STABILITY AND CHARACTERISTICS ANALYSIS OF NON LINEAR UNDAMPED ASYMMETRIC OSCILLATION

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ABSTRACT

The stability of the solutions near the equilibrium points of the non linear undamped oscillation of the form $\ddot{x} + f(x) = 0$ where f(x) is an asymmetric non linear function of x is analysed. The main characteristic of the oscillation namely, the frequency p, the shift of the center Δ from the origin and the average amplitude a are also obtained. Our results agreed with those of [1] where a series solution of an asymmetric oscillation of the form $\ddot{x} + 3x^2 - 6x + 2 = 0$ was obtained. We also analysed the stability conditions of the solutions near the equilibrium points of two other asymmetric oscillations namely when $f(x) = \alpha^2 x^2 - \beta^2 x^4$ and $f(x) = \alpha^2 x^2 + \beta x^3$ where α, β are constants, whose solutions were obtained in series form also in [2].

Keywords: Nonlinear, oscillations, characteristics, linear stability analysis.

INTRODUCTION

A non linear undamped oscillation given by the equation

$$\ddot{x} + f(x) = 0 \tag{1}$$

is asymmetric if the characteristic function f(x) is asymmetric i.e if

$$f(x) \neq - f(-x) \tag{2}$$

and it is symmetric if

$$f(x) = - f(-x) \tag{3}$$

Non linear symmetric oscillations are studied in many well known equations such as van Der Pole equation [3], the Rayleigh equation [3], Duffing's equation [4] and Lashink's equation [5]. In all these equations, the attenuation and the restoring force are symmetric. Non linear asymmetric oscillations are much difficult than symmetric ones. This is because there is a direct relation between the characteristic function f(x) of the undamped oscillation and its periodic time T, the linear characteristic that approximates f(x) passes through the origin and there is equal excursion on both sides of the center of the symmetric oscillation [6]. In section 2 we analyse the stability of an asymmetric non linear undamped oscillation whose characteristic function is of a general form f(x). We used the linearlied theory of stability since the results of Lyapunov analysis and linearlized stability analysis are proved to be the same [7]. In section 3 we calculated the average amplitude $\bar{a}$, the shift of the center of the oscillation Δ from the origin and the frequency p of the oscillations. In section 4 we present the results of both the analysis of stability and calculation of a, Δ and p when the characteristic function f(x) is of the form

$$f(x) = ax^2 + bx + c \tag{4}$$

where a, b, c are constant. We got numerical results when a,b, and c have the values of 3, -6, and 2 respectively. These results agreed with those of Eid [1] where a series solution of an asymmetric non linear oscillation of the form $\ddot{x} + 3x^2 - 6x + 2 = 0$ was obtained. In sections 5, and 6, the stability of two other non linear asymmetric oscillations given by $\ddot{x} + \alpha^2 x^2 - \beta^2 x^4 = 0$ and $\ddot{x} + \alpha^2 x^2 - \beta x^3 = 0$ are analysed. Conclusion is presented in section 7.

2. STABILITY

To use the linearlized analysis of equilibria [3], [4] we begin by defining x_1, x_2 such that

$$x_1 \equiv x$$

and (5)

$$x_2 \equiv \dot{x}$$

The 2nd order d.e. given by (1) is now transformed into two 1st order d.e. in the from

$$\dot{x}_1 = Q(x_1,x_2)$$

and (6)

$$\dot{x}_2 = P(x_1, x_2)$$

where

$$Q(x_1, x_2) = x_2, \quad P(x_1, x_2) = -f(x_1) \tag{7}$$

Let the equilibrium points be defined as the points at which $\dot{x} = 0$ and $\ddot{x} = 0$. This means that the values of equilibrium points x_o satisfies the equation

$$f(x_o) = 0 \tag{8}$$

Now we can take the origin of coordinates to be at any one of the equilibrium points by merely a translation of coordinate axes. By expanding $Q(x_1,x_2)$ and $P(x_1, x_2)$ in a Taylor series around the origin, we get from equations (6)

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \epsilon_1 (x_1,x_2)$$

and (9)

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \epsilon_2 (x_1,x_2)$$

where the coefficients a_{ij} are evaluated at the origin and have the values

$$a_{11} = \frac{\partial Q}{\partial x_1}, \; a_{12} = \frac{\partial Q}{\partial x_2}, \; a_{21} = \frac{\partial P}{\partial x_1}, \; a_{22} = \frac{\partial P}{\partial x_2} \tag{10}$$

while $\epsilon_1 (x_1, x_2)$ and $\epsilon_2 (x_1, x_2)$ are non linear functions of x_1, x_2 which can be neglected near the origin.

So, equations (9) can be written in matrix form as

$$\{\dot{x}\} = [a]\{x\} \tag{11}$$

where

$$[a] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\{\dot{x}\} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}, \quad x = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{12}$$

let the solution of (11) be of the form

$$\{x\} = e^{\lambda t}\{c\} \tag{13}$$

where λ is a constant and $\{c\}$ is a constant column matrix given by

$$\{c\} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \tag{14}$$

To get a nontrivial solution of equation (11), the characteristic equation given by

$$\det([a] - \lambda[I]) = 0 \tag{15}$$

where [I] is the unit matrix must be satisfied.

Substituting from equations (10) and (5) in (15) the characteristic equation can be written as

$$\lambda^2 - \left(\frac{\partial P}{\partial \dot{x}} + \frac{\partial Q}{\partial x}\right)\lambda + \frac{\partial P}{\partial \dot{x}}\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x}\cdot\frac{\partial Q}{\partial \dot{x}} = 0 \tag{16}$$

For each equilibrium point of the system, λ has two values. The behavior of the solution given by (13) in the neighborhood the equilibrium point depends on the two values of λ. When both roots of (16) are conjugate complex then the solution x(t) is oscillatory and if they are purely imaginary then the solution is stable periodic [3]. This is the kind of solution we are searching for.

3. AVERAGE AMPLITUDE, CENTER AND FREQUENCY OF OSCILLATION

Let initial condition be given such that at $t = 0$,

$$x(0) = a_1, \; \dot{x}(0) = 0 \tag{17}$$

Due to unsymmetry of the function f(x), let $x = -a_2$

be the other point of maximum deflection of the oscillation at which $\dot{x} = 0$. The relation between a_1 and a_2 is given by

$$\int_{-a_2}^{a_1} f(x)dx = 0 \tag{18}$$

This can be obtained from equation (1) by substituting $\ddot{x}$ by $\dot{x}\frac{d\dot{x}}{dx}$ and integrating from $x = -a_2$ to $x = a_1$.

Due to unsymmetry of the function f(x), let the central position about which the system is vibrating by shifted to the left of the origin by the distance Δ such that

$$\Delta = \frac{a_2 - a_1}{2} \tag{19}$$

let $\bar{a}$ be the half of the total excursion, i.e.,

$$\bar{a} = \frac{a_1 + a_2}{2} \tag{20}$$

Thus for any initial condition given by equation (17) there are certain distinct values of the deflection a_2, the shift of the center Δ and average amplitude $\bar{a}$ that can be obtained from equations (18), (19) and (20) respectively. Let an approximating linear characteristic drawn from the center of oscillation be given by

$$f^*(x) = p^2(x + \Delta) \tag{21}$$

where p is a frequency to be obtained.

Figure (1) shows the assumed location of a_1, and that of a_2 and Δ as given by equations (17), (18), (19) the hatched area in this figure shows the sum of the differences between some asymmetric function f(x) and $f^*(x)$ during a half of the oscillation. (i.e. from $x = -a_2$ to $x = a_1$). To relate these differences to the coordinate x we define the weighed deviation r between f(x) and $f^*(x)$ by:

$$r = (f(x) - f^*(x))\,(x + \Delta) \tag{22}$$

Then the sum I along a half oscillation of the square of the weighed deviation r is given by

$$I = \int_{-a_2}^{a_1} r^2\,dx \tag{23}$$

We choose p such that the integral I has a minimum value. This is obtained from the relation $\frac{dI}{dp^2} = 0$ from which p^2 has the value

$$p^2 = \frac{5}{(a_1 + a_2)^5}\int_{-a_2}^{a_1} f(x)(x+\Delta)^3 dx \tag{24}$$

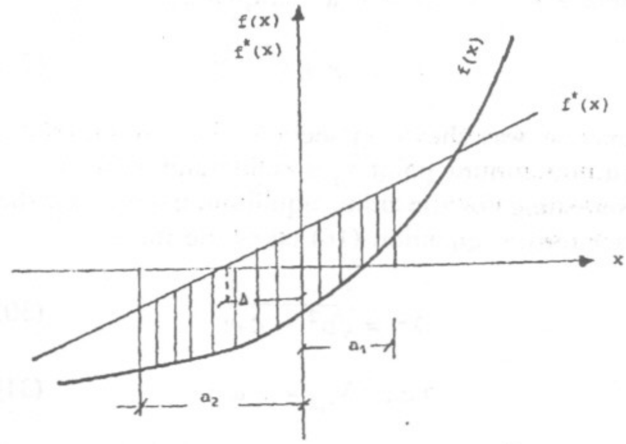

Figure 1. The assumed locations of a_1, a_2 and Δ. The hatched area shows the sum of the differences between f(x) and $f_*(x)$ during a half oscillation.

4. RESULTS

4.1. *1st Kind of Asymmetric Non Linear Undamped Oscillation*

4.1.1. *Stability*

We begin by an undamped non linear asymmetric oscillation for which f(x) has the value

$$f(x) = ax^2 + bx + c \tag{25}$$

where a, b and c are any constants. The two equilibrium points $x_{1,2}$ as obtained from equation (8), and (25) are given by

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{26}$$

Thus if $\sqrt{b^2 - 4ac} > 0$, then for the first equilibrium point x_1 , after substituting from equations (5),(7) and (25) the characteristic equation (16) takes the form:

$$\lambda^2 = -\sqrt{b^2 - 4ac} \tag{27}$$

$$\text{i.e., } \lambda^2 = \omega^2 i^2 \tag{28}$$

where $i = \sqrt{-1}$ and ω is a real quantity i.e.,

$$\lambda_{1.2} = \pm \omega i \tag{29}$$

Hence as was shown in section 2, the solution x(t) near equilibrium point x_1 is stable and periodic.

Repeating for the other equilibrium point x_2, the characteristic equation (16) takes the form:

$$\lambda^2 = \sqrt{b^2 - 4ac} \tag{30}$$

$$\text{i.e., } \lambda_{1,2} = \pm \omega \tag{31}$$

Thus since the roots $\lambda_{1,2}$ are real of different sign, the solution x(t) near equilibrium point x_2 is non oscillatory and unstable [3]. Again if $\sqrt{b^2 - 4ac} = 0$, we have only one equilibrium point for which the characteristic equation (16) is given by

$$\lambda_{1,2} = 0 \tag{32}$$

and hence no oscillattory solution is expected near equilibrium point for this case [3].

Thus $\sqrt{b^2 - 4ac} > 0$ is necessary and sufficient for the oscillation for which f(x) is given by equation (25) to have a stable periodic solution near equilibrium point x_1 given by equation (26).

These results are in agreement with the results of Eid [1] where the solution x(t) of the oscillation

$$\ddot{x} + 3x^2 - 6x + 2 = 0 \tag{33}$$

was studied. It obvious that $\sqrt{b^2 - 4ac}$ is > 0 and the obtained solution is stable around the equilibrium point $x_1 = 1 + \frac{1}{\sqrt{3}}$ and is unstable around the equilibrium point $x_2 = 1 - \frac{1}{\sqrt{3}}$.

4.1.2. *Average Amplitude, Center and Frequency of Oscillation*

To get a_2 the other point at which $\dot{x} = 0$, we substitute from equation (25) in equation (18) we get an algebraic equation of the third degree in a_2 that is

$$a_2^3 - \frac{3b}{2a}a_2^2 + \frac{3c}{a}a_2 + D = 0 \tag{34}$$

where

$$D = D(a_1) = a_1^3 + \frac{3b}{2a}a_1^2 + \frac{3c}{a}a_1 \tag{35}$$

Equation (34) has at least one real root and at most three different real roots [8]. Hence the number and value of a_2 depends on initial value of a_1 and the coefficients a,b,c of the function f(x).

If b = 0 and c and a are of the same sign then only one of the three roots of equation (34) is real [8] and is given by

$$a_2 = \{ -\frac{1}{2}[a_1^3 + \frac{3c}{a}a_1 + (a_1^6 + \frac{6c}{a}a_1^4 + \frac{9c^2}{a^2}a_1^2 + \frac{4c^3}{a^3})^{1/2}]\}^{1/3} + \{ -\frac{1}{2}[a_1^3 + \frac{3c}{a}a_1 - (a_1^6 + \frac{6c}{a}a_1^4 + \frac{9c^2}{a^2}a_1^2 + \frac{4c^3}{a^3})^{1/3}]\}^{1/3} \tag{36}$$

Also it is obvious that one of the three roots of equation (34) is equal to - a_1. The two other values of a_2 are real and different if L defined by

$$L = (\frac{3b}{2a} + a_1)(\frac{3b}{2a} - 3a_1) - \frac{12ca_1}{a} \tag{37}$$

is + ve.

However, they are real and equal if

$$L = 0 \tag{38}$$

and they are complex conjugate if

$$L < 0 \tag{39}$$

Hence we may note that there is two centers, one center and no center of oscillation when conditions (37), (38) and (39) are satisfied respectively.

Thus once a_2 is found, the shift of the centers of oscillation Δ, the average amplitude $\bar{a}$, and the approximated frequency p are calculated from equations (19), (20) and (24) respectively.

Table (1) shows values of a_2 for different values of a_1 and the corresponding values of Δ, a, and p^2 for the asymmetric oscillation given by equation (33).

It was found that L is < 0 when the initial value of a_1 is greater than or approximately equal to equilibrium point so there is no real values of a_2 and consequently no values of Δ, $\bar{a}$ and p^2 can be found. This is shown in the base of table (1). However L is > 0 for all other values of a_1 and hence two values for a_2 are found. In the table if only one value of a_2 is recorded this means that the frequency p^2 which correspond to the unrecorded values of a_2 is -ve and hence this value of a_2 is rejected. The frequency $p^2 = 3$ that corresponds to $a_1 = 1$ is identical with the frequency of the solution of [1]. Figure (2) shows the extreme excursion a_2, the shift of the center of the oscillation Δ, the amplitude $\bar{a}$ for this case.

It is obvious that x(t) has only +ve values. This agrees with the results in [1] where only +ve values of x(t) are found in the solution. Also, Figure (3) shows the function $f(x) = 3x^2 - 6x + 2$ and the approximated linear characteristic function $f^*(x) = 3(x+1.5)$ when $a_1 = 1$.

The stable equilibrium point $x_1 = 1 + \dfrac{1}{\sqrt{3}} = 1.5772$

and the unstable equilibrium point $x_2 = 1 - \dfrac{1}{\sqrt{3}} = 0.4228$

are also shown, in this figure. The sum of the differences between f(x) and $f^*(x)$ during a half period (from $x = a_1 = 1$ to $x = a_2 = 2$) is a minimum because the value of $p^2 = 3$ is calculated from equation (24).

Table 1. Values of a_2 for different values of a_1 and the corresponding values of Δ, a and p^2 for the oscillation given by $\ddot{x} + 3x^2 - 6x + 2 = 0$ studied in [1].

a_1	a_2	Δ	$\bar{a}$	p^2
0.1	-2.9307	-1.51535	1.41505	3.09210
0.2	-2.856	-1.528	1.328	3.168
0.3	-2.77566	-1.53779	1.2378	3.228
0.4	-2.6892	-1.5446	1.1446	3.2676
0.5	-2.5963	-1.5481	1.04815	3.2786
0.6	-2.49615	-1.5483	0.9481	3.2886
0.7	-2.3879	-1.5439	0.8439	3.2634
0.8	-2.2705	-1.5352	0.73523	3.2112
0.9	-2.1420	-1.521	0.621	3.126
1	-2	-1.5	0.5	3
1.1	-1.8402	-1.47	0.3701	2.82
1.2	-1.6549	-1.4275	0.2275	2.5649
1.3	-1.4266	-1.3633	0.0633	2.1798
1.4	-1.0828	-1.2414	0.1586	1.4484
1.41	-1.0275	-1.2186	0.1914	1.3116
1.42	-0.9564 -0.62356	-1.1882 -1.02178	0.2318 0.39822	1.1292 0.13068
1.43	-0.8214 -0.7486	-1.1257 -1.0893	0.3043 0.3407	0.7542 0.5358
1.44	---	---	----	---
1.45				
1.5				

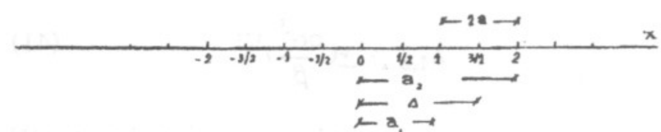

Figure 2. a_2, Δ, $\bar{a}$ for the nonlinear oscillation studied in [1] when $a_1 = 1$.

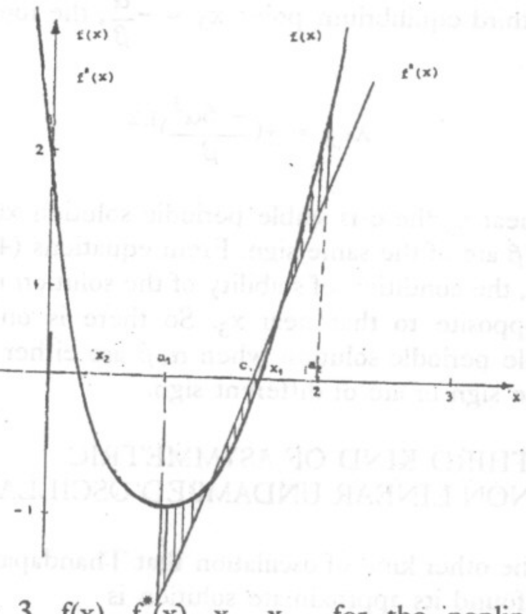

Figure 3. f(x), $f^*(x)$, x_1, x_2 for the nonlinear oscillation studied in [1] when $a_1 = 1$.

5. 2nd KIND OF ASYMMETRIC NON LINEAR UNDAMPED OSCILLATION

Thandapani et al [2] found an approximate solution of the oscillation

$$\ddot{x} + \alpha^2 x^2 = \beta^2 x^4 \tag{40}$$

where α and β are constants.

To study the stability of this oscillation, we repeat the same procedure of section 2. It is easily proved that there are three equilibrium points $x_{1,2,3}$ which are 0, $\pm\frac{\alpha}{\beta}$.

Substituting from equations (31), (3), (4), (7) in (6), we find that there is no stable solution near equilibrium point $x_1 = 0$ since $\lambda_{1,2} = 0$.

Repeating again for the equilibrium point $x_2 = \frac{\alpha}{\beta}$, we find that the roots $\lambda_{1,2}$ are

$$\lambda_{1,2} = \pm(\frac{6\alpha^3}{\beta})^{1/2} \tag{41}$$

hence near x_2 there is a stable periodic solution x(t) if α and β are of different sign since this makes $\lambda_{1,2}$ complex conjugate purely imaginary. Once again for the third equilibrium point $x_3 = -\frac{\alpha}{\beta}$, the roots $\lambda_{1,2}$ are

$$\lambda_{1,2} = \pm(\frac{-6\alpha^3}{\beta})^{1/2} \tag{42}$$

So near x_3 there is stable periodic solution x(t) if α and β are of the same sign. From equations (41) and (42), the condition of stability of the solution near x_2 is opposite to that near x_3. So there is only one stable periodic solution when α, β are either of the same sign or are of different sign.

6. THIRD KIND OF ASYMMETRIC NON LINEAR UNDAMPED OSCILLATION

The other kind of oscillation that Thandapani et al [2] found its approximate solution is

$$\ddot{x} + \alpha^2 x^2 = -\beta x^3 \tag{43}$$

The equilibrium point $x_{1,2}$ are 0, $\frac{-\alpha^2}{\beta}$.

Repeating the same procedure as in section 5, we find that near equilibrium point $x_1 = 0$ there is no stable solution since $\lambda_{1,2}=0$.

However near equilibrium point $x_2 = \frac{-\alpha^2}{\beta}$, $\lambda_{1,2}$ have the values

$$\lambda_{1,2} = \pm(\frac{-\alpha^4}{\beta})^{1/2} \tag{44}$$

hence there is stable periodic solution near x_2 if $\beta>0$. Hence the oscillation presented in equation (43) has only one equilibrium point and the condition of stability of the solution is quite simple.

We postpone the calculations of a_2, Δ and p^2 for the two oscillations given by equations (31) and (34) that were studied by Thandapani et al [2] to future work. This is because the approximated solution presented there is not calculated numerically. Hence more laborious lengthy work is needed to compare the results of our work with those of [2].

7. CONCLUSION

Solutions of equations of non linear oscillations are difficult. Whether or not these difficulties are overcome, the study of the stability of the solution near equilibrium points is a must. Mean while, in many practical applications, it will suffice to determine the frequency of the oscillation, its average amplitude and the shift of the center of the oscillation from the origin without going into the details of the process of oscillation. Since our results concerning both the stability and characteristics analysis agreed with those of [1], we believe that the method used in this work has proved its consistency.

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