Appendix

To get a solution of equation (8) using Bubnov-Galerkin method [7], we require that

$$\int_0^{\frac{2\pi}{\omega}} (\ddot{u}_o + au_o^2 + d - P_o \sin\omega t)u_o dt = 0 \qquad \text{(A1)}$$

where u_o (t) is an assumed suitable form of solution of the equation. Assume this solution to be:

$$u_o = A + \frac{b}{2a} + a_o \sin^2 \omega t \qquad \text{(A2)}$$

where the value of a_o has to be found, such form of (A2) has been assumed inorder to satisfy initial conditions of equations (10) and (11) and to be compared with the solution given by equation (25) when the first two terms are only considered.

Substituting from equation (A2) in equation (A1) and carrying out the integration we get an algebraic equation of the third degree for the amplitude a_o; it can be written in expanded form as

$$\frac{5a}{8}a_o^3 + [\frac{3}{4}a(A + \frac{b}{2a}) + \frac{\omega^2}{2} + \frac{3}{2}a]a_o^2 + [\frac{3a}{2}(A + \frac{b}{2a})^2 + \frac{d}{2}]a_o + [a(A + \frac{b}{2a})^3 + d(A + \frac{b}{2a})] = 0 \qquad \text{(A3)}$$

when a,b,d,ω are equalled to the same forementioned values, and A=1, a_o has only one real value [18] given by

$$a_o = -3.92571 \qquad \text{(A4)}$$

The solution given by (A2) after substituting the value of a_o from equation (A4) proved to be much less accurate than that obtained from equation (25) for the same value of A and when only the first two terms are considered.

To get better accuracy by this method we try again a solution of the form

$$u_o = A + \frac{b}{2a} + a_o \sin^2 \omega t + a_1 \sin^3 \omega t \qquad \text{(A5)}$$

where the value of a_1 has to be found by substituting from eqn (A5) in (A1) and carrying out the integration. We shall get a new third degree algebraic equation in a_1. Thus to get the same accuracy as that obtained from equation (25) by considering more than 3 terms requires much more effort. Other kinds of mathematical difficulties are present when equation (8) is solved by other methods such as direct linearization method or the Duffing's method [7].

It is of importance to note that assuming the solution u(t) of equation (8) in the form given by equation (12) permits higher harmonics to be constructed in the obtained solution given by equation (25). This is acheived by merely substitution of $\frac{1}{2}(1 - \cos 2\omega t)$, $(\frac{3}{4} \sin \omega t - \frac{1}{4} \sin 3 \omega t)$, ... instead of $\sin^2 \omega t$, $\sin^3 \omega t$, ... Also higher harmonics are obviously obtained in the solution if we assume u(t) in the form

$$u(t) = \sum_{n=0}^{\infty} C_n \sin n\omega t \qquad \text{(A5)}$$

But in this case it is much more difficult to get a recurrence relation for the C_n coefficients and to prove convergence of the series (A5).