ON THE APPLICATION OF KRYLOV-BOGOLIUBOV- MITROPOLSKI TECHNIQUE FOR TREATING THE MOTION ABOUT A FIXED POINT OF A FAST SPINNING HEAVY SOLID

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ABSTRACT

Poincaré's small parameter method and the Krylov - Bogoliubov asymptotic method are among the number of basic methods used for the study of nonlinear oscillations. Poincaré's method was developed in conformity with stationary (periodic) oscillations [1], although it may be extended to nonstationary oscillations [2]. The krylov - Bogoliubov method may be used, first of all, for a study of nonstationary oscillations, but it is, of course, completely applicable to periodic oscillations [3]. In the present paper, the method of Krylov-Bogoliubov-Mitropolski [4] is modified to investigate the periodic solutions for the equations of motion of a heavy solid, with one fixed point, rapidly spinning about the major or the minor axis of the ellipsoid of inertia.

Keywords: Rigid body motion, Perturbation techniques, Nonlinear Oscillations, Krylov-Mitropolski technique.

1. INTRODUCTION

Consider a heavy solid of mass (M), with one fixed point, whose ellipsoid of inertia is arbitrary and its center of gravity is arbitrarily located. Selecting OX, OY and OZ to represent the fixed frame in space and Ox, Oy and Oz to represent the principal axes of the ellipsoid of inertia which are fixed in the body. Assuming that at the initial instant of time the principal axis z of the ellipsoid of inertia makes an angle $\theta_o \neq k\pi/2$; k=0, 1, 2; with the vertical, and that the body spins about this principal axis with a high angular velocity r_o. Taking p, q and r to represent the projections of the angular velocity vector of the body on the principal axes of inertia, γ, γ' and γ'' to be the direction cosines of the Z-axis.

Remembering that g is the acceleration of gravity; A, B and C are the principal moments of inertia; x_o, y_o and z_o are the coordinates of the center of mass in the moving coordinate system. The general differential equations of motion and their first integrals can be reduced to the following autonomous system with one first integral [5]:

$$\ddot{p}_2+\omega^2 p_2=\mu^2 F(p_2,\dot{p}_2,\gamma_2,\dot{\gamma}_2,\mu),$$
$$\ddot{\gamma}_2+\gamma_2=\mu^2\Phi(p_2,\dot{p}_2,\gamma_2,\dot{\gamma}_2,\mu); \qquad (1)$$

$$\gamma_2^2+\dot{\gamma}_2^2+2\mu(\nu p_2\gamma_2+\nu_2\dot{p}_2\dot{\gamma}_2+s_{21})+\mu^2(\ldots)=\gamma_o''^{-2}-1; \qquad (2)$$

where:

$$F=C_1A_1^{-1}p_2\dot{p}_2^2+x_o'\dot{p}_2\dot{\gamma}_2-y_o'a^{-1}p_2\dot{\gamma}_2-y_o'A_1^{-1}(A_1+a^{-1})\gamma_2\dot{p}_2-z_o'a^{-1}p_2-\nu e_1(1-\omega^2)p_2-\omega^2p_2s_{11}+A_1b^{-1}x_o's_{21}+O(\mu)+\cdots,$$
$$\Phi=-(1-C_1)A_1^{-1}p_2\dot{p}_2\dot{\gamma}_2+x_o'\dot{\gamma}_2^2-y_o'\gamma_2\dot{\gamma}_2-z_o'b^{-1}\gamma_2+x_o'b^{-1}-A_1^{-2}\gamma_2\dot{p}_2^2$$
$$+\nu(1-\omega^2)(e+e_1\gamma_2)-\gamma_2s_{11}+(1+B_1)p_2s_{21}+O(\mu)+\cdots; \qquad (3)$$

$$p_2=p_1-\mu e-\mu e_1\gamma_2,\gamma_2=\gamma_1-\mu\nu p_2,\; q_1=-A_1^{-1}\dot{p}_2+\mu A_1^{-1}(y_o'a^{-1}-e_2\dot{y}_2)+\cdots,$$
$$\gamma_1'=\dot{\gamma}_2+\mu\nu_2\dot{p}_2+\cdots,\; r_1=1+\frac{1}{2}\mu^2s_{11}+\cdots,\gamma_1''=1+\mu s_{21}+\mu^2(s_{22}-\frac{1}{2}s_{11})+\cdots; \qquad (4)$$

$$p_1 = p/c\sqrt{\gamma_o''}, q_1 = q/c\sqrt{\gamma_o''}, r_1 = r/r_o, \gamma_1 = \gamma/\gamma_o'', \gamma_1' = \gamma'/\gamma_o'', \gamma_1'' = \gamma''/\gamma_o'', \tau = r_o t, (\equiv d/d\tau); \tag{5}$$

$$\begin{aligned} s_{11} &= a(p_{2o}^2 - p_2^2) + b(\dot{p}_{2o}^2 - \dot{p}_2^2)/A_1^2 - 2[x_o'(\gamma_{2o} - \gamma_2) + y_o'(\dot{\gamma}_{2o} - \dot{\gamma}_2)], \\ s_{21} &= a(p_{2o}\gamma_{2o} - p_2\gamma_2) - bA_1^{-1}(\dot{p}_{2o} - \dot{p}_2\dot{\gamma}_2), \\ s_{22} &= a[\nu(p_{2o}^2 - p_2^2) + e(\gamma_{2o} - \gamma_2) + e_1(\gamma_{2o}^2 - \gamma_2^2)] + bA_1^{-1}[-\nu_2(\dot{P}_{2o}^2 - \dot{p}_2^2) + a^{-1}y_o'(\dot{\gamma}_{2o} - \dot{\gamma}_2) - e_2(\dot{\gamma}_{2o}^2 - \dot{\gamma}_2^2)]; \end{aligned} \tag{6}$$

$$A_1 = \frac{C-B}{A}, B_1 = \frac{A-C}{B}, C_1 = \frac{B-A}{C}, \gamma_o \geq 0, 0 < \gamma_o'' < 1, a = \frac{A}{C}, b = \frac{B}{C}, c^2 = \frac{Mg\ell}{C}, \mu = \frac{c\sqrt{\gamma_o''}}{r_o}, x_o = \ell x_o', y_o = \ell y_o',$$

$$z_o = \ell z_o', \ell^2 = x_o^2 + y_o^2 + z_o^2, \omega^2 = -A_1 B_1, e = \frac{x_o' A_1}{b\omega^2}, e_1 = \frac{z_o'(A_1 b^{-1} - a^{-1})}{1-\omega^2}, \nu = \frac{1+B_1}{1-\omega^2}, e_2 = e_1 + a^{-1} z_o', \nu_2 = \nu - A_1^{-1}; \tag{7}$$

here $p_o, q_o, \gamma_o, \gamma_o'$ and γ_o'' are the initial values of the corresponding variables. Considering r_o, is large then μ is small.

2. PROPOSED METHOD

In this section, the proposed method is applied to investigate the periodic solutions, with zero basic amplitudes, for the system (1) when ω^2 is positive. Since the system (1) is autonomous, the following condition does not affect the generality of the solutions [5]:

$$p_2(0,0) = \dot{p}_2(0,0) = \dot{\gamma}_2(0,\mu) = 0. \tag{8}$$

The generating system of (1) is:

$$\ddot{p}_2^{(o)} + \omega^2 p_2^{(o)} = 0, \qquad \ddot{\gamma}_2^{(o)} + \gamma_2^{(o)} = 0, \tag{9}$$

with frequencies ω and I. Let us consider the case when $\omega = m/n$ where m and n are relative primes, for this case, the solutions of (9) [with the period $T_o = 2\pi n$] are:

$$p_2^{(o)} = a_o^* \cos\omega\tau, \qquad \gamma_2^{(o)} = b_o^* \cos\tau, \tag{10}$$

where a_o^* and b_o^* are constants to be determined.

Supposing that the system (1) has periodic solutions with the period $T_o + \alpha$, which reduce to the generating solutions (10) at $\mu = 0$, that is; α is a function of μ such that $\alpha(0)=0$. Assuming the required solutions in the forms:

$$p_2 = a^* \cos\psi + \sum_{n=1}^{N} \mu^n p_n^*(a^*, \psi) + O(\mu^{N+1}),$$

$$\gamma_2 = b^* \cos\phi + \sum_{n=1}^{N} \mu^n \gamma_n^*(a^*, \phi) + O(\mu^{N+1}) \tag{11}$$

with the initial conditions:

$$\begin{aligned} p_2(0,\mu) &= a^* = a_o^* + a^*(\mu), \\ \gamma_2(0,\mu) &= b^* = b_o^* + b^*(\mu), \quad \dot{\gamma}_2(0,\mu) = 0, \end{aligned} \tag{12}$$

where $a^*(\mu)$ and $b^*(\mu)$ are equal to zero when μ=0. Taking into consideration the first integral (2) with the initial conditions (12) to obtain:

$$\begin{aligned} 0 &< b_o^* = (1 - \gamma_o''^2)^{1/2} (\gamma_o'')^{-1} < \infty, \\ b^*(\mu) &= -\mu\nu[a_o^* + a^*(\mu)] + \ldots \end{aligned} \tag{13}$$

Assuming a^*, ψ and ϕ vary with time according to:

$$\dot{a}^* = \sum_{n=1}^{N} \mu^n A_n^*(a^*) + O(\mu^{N+1}), \tag{14}$$

$$\dot{\psi} = \omega + \sum_{n=1}^{N} \mu^n \psi_n(a^*) + O(\mu^{N+1}), \tag{15}$$

$$\dot{\phi} = 1 + \sum_{n=1}^{N} \mu^n \phi_n(a^*) + O(\mu^{N+1}). \tag{16}$$

The derivatives are obtained in the forms:

$$\dot{p}_2 = -a^* \omega \sin\psi + O(\mu), \qquad \dot{\gamma}_2 = -b^* \sin\phi + O(\mu),$$

$$\ddot{p}_2 = -a^* \omega^2 \cos\psi + \mu\left[\omega^2 \frac{\partial^2 p_1^*}{\partial\psi^2} - 2a^* \omega\psi_1 \cos\psi - 2\omega A_1^* \sin\psi\right]$$
$$+\mu^2\Big[2\omega A_1^* \frac{\partial^2 p_1^*}{\partial a^* \partial\psi} - 2(\omega A_2^* + A_1^* \psi_1)\sin\psi + A_1^* \frac{dA_1^*}{da^*}\cos\psi + \omega^2 \frac{\partial^2 p_2^*}{\partial\psi^2}$$
$$+2\omega\psi_1 \frac{\partial^2 p_1^*}{\partial\psi^2} - a^*(\psi_1^2 + 2\omega\psi_2)\cos\psi - a^* A_1^* \sin\psi \frac{d\psi_1}{da^*}\Big] + O(\mu^3),$$

$$\ddot{\gamma}_2 = -b^* \cos\phi + \mu\left[\frac{\partial^2 \gamma_1^*}{\partial\phi^2} - 2b^* \phi_1 \cos\phi\right] + \mu^2\Big[\frac{\partial^2 \gamma_2^*}{\partial\phi^2} + 2\phi_1 \frac{\partial^2 \gamma_1^*}{\partial\phi^2}$$
$$-b^*(\phi_1^2 + 2\phi_2)\cos\phi + 2A_1^* \frac{\partial^2 \gamma_1^*}{\partial a^* \partial\phi} - b^* A_1^* \frac{d\phi_1}{da^*}\sin\phi\Big] + O(\mu^3). \tag{17}$$

Making use of (6), (11) and (17) leads to:

$$s_{11}^{(o)} = a a_o^{*2}(\cos^2\psi_o - \cos^2\psi) - bA_1^{-2} a_o^{*2}\omega^2 \sin^2\psi - 2b_o^* [x_o' (\cos\phi_o - \cos\phi) + y_o' \sin\phi],$$

$$s_{21}^{(o)} = a_o^* b_o^* [a(\cos\psi_o \cos\phi_o - \cos\psi\cos\phi) + bA_1^{-1}\omega \sin\psi \sin\phi],$$

$$s_{22}^{(o)} = a[v a_o^{*2}(\cos^2\psi_o - \cos^2\psi) + e b_o^* (\cos\phi_o - \cos\phi) + e_1 b_o^{*2}(\cos^2\phi_o - \cos^2\phi)]$$
$$+bA_1^{-1}[v_2 a_o^{*2}\omega^2 \sin^2\psi + a^{-1} y_o' b_o^* \sin\phi + e_2 b_o^{*2}\sin^2\phi], \tag{18}$$

where ψ_o and ϕ_o are the initial values of the corresponding functions. Using (3), (11), (17) and (18) the functions $F^{(o)}$ and $\Phi^{(o)}$ are obtained in the following forms:

$$F^{(o)} = C_1 A_1^{-1} a_o^{*3}\omega^2 \cos\psi \sin^2\psi + \omega a_o^* b_o^* x_o' \sin\psi \sin\phi + a^{-1} a_o^* b_o^* y_o' \cos\psi \sin\phi$$
$$+\omega A_1^{-1}(A_1 + a^{-1}) a_o^* b_o^* y_o' \sin\psi \cos\phi - z_o' a^{-1} a_o^* \cos\psi$$
$$-v e_1 (1-\omega^2) a_o^* \cos\psi - \omega^2 a_o^* \cos\psi\{a a_o^{*2}(\cos^2\psi_o - \cos^2\psi)$$
$$-bA_1^{-2} a_o^{*2}\omega^2 \sin^2\psi - 2b_o^* [x_o' (\cos\phi_o - \cos\phi) + y_o' \sin\phi]\}$$
$$+A_1 b^{-1} x_o' a_o^* b_o^* [a(\cos\psi_o \cos\phi_o - \cos\psi\cos\phi) + bA_1^{-1}\omega \sin\psi \sin\phi],$$

$$\Phi^{(o)} = \frac{1}{2}(C_1 - 1)A_1^{-1}\omega a_o^{*2} b_o^* \sin 2\psi \sin\phi + \frac{1}{2}x_o' b_o^{*2}(1 - \cos 2\phi) + \frac{1}{2}y_o' b_o^{*2}\sin 2\phi$$
$$-z_o' b^{-1} b_o^* \cos\phi + x_o' b^{-1} - \frac{1}{2}A_1^{-2}\omega^2 a_o^{*2} b_o^* (1 - \cos 2\psi)\cos\phi + ve(1-\omega^2)$$
$$+ve_1(1-\omega^2) b_o^* \cos\phi - a a_o^{*2} b_o^* \cos^2\psi_0 \cos\phi + \frac{1}{2}a a_o^{*2} b_o^* (1 + \cos 2\psi)\cos\phi$$
$$+\frac{1}{2}bA_1^{-2}\omega^2 a_o^{*2} b_o^* (1 - \cos 2\psi)\cos\phi + 2x_o' b_o^{*2}\cos\phi_o \cos\phi$$
$$-x_o' b_o^{*2}(1 + \cos 2\phi) + y_o' b_o^{*2}\sin 2\phi + a_o^{*2} b_o^* (1 + B_1)[bA_1^{-1}\omega \sin\psi \sin\phi$$
$$+a(\cos\psi_o \cos\phi_o - \cos\psi\cos\phi)]\cos\psi \quad . \tag{19}$$

Substituting (11), (17) and (19) into the initial system (1), and equating coefficients of each power of μ, to get:

$$\frac{\partial^2 p_1^*}{\partial\psi^2}+P_1^* = \frac{2a_o^*}{\omega}\psi_1\cos\psi+\frac{2A_1^*}{\omega}\sin\psi, \qquad \frac{\partial^2\gamma_1^*}{\partial\phi^2}+\gamma_1^* = 2b_o^*\phi_1\cos\phi,$$

$$\frac{\partial^2 p_2^*}{\partial\psi^2}+p_2^* = \frac{2A_2^*}{\omega}\sin\psi+\frac{a_o^*}{\omega^2}[2\omega\psi_2+\frac{1}{4}\omega^2C_1A_1^{-1}a_o^{*2}+\frac{3}{4}\omega^2aa_o^{*2}-aa_o^{*2}$$
$$-z_o^{\prime}a^{-1}-ve_1(1-\omega^2)+\frac{1}{4}bA_1^{-2}a_o^{*2}\omega^4+2\omega^2x_o^{\prime}b_o^*\cos\phi_o]\cos\psi$$
$$+\frac{1}{4}a_o^{*3}(a-C_1A_1^{-1}-\omega^2bA_1^{-2})cos3\psi+\frac{aa_o^*}{\omega^2}x_o^{\prime}A_1b^{-1}b_o^*\cos\phi_o\cos\psi_o$$
$$+x_o^{\prime}a_o^*b_o^*(\frac{1}{\omega}-\frac{aA_1}{2b\omega^2}-1)\cos(\phi-\psi)-x_o^{\prime}a_o^*b_o^*(1+\frac{1}{\omega}+\frac{aA_1}{2b\omega^2})\cos(\phi+\psi)$$
$$+y_o^{\prime}a_o^*b_o^*[1+\frac{1}{2a\omega^2}-\frac{A_1+a^{-1}}{2A_1\omega}]\sin(\phi-\psi)$$
$$+y_o^{\prime}a_o^*b_o^*[1+\frac{1}{2a\omega^2}+\frac{A_1+a^{-1}}{2A_1\omega}]\sin(\phi+\psi),$$

$$\frac{\partial^2\gamma_2^*}{\partial\phi^2}+\gamma_2^* = [2\phi_2-z_o^{\prime}b^{-1}+\frac{1}{2}A_1^{-2}\omega^2a_o^{*2}(b-1)+ve_1(1-\omega^2)-aa_o^{*2}\cos^2\psi_o$$
$$-\frac{1}{2}aB_1a_o^{*2}+2x_o^{\prime}b_o^*\cos\phi_o]b_o^*\cos\phi-\frac{1}{2}x_o^{\prime}b_o^{*2}+x_o^{\prime}b^{-1}+ve(1-\omega^2)$$
$$+(1+B_1)aa_o^{*2}b_o^*\cos\psi_o\cos\phi_o\cos\psi-\frac{3}{2}x_o^{\prime}b_o^{*2}cos2\phi+\frac{3}{2}y_o^{\prime}b_o^{*2}\sin 2\phi$$
$$+a_o^{*2}\{[\frac{1}{2}A_1^{-2}\omega^2(1-b)-\frac{1}{2}aB_1+A_1^{-1}\omega b_o^*(b-1)]\cos(2\psi-\phi)\}$$
$$+[\frac{1}{2}A_1^{-2}\omega^2(1-b)-\frac{1}{2}aB_1-A_1^{-1}\omega b_o^*(b-1)]\cos(2\psi+\phi)\}. \tag{20}$$

Eliminating secular terms [6] from (20), yields:

$$\psi_1=A_1^*=\phi_1=A_2^*=0,$$
$$\psi_2=\frac{1}{2\omega}[-\frac{1}{4}\omega^2C_1A_1^{-1}a_o^{*2}-\frac{3}{4}\omega^2aa_o^{*2}+aa_o^{*2}+z_o^{\prime}a^{-1}+ve_1(1-\omega^2)-\frac{1}{4}bA_1^{-2}\omega^4a_o^{*2}-2\omega^2x_o^{\prime}b_o^*\cos\phi_o],$$
$$\phi_2=\frac{1}{2}[z_o^{\prime}b^{-1}-\frac{1}{2}A_1^{-2}\omega^2a_o^{*2}(b-1)-ve_1(1-\omega^2)+aa^{*2}(\frac{1}{2}B_1+\cos^2\psi_o)-2x_o^{\prime}b_o^*\cos\phi_o]. \tag{21}$$

Substituting (21) into (14), (15) and (16), and integrating, to obtain (up to the second approximation of μ):

$$a^* = cost. = a_o^*,$$
$$\psi=\omega\tau+\frac{1}{2}\mu^2[-\frac{1}{4}\omega C_1A_1^{-1}a_o^{*2}-\frac{3}{4}\omega aa_o^{*2}\omega^{-1}+z_o^{\prime}a^{-1}\omega^{-1}+ve_1(\omega^{-1}-\omega)-\frac{1}{4}bA_1^{-2}\omega^3a_o^{*2}-2\omega x_o^{\prime}b_o^*\cos\phi_o]\tau,$$
$$\phi=\tau+\frac{1}{2}\mu^2[z_o^{\prime}b^{-1}-\frac{1}{2}A_1^{-2}\omega^2a_o^{*2}(b-1)-ve_1(1-\omega^2)+aa_o^{*2}(1+\frac{1}{2}B_1)-2x_o^{\prime}b_o^*]\tau. \tag{22}$$

From the above results, yields:

$$\psi(0)=\psi_o=0, \qquad \phi(0)=\phi_o=0. \tag{23}$$

On the use of (22) and (12), then $a^*(\mu)$ begins from terms of order greater than μ^2. Substituting (21) and (22) into (20) and solving the obtained equations, then using (11) and (13) to construct p_2 and γ_2.

Starting p_2 and γ_2, using (4, (18), (22) and (23) to obtain p_1, q_1, r_1, γ_1, γ_1' and γ_1'', taking into consideration the initial conditions (8) and (12) with ω is rational and does not equal to ½, 1, 2, the first terms in the expansions of the periodic solutions (with zero basic amplitudes) of the system (1) can be expressed in the following forms:

$$p_1 = -\frac{\mu x_o'}{bB_1} + \mu e_1 b_o^* \cos\tau + \cdots,$$

$$q_1 = \frac{\mu y_o'}{aA_1} + \mu e_2 A_1^{-1} b_o^* \sin\tau + \cdots,$$

$$r_1 = 1 - \mu^2 b_o^* [x_o'(1-\cos\tau) + y_o' \sin\tau] + \cdots,$$

$$\gamma_1 = b_O^* \cos\tau + \cdots,$$

$$\gamma_1' = -b_o^* \sin\tau + \cdots,$$

$$\gamma_1'' = 1 + \mu^2 [b_o^* (1-a)^{-1} x_o' + \frac{1}{2} b_o^{*2} z_o' (\frac{a-b}{a+b-1}) + b_o^* (1-b)^{-1} y_o' \sin\tau - b_o^* (1-a)^{-1} x_o' \cos\tau - \frac{1}{2} b_o^{*2} z_o' (\frac{a-b)}{a+b-1}) \cos 2\tau] + \cdots, \qquad (24)$$

and the correction α of the period is:

$$\alpha(\mu) = 2\mu^2 \pi n [b_o^* x_o' - z_o'] + \cdots. \qquad (25)$$

The expansions (24) and (25) agree with (2.30) and (2.31) of Arkhangel'skii [5], that is; the method proposed here for treating a quasilinear autonomous system of two degrees of freedom and one first integral gives asymptotic representations in agreement with those obtained by using Poincaré's small parameter method. The geometric interpretation of motion, by means of Euler's angles, showed that it is possible to determine the six initial conditions at which the body will perform a pseudo-regular precession [5]. The expressions of Euler's angles depend on four arbitrary constants.

3. CONCLUSION

The proposed method has an advantage in that:

1. The dependence of the solutions on the independent periodicity conditions [7] need not be specified a priori, and also, in that the criterion of eliminating of secular producing terms introduced for the determination of the arbitrary functions that appear in the expansions.
2. Poincaré's method requires the convergence of series in a small parameter which represent periodic solutions but in the description of the Krylov-Bogoliubov -Mitropolski method it is emphasized that the question of the convergence of small parameter expansions does not arise at all.

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