"ON THE MOTION OF A VARIABLE-LENGTH PENDULUM"

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ABSTRACT

The governing equation that determines the motion of a pendulum length varies in an exponential manner has been derived in a form of a nonlinear nonautonomous differential equation with variable coefficients. A perturbation analysis has been carried out to determine the motion characteristics. Results reveal that the motion is of the "orbiting" type and the pendulum acts as an "Energy Pump".

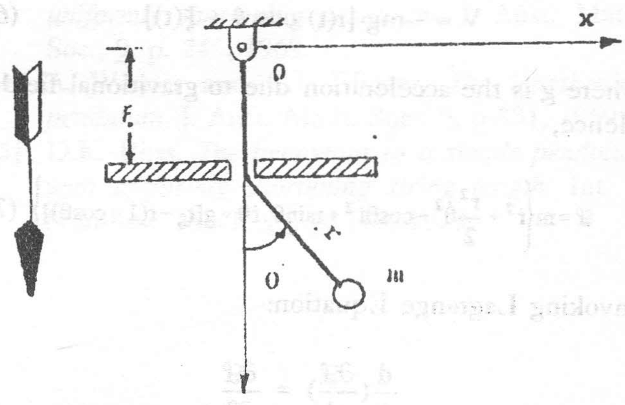
Keywords: Lagrange's equations, nonlinear differential equations with variable coefficients, perturbation analysis.

1. INTRODUCTION

The problem of plane motion of the simple pendulum is one of the oldest problems of the science of mechanics. Quating Sir Edmund Whittaker in his famous book "Analytical Dynamics" [Ref. (1), p. 72] he stated: "In this case the particle is free to move in the interior of a given fixed smooth circular tube of small bore whose plane is vertical and the only external force acting on the particle is gravity. There are two distinct types of motion, namely the "oscillatory" in which the particle swing to and from about the lowest point of the tube, and the "orbiting", in which the velocity of the particle is large enough to carry it over the highest point of the tube, so that it moves round and round the tube, always in the same sense.

In this paper we study the plane motion of a pendulum whose length is made to vary with time. Thus, in terms of Whittaker parlance, the bob of the pendulum is going to move in a noncircular smooth tube whose form is deformed with time. This presents what is known as a "rheonomous" dynamical system. The case of uniform rate of varying length with respect to time has been studied for linear systems in [2-5] via Bessel functions. Here, we give a treatment for an exponential length variation accompanied by a nonlinear treatment via perturbation analysis.

2. FORMULATION



Consider a bob of mass "m" moving in the vertical (x,y) plane, at the end of a light taut chord of fixed length r_o. The chord passes through a slot in moving board whose motion is constrained in a vertical straight line and the other end of the chord is fixed at O, Denoting $\xi(t)$ as the vertical displacement of the board at any instant t and setting Oxy as a fixed frame of reference one can write the coordinates of the bob as

$$x(t) = r(t) \sin \theta(t) \quad (1)$$

$$y(t) = \xi(t) + r(t). \cos \theta(t) \quad (2)$$

in which

$\theta(t)$ is the angle between the chord and the y-axis and

$$r(t) = r_o - \xi(t) \qquad (3)$$

is the distance between the slot and the bob.
The initial conditions for the motion are

$$\theta = \alpha \quad \text{and} \quad d\theta/dt = 0 \qquad (4)$$

To find the governing equation of the motion, we start by forming the Lagrangian "$\mathcal{L}$" of the system by constructing the kinetic energy "T" and the potential energy "V".
Thus,

$$T = \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{\xi}^2 + 2\cos\theta . \dot{\xi}\dot{r} - 2r\sin\theta \, . \, \dot{\xi}\dot{\theta}\right] \qquad (5)$$

and

$$V = - mg\,[r(t)\cos\theta + \xi(t)] \qquad (6)$$

where g is the acceleration due to gravitional field.
Hence,

$$\mathcal{L} = m\left\{\dot{r}^2 + \frac{r^2}{2}\dot{\theta}^2 - \cos\theta\dot{r}^2 + r\sin\theta . \dot{r}\dot{\theta} + g[r_o - r(1-\cos\theta)]\right\} \qquad (7)$$

Invoking Lagrange Equation:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathcal{L}}{\partial\theta}$$

One can obtain the equation of motion in the following form:

$$\ddot{\theta} + \frac{d}{dt}(\ln r^2) \, . \, \dot{\theta} + \frac{(\ddot{r}+g)}{r}\sin\theta = 0 \qquad (8)$$

subject to the initial conditions Eqns. (4).
Eq. (8) is a nonlinear nonautonomous differential equation in $\theta(t)$. This equation is very difficult to be handled but if we approximate $\sin\theta$ by $(\theta - \sigma^2\theta^n)$ where n and σ are suitable positive numbers, the resulting equation is

$$\ddot{\theta} + \frac{d}{dt}(\ln r^2) \, . \, \dot{\theta} + \frac{(\ddot{r} + g)}{r}(\theta - \sigma^2\theta^3) = 0 \qquad (9)$$

which is amenable to analysis

3. MOTION CHARACTERISTICS

Let the length of the chord vary in an exponential way according to the law $r = r_o \exp(-\frac{\varepsilon}{2}t)$ which means that the length of the pendulum is shortening in a slowly varying manner. This suggests a perturbation route for the analysis of the problem. Thus, we set

$$\theta(t) = \theta_o(t) + \varepsilon\theta_1(t) \qquad (10)$$

substituting (10) in (9), one can get

$$\ddot{\theta}_o + \varepsilon\ddot{\theta}_1 - \varepsilon\dot{\theta}_o + \frac{g}{r_o}\left(1 + \frac{\varepsilon t}{2} + \ldots\right)(\theta_o + \varepsilon\theta_1 - \varepsilon\sigma'^2\theta_o^3) = 0 \qquad (11)$$

Dropping the prime in σ'^2 for convenience in writing and equating coefficients of like powers of ϵ, lead to

<u>Order ϵ^o</u>

$$\ddot{\theta}_o + \omega^2\theta_o = 0 \qquad (12)$$

<u>Order ϵ</u>

$$\ddot{\theta}_1 + \omega^2\theta_1 = \dot{\theta}_o - \omega^2\left(\frac{t}{2}\theta_o - \sigma^2\theta_o^3\right) \qquad (13)$$

in which ω stands for $(g/r_o)^{1/2}$ accompanied by the initial conditions

$$\theta_o = \alpha \quad , \quad \dot{\theta}_o = 0 \qquad (14)$$

and

$$\theta_1 = 0 \quad , \quad \dot{\theta}_1 = 0 \qquad (15)$$

The solution of (12) that satisfies the initial conditions (14) is

$$\theta_o = \alpha\cos\omega t \qquad (16)$$

Substituting for θ_o into (13) yields

$$\ddot{\theta}_1 + \omega^2\theta_1 = -\alpha\omega\sin\omega t - \frac{\omega^2\alpha}{2}t\cos\omega t + \frac{\alpha^3\omega^2\sigma^2}{4}(3\cos t\omega t + \cos 3\omega t) \tag{17}$$

The solution of (17) is

$$\theta_1 = (A + \frac{3\alpha}{8}t)\cos\omega t - \frac{\alpha^3\sigma^2}{32}\cos 3\omega t + [B - \frac{\alpha\omega^2}{2}(\frac{t^2}{4\omega} - \frac{1}{8\omega^3}) + \frac{3\alpha^3\sigma^2\omega}{8}t]\sin\omega t \tag{18}$$

where A,B are integration constants. To determine them we appeal to Eqns (15) that give

$$A = \frac{\sigma^2\alpha^3}{32} \quad \text{and} \quad B = \frac{-7\alpha}{16\omega}$$

Thus, the final form of θ_1, may take the following form

$$\theta_1 = \frac{\alpha}{32}(\sigma^2\alpha^2 + 12t)\cos\omega t - \frac{\alpha^3\sigma^2}{32}\cos 3\omega t - \frac{\alpha}{8\omega}(3 - 3\omega^2\sigma^2\alpha^2 t + \omega^2 t^2)\sin\omega t \tag{19}$$

and the deflection angle θ is

$$\theta(t) = \frac{\alpha}{32}(32 + \varepsilon\sigma^2\alpha^2 + 12\varepsilon t)\cos\omega t - \frac{\varepsilon\alpha^3\sigma^2}{32}\cos 3\omega t - \frac{\alpha\varepsilon}{8\omega}(3 - 3\omega^2\sigma^2\alpha^2 t + \omega^2 t^2)\sin\omega t \tag{20}$$

Scrutinizing (20), it can be clearly seen that the amplitude of the motion is increasing monotonically with time. This will imply that the pendulum motion is of the "orbiting" type. This "orbiting" motion is due to the continuous decrease of the pendulum length. Since the pendulum in this case is a rheonomous system in which the tension of the chord does work that accumulates in a continuous increase of kinetic energy. This simply means that this pendulum device is now acting as "Energy Pump".

This will never happen for ordinary pendulum !.

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