ON THE MOTION OF A PENDULUM WHOSE SUPPORT MOVES RAPIDLY

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ABSTRACT

In the present study, the problem of the plane pendulum with a rapidly moving point of suspension has been considered. The study is made for arbitrary angular displacements of the pendulum. By applying a Kapitza type transformation, the canonical equations have been reduced to an autonomous system of differential equations. A general solution of this autonomous system has been obtained in terms of elliptic functions.

Keywords: *Hamiltonians theory, non linear differential equations, Kapitza transformation, method of averaging, Elliptic functions.*

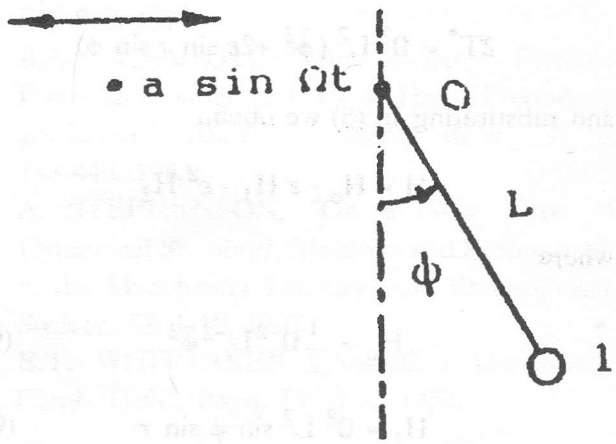
INTRODUCTION

Interest in the motion of a pendulum with a moving support has attracted the attention of researchers for a very long period of time. In the beginning of this century, STEPHENSON [8] studied the plane pendulum when the support is allowed to oscillate vertically. A linearised analysis has been made for small angular displacements of the pendulum. LOWENSTERN [4] investigated the spherical pendulum also for vertical oscillations of the support. When the angular displacement of the pendulum is arbitrary and the support is excited vertically with high frequency, this case has been treated by KAPITZA [3]. RYLAND & MEIROVICH [6] considered the stability of the vertical position of the plane flexible pendulum with a vertical harmonic oscillation at an unrestricted frequency. SCHMIDT [7] presented solutions for a pendulum with excitation along the radial direction.

In the present paper, the problem of the plane pendulum with a horizontally oscillating support will be studied. The excitation applied at the support is restricted so that the frequency is high and the amplitude of oscillation is small.

EVOLUTION EQUATIONS

Let us consider the plane pendulum shown in the annexed figure that consists of a particle of unit mass connected with a massless rigid rod of length "L". The point of suspension O moves horizontally according to the law a sin ω t in which "a" is the amplitude of oscillation and it is small compared with the pendulum length "L" and "ω" is the excitation frequency that is very large compared with the natural frequency of the pendulum.



If "ϕ" is the angular displacement of the pendulum at any instant "t" then kinetic and potential energies "T, V" take the forms:

$$2T = L^2\left(\frac{d\phi}{dt}\right)^2 + 2aL\Omega\cos\Omega t\cos\phi\left(\frac{d\phi}{dt}\right) + a^2\Omega^2\cos^2\Omega t \quad (1)$$

$$V = -gL\cos\phi \quad (2)$$

Defining the fast time "τ" by $\tau = \Omega t$ and introducing the following constants $\varepsilon = a\,L^{-1}$, $\lambda = (gL)^{1/2}/a\Omega$, Eqs. (1,2) may be written as:

$$2T = \Omega^2 L^2 \left(\dot{\phi}^2 + 2\varepsilon\cos\tau\cos\phi\,\dot{\phi} + \varepsilon^2\cos^2\tau\right) \quad (3)$$

$$V = -\varepsilon^2\,\Omega^2 L^2 \lambda^2 \cos\phi \quad (4)$$

in which the dot denotes differentiation with respect to the fast time "τ".

We construct the Hamiltonian $H(\phi, \hat{\phi}, t)$ of the system as:

$$H = \hat{\phi}\dot{\phi} - T^* + V \quad (5)$$

Where $\hat{\phi}$ is the momentum conjugate to the angular displacement ϕ and T^* is defined by (ref. [1])

$$T^* = T - \frac{dF}{dt} \quad (6)$$

in which F is arbitrary function of time. Choosing T^* as

$$2T^* = \Omega^2 L^2 (\dot{\phi}^2 + 2\varepsilon\sin\tau\sin\phi) \quad (7)$$

and substituting in (5) we obtain

$$H = H_o - \varepsilon H_1 - \varepsilon^2 H_2 \quad (8)$$

where

$$H_o = \frac{1}{2}\Omega^{-2}L^{-2}\hat{\phi}^2 \quad (9\text{-a})$$

$$H_1 = \Omega^2 L^2 \sin\phi\sin\tau \quad (9\text{-b})$$

$$H_2 = \Omega^2 L^2 \lambda^2 \cos\phi \quad (9\text{-c})$$

$$\frac{d\phi}{d\tau} = \Omega^{-2}L^{-2}\hat{\phi} \quad (10\text{-a})$$

$$\frac{d\hat{\phi}}{d\tau} = \varepsilon\Omega^2 L^2(\cos\phi\sin\tau - \varepsilon\lambda^2\sin\phi) \quad (10\text{-b})$$

Invoking Hamilton's principle (WHITTAKER [9]) we get the canonical equations

MOTION CHARACTERISTICS

For the determination of the solution of Eqs. (10) we introduce the new variables $\theta(\tau)$ and $\psi(\tau)$ in the following manner

$$\phi = \theta - \varepsilon\cos\theta\sin\tau \quad (11\text{-a})$$

$$\dot{\phi} = \varepsilon(\psi - \cos\theta\cos\tau) \quad (11\text{-b})$$

The transformation (11) is motivated by ideas due to KAPITZA [3]. Substituting (11) in (10) and retaining terms of the lowest order of magnitude in ε lead to

$$\frac{d\theta}{d\tau} = \varepsilon\Psi \quad (12\text{-a})$$

$$\frac{d\Psi}{d\tau} = \varepsilon(\cos\theta\sin^2\tau - \Psi\cos\tau - \lambda^2)\sin\theta \quad (12\text{-b})$$

Now, applying the method of averaging (NAYFEH [5], Chap. (5)) we postulate

$$\theta = \bar{\theta} + \varepsilon H(\bar{\theta}, \bar{\Psi}, \tau) \quad (13\text{-a})$$

$$\psi = \bar{\Psi} + \varepsilon N(\bar{\theta}, \bar{\Psi}, \tau) \quad (13\text{-b})$$

In which H,N are functions that can be obtained at later stages of approximation and the variables $\bar{\theta}, \bar{\Psi}$ satisfy the following autonomous system of differential equations:

$$\frac{d\bar{\theta}}{d\tau} = \varepsilon\bar{\Psi} \quad (14\text{-a})$$

$$\frac{d\bar{\Psi}}{d\tau} = \frac{\varepsilon}{4}(\sin 2\bar{\theta} - 4\lambda^2 \sin\bar{\theta}) \qquad (14\text{-b})$$

To integrate (14), we notice that

$$\frac{d\bar{\theta}}{d\bar{\Psi}} = \frac{4\bar{\Psi}}{\sin 2\bar{\theta} - 4\lambda^2 \sin\bar{\theta}} \qquad (15)$$

This leads to

$$\bar{\Psi}^2 + \frac{1}{4}\cos 2\bar{\theta} - 2\lambda^2 \cos\bar{\theta} = \text{const.} \qquad (16)$$

Substituting (16) into (14-a) we obtain

$$\left[\frac{d\bar{\theta}}{d\tau}\right]^2 = \frac{\varepsilon^2}{4}(A - \cos 2\bar{\theta} + 8\lambda^2 \cos\bar{\theta}) \qquad (17)$$

in which the constant "A" depends on the initial conditions of the motion.

For finding an explicit solution of (17), we take $A = 10\,\lambda^4 - 1$ and we apply the standard transformation (DAVIS [2], Chap. 7).

$$u^2 = \frac{1+5\lambda^2}{1-5\lambda^2}\;\frac{1-\cos\bar{\theta}}{1+\cos\bar{\theta}} \qquad (18)$$

Substituting (18) into (17) we obtain

$$\left[\frac{du}{d\tau}\right]^2 = \frac{\varepsilon^2}{2}\, M^2\, (1-u^2)\, (1-k^2 u^2) \qquad (19)$$

where

$$M^2 = \frac{1}{4}(1+5\,\lambda^2)\,(1+\lambda^2)$$

and

$$k^2 = \frac{(1-\lambda^2)(1-5\lambda^2)}{(1+\lambda^2)(1+5\lambda^2)}$$

The solution of Eq. (19) is the well - known Jacobian sn elliptic function:

$$u = \text{sn}\left[\frac{\varepsilon M}{\sqrt{2}}\tau, k\right] \qquad (20)$$

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