MATHEMATICAL STUDY OF EARTH DAM WITH UPSTREAM BLANKET

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ABSTRACT

In the present study, the problem of seepage through an earth dam with upstream blanket is mathematically treated using the complex function theory. The main function of the impervious blanket appears in minimizing the quantity of seepage discharge. Also a loss of head due to blanket is achieved and increased with increasing the upstream length of the blanket. Effect of depth to the impervious stratum, base width of the dam, and the upstream retained water head on loss of head due to blanket are studied and are put in a graphical dimensionless form. The optimum length of both the upstream blanket and the base width of the dam are recommended for the case study.

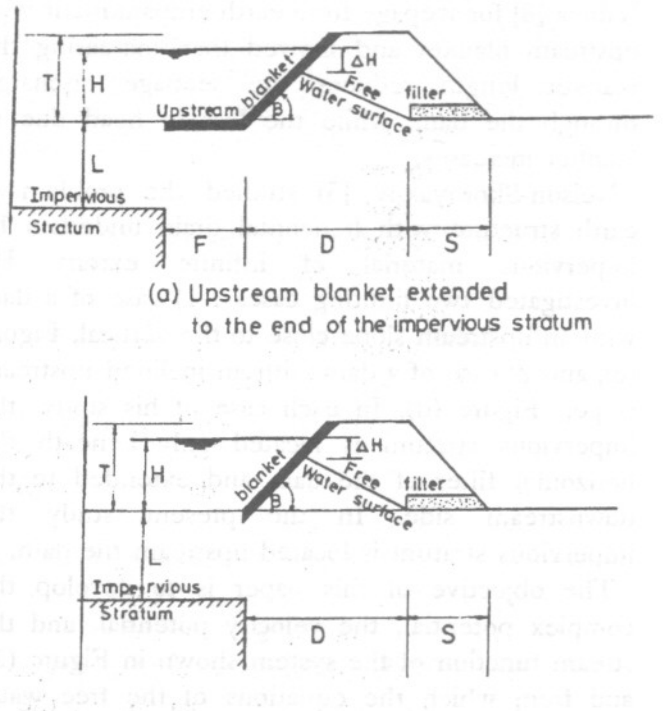
Keywords: Seepage, Earth dam, Complex function theory, Impervious blanket, Impervious stratum, Waterhead, filter.

NOTATION

D	base width of the dam,
F	length of the upstream blanket,
H	the upstream retained water head,
i	$\sqrt{-1}$
K	hydraulic conductivity of the soil,
L	depth to the impervious stratum measured from the upstream bed level of the dam,
m	strength of the filter [ref. 1],
q	seepage discharge to the filter,
S	length of the filter,
T	minimum height of the dam,
W	complex potential = $\phi + i\Psi$,
Z	complex number = x + iy,
ϕ	the velocity potential,
Ψ	the stream function,
β	angle of inclination of the upstream face of the dam,
ΔH	loss of head due to blanket, and
$\Delta H/H$	relative head loss.

INTRODUCTION

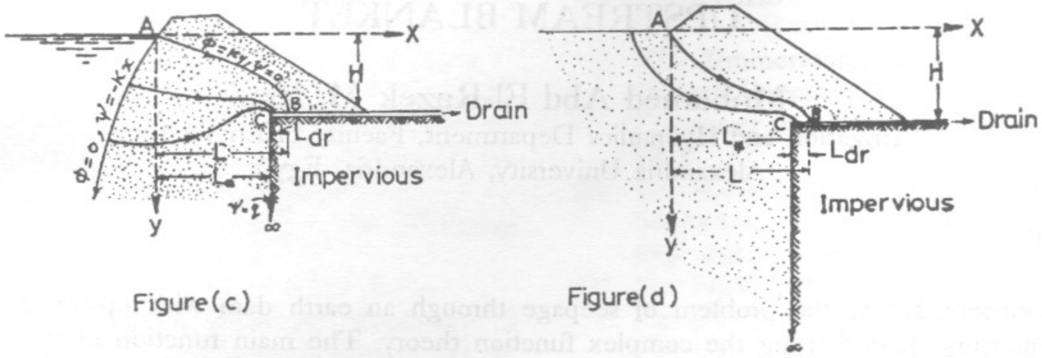
The problem of seepage through an earth dam with upstream blanket, is studied in this paper. Two cases are to be considered in the present study and shown in Figure (1).



(a) Upstream blanket extended to the end of the impervious stratum

(b) The impervious stratum vanishes at the beginning of the dam (F = 0.0).

Figure 1. The two cases under study.



The two cases of study as given by Nelson Shornyakov [5]

A mathematical model based on complex function theory is derived and used in solving of the two assumed cases.

The same theory of the complex function has been successfully used before in the problems of seepage as given by Hathoot [3,4] and Rezk [7]. An experimental study was carried out by Rezk and Rabiea [8] for seepage from earth embankment with upstream blanket and showed that, increasing the blanket length reduces the seepage discharge through the dam, while the loss of head due to blanket increases.

Nelson-Skornyakov [5] studied the problem of earth structure with horizontal drain underlain by impervious material of infinite extent. He investigated two limiting cases : 1) case of a dam with an upstream slope close to the vertical, Figure (c), and 2) case of a dam with an inclined upstream slope, Figure (d). In each case of his study, the impervious stratum is located only beneath the horizontal filter of the dam and extended to the downstream side. In the present study the impervious stratum is located upstream the dam.

The objective of this paper is to develop the complex potential, the velocity potential, and the stream function of the system shown in Figure (2), and from which the equations of the free water surface and the seepage discharge to the filter can be derived. In addition to determine the optimum lengths of the upstream blanket, and the base width of the dam.

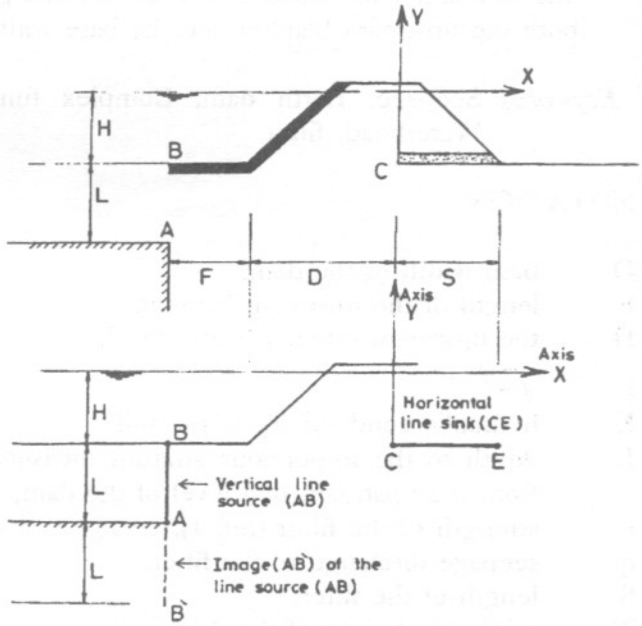


Figure 2. Mathematical model.

MATHEMATICAL MODEL AND ASSUMPTIONS

The two cases under study in this paper are shown in Figure (1). The flow is hydrodynamically represented by the vertical line Source (AB) with image (AB′). The horizontal filter is represented by a horizontal line sink (CE), Figure (2). The free water surface intersects with the filter at its beginning at point (C). The upstream blanket starts where the impervious stratum vanishes. The upstream blanket is considered impervious which results to loss of head (ΔH).

The Complex Potential of the vertical line source (AB)

The general form of the complex potential of the line source at the origin with an inclination "α" to the horizontal can be expressed as follow Ref [7]:

$$W = \frac{m}{L} e^{-i\alpha}[(Z - Le^{i\alpha})\ln(Z - Le^{i\alpha}) - (Z - Le^{i\alpha}) - Z\ln Z + Z]$$

For the case studied in this paper $\alpha = 90°$, $e^{i\alpha} = i$ and $e^{-i\alpha} = -i$, then the above equation can be written for the vertical line source as follow:

$$W_1 = \frac{im}{L}[Z\ln Z - (Z - iL)\ln(Z - iL) - iL] \quad (1)$$

Transforming the axes to point "B" (-(F+D), - iH) shown in Figure (2) then, put $Z = Z - [-(D+F) - iH]$ and $Z = x + iy$ in equ. (1).

Equation (1) can be put in the following form:

$$W_1 = \frac{im}{L}[(x_1 + iy_1)(\ln r_1 + i\theta_1) - (x_2 + iy_2)(\ln r_2 + i\theta_2) - iL]$$

$$W_1 = \frac{im}{L}[x_1 \ln r_1 + i x_1 \theta_1 + i y_1 \ln r_1 - y\theta_1$$

$$- x_2 \ln r_2 - ix_2 \theta_2 - iy_2 \ln r_2 + y_2 \theta_2 - iL] \quad (2)$$

where:

$x_1 = x + F + D$, $x_2 = x + F + D$,
$y_1 = y + H$ $y_2 = y + H - L$
$r_1 = \sqrt{x_1^2 + y_1^2}$, $r_2 = \sqrt{x_2^2 + y_2^2}$
$\theta_1 = \tan^{-1} y_1/x_1$ $\theta_2 = \tan^{-1} y_2/x_2$

Substituting $W_1 = \phi_1 + i\Psi_1$ in equation (2) from which, the velocity potential of the vertical line source is given as follow:

$$\phi_1 = \frac{m}{L}[-x_1 \theta_1 - y_1 \ln r_1 + x_2 \theta_2 + y_2 \ln r_2 + L] \quad (3)$$

The stream function of the vertical line source is written in the following form:

$$\Psi_1 = \frac{m}{L}[x_1 \ln r_1 - y_1 \theta_1 - x_2 \ln r_2 + y_2 \theta_2] \quad (4)$$

The Complex potential of the image (AB') of the vertical line source (AB)

As mentioned before the general form of the complex potential of the image of the line source at the origin is given as follow Ref [7]:

$$W = \frac{m}{L} e^{-i\alpha}[Z\ln Z - (Z + Le^{i\alpha})\ln(Z + Le^{i\alpha}) + Le^{i\alpha}]$$

To get the Complex potential of the image in the vertical position

put $\alpha = 90°$, $e^{i\alpha} = i$ and $e^{-i\alpha} = -i$, then

$$W_2 = \frac{im}{L}[(Z + iL)\ln(Z + iL) - Z\ln Z - iL] \quad (5)$$

Transforming the axes to point A (-(F+D), - i (H+L)) shown in Figure (2) then, put $Z = Z - [-(F+D) - (iL + iH)]$ and $Z = x + iy$ in equ. (5). Equation (5) can be written in the following form:

$$W_2 = \frac{im}{L}[(x_3 + iy_3)(\ln r_3 + i\theta_3) - (x_4 + iy_4)(\ln r_4 + i\theta_4) - iL]$$

$$W_2 = \frac{im}{L}[x_3 \ln r_3 + i x_3 \theta_3 + i y_3 \ln r_3 - y_3 \theta_3$$

$$- x_4 \ln r_4 - ix_4 \theta_4 - iy_4 \ln r_4 + y_4 \theta_4 - iL] \quad (6)$$

where:

$x_3 = x + F + D$, $x_4 = x + F + D$, $y_3 = y + H + 2L$, $y_4 = y + H + L$

$r_3 = \sqrt{x_3^2 + y_3^2}$, $r_4 = \sqrt{x_4^2 + y_4^2}$, $\theta_3 = \tan^{-1} y_3/x_3$,

$\theta_4 = \tan^{-1} y_4/x_4$

Substituting $W_2 = \phi_2 + i\,\Psi_2$ in equation (6), then the velocity potential of the image:

$$\phi_2 = \frac{m}{L}\left[-x_3\,\theta_3 - y_3 \ln r_3 + x_4\,\theta_4 + y_4 \ln r_4 + L\right] \quad (7)$$

The stream function of the image:

$$\Psi_2 = \frac{m}{L}\left[x_3 \ln r_3 - y_3\,\theta_3 - x_4 \ln r_4 + y_4\,\theta_4\right] \quad (8)$$

The Complex potential of the horizontal line sink (CE)

The general form of the complex potential of the line sink at the origin and has an inclination "γ" to the horizontal can be written as follow Ref [3]:

$$W_3 = \frac{m}{S} e^{-i\gamma}\left[Z \ln Z - (Z - Se^{i\gamma}) \ln (Z - Se^{i\gamma}) - Se^{i\gamma}\right]$$

Substituting $\gamma = 0.0$, $e^{i\gamma} = 1$ and $e^{-i\gamma} = 1$ to get the complex potential of the horizontal line sink as follow:

$$W_3 = \frac{m}{S}\left[Z \ln Z - (Z - S) \ln (Z - S) - S\right] \quad (9)$$

Transforming the axes to point "C" (0.0, iH) shown in Figure (2), then put $Z = Z - (-iH) = Z + iH$ and $Z = x + iy$ in equ. (9), equation (9) can be expressed as follow:

$$W_3 = \frac{m}{S}\left[(x_5 + iy_5)(\ln r_5 + i\theta_5) - (x_6 + iy_6)(\ln r_6 + i\theta_6) - S\right]$$

$$W_3 = \frac{m}{S}\left[x_5 \ln r_5 + i_5\, x_5\,\theta_5 + iy_5 \ln r_5 - y_5\,\theta_5\right.$$

$$\left. - x_6 \ln r_6 - ix_6\,\theta_6 - iy_6 \ln r_6 + y_6\,\theta_6 - S\right] \quad (10)$$

where:

$x_5 = x$, $x_6 = x - S$, $y_5 = y + H$,

$y_6 = y + H$ $r_5 = \sqrt{x_5^2 + y_5^2}$, $r_6 = \sqrt{x_6^2 + y_6^2}$,

$\theta_5 = \tan^{-1} y_5/x_5$, $\theta_6 = \tan^{-1} y_6/x_6$

Substituting $W_3 = \phi_3 + i\Psi_3$ in equation (10) then we get: the velocity potential

$$\phi_3 = \frac{m}{S}\left[x_5 \ln r_5 - y_5\,\theta_5 - x_6 \ln r_6 + y_6\,\theta_6 - S\right] \quad (11)$$

The stream function

$$\Psi_3 = \frac{m}{S}\left[x_5\,\theta_5 + y_5 \ln r_5 - x_6\,\theta_6 - y_6 \ln r_6\right] \quad (12)$$

The complex potential of the System (W_s)

$$W_s = W_1 + W_2 + W_3 \quad (13)$$

where W_1, W_2 and W_3 are denoted by the equations (2), (6) and (10) respectively,

The velocity potential of the system (ϕ_s)

$$\phi_s = \phi_1 + \phi_2 + \phi_3 \quad (14)$$

where ϕ_1, ϕ_2 and ϕ_3 are denoted by the equations (3), (7) and (11) respectively.

The Stream function of the system (Ψ_s)

$$\Psi_s = \Psi_1 + \Psi_2 + \Psi_3 \quad (15)$$

where Ψ_1, Ψ_2 and Ψ_3 are denoted by the equations (4), (8) and (12) respectively.

BOUNDARY CONDITION

$$\phi_s = -k\left(\frac{p}{\rho g} + y\right) \text{ Ref. [6]} \quad (16)$$

where: ϕ is the velocity potential, K is the hydraulic conductivity of the soil, p is the gauge pressure, ρ is the fluid density and g is the acceleration due to gravity.

Applying equations (14) and (16) to point (C) where the pressure is atmospheric, Figure (2), at which x = 0.0 and y = - H to find value of the strength "m"

$$\therefore KH = \phi'_1 + \phi'_2 + \phi'_3 \quad (17)$$

where ϕ'_1, ϕ'_2, ϕ'_3 equal ϕ_1, ϕ_2, ϕ_3 respectively at x = 0, and y = -H.

EQUATION OF THE FREE WATER SURFACE

Along the free water surface, the pressure is atmospheric. The velocity potential can be expressed as follow:

$$\phi_s = - k\, y \qquad (18)$$

To find equation of the free water surface applying equations (14) and (18).

$$\therefore\ y = - \phi_s / k$$

$$\therefore\ y = \frac{-m}{K} \{ \frac{1}{L} [-x_1\theta_1 - y_1 \ln r_1 + x_2\theta_2 + y_2 \ln r_2 - x_3\,\theta_3 - y_3 \ln r_3 + x_4\,\theta_4 + y_4 \ln r_4 + 2\,L] + \frac{1}{S} [x_5 \ln r_5 - y_5\,\theta_5 - x_6 \ln r_6 + y_6\,\theta_6 - S]\} \qquad (19)$$

Equation (19) is used for plotting the free water surface in case of the problem studied in this paper.

The seepage discharge to the filter (q)

The strength "m" can be calculated at point "C", using equation (17), where the seepage water to the filter enters. The seepage discharge to the filter can be deduced from equation (15) by substituting x = 0.0 and y = - H;

$$\Psi_s = \Psi_c = q_{filter} \qquad (20)$$

ANALYSIS OF RESULTS

The complex function theory is used to establish the complex potential of the system shown in Figure (2) (equation No. 13), the velocity potential of the system (equation No. 14), and the stream function of the system (equation No. 15). Also, equation (No. 19) of the free water surface is deduced, and used to draw free water surface, and from which head loss is measured for different values of the upstream blanket length (F), depth to the impervious stratum (L) and angle of inclination of the upstream face of the dam (β). In addition to, equation of the discharge entering the filter is established (equation No. 20), and a sample of its results are recorded in table (1).

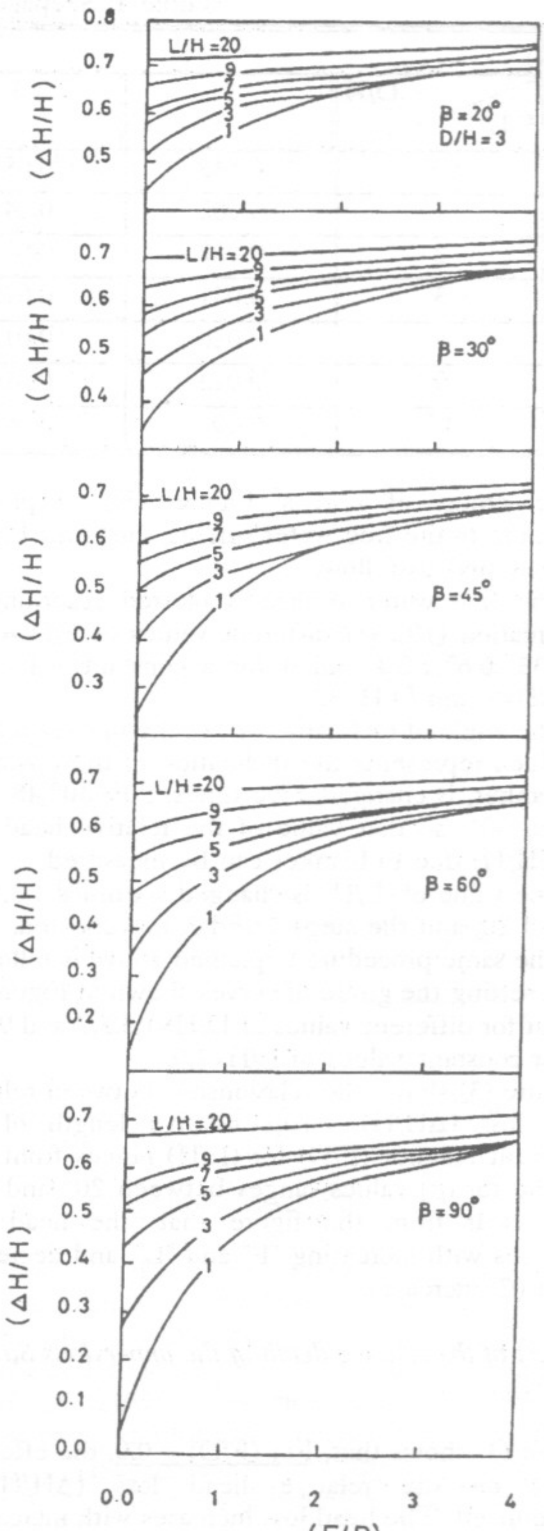


Figure 3. Relative head loss (ΔH/H) versus relative length of the blankets (F/D).

Table 1. Seepage discharge entering the filter.

	(q/KH)				
F/L \ D/H	1	3	5	7	9
0.0	0.149	0.057	0.031	0.020	0.015
1	0.088	0.041	0.025	0.017	0.013
2	0.057	0.031	0.021	0.015	0.011
3	0.041	0.025	0.017	0.013	0.010
6	0.020	0.015	0.011	0.009	0.008
9	0.013	0.010	0.008	0.007	0.006
12	0.009	0.008	0.007	0.006	0.005

The group of curves Figure (3), is plotted according to the free water surface shown in Figure (I-Appendix) as follow:

1- The free water surface is plotted according to equation (19), for different values of F/D = 0.0, 0.33, 0.67,1,2,3 and 4 for a constant values of L/H=1 and D/H=3.
2- The angle of inclination "β" of the upstream face, which represents the inclination of the upstream blanket, is changed Five times; 20°, 30°,45°,60° and 90°, so that value of the relative head loss (ΔH/H) due to blanket can be measured.
3- The value of (L/H) is changed six times 1,3,5,7,9 and 20, and the steps 1 and 2 are repeated.
4- The same procedure explained above is followed in setting the group of curves shown in Figure (4) but for different values of D/H=1,3,5,7 and 9 and for constant values of L/H=1.0.

Figure (3) shows the relationship between relative head loss (ΔH/H) versus relative length of the upstream blanket (F/D) for (L/H) ranges from 1 to 20, and for (β) values ranges between 20° and 90°. It is clear from the figure that, the head loss increases with increasing "F" and "L", and decreases when (β) increases.

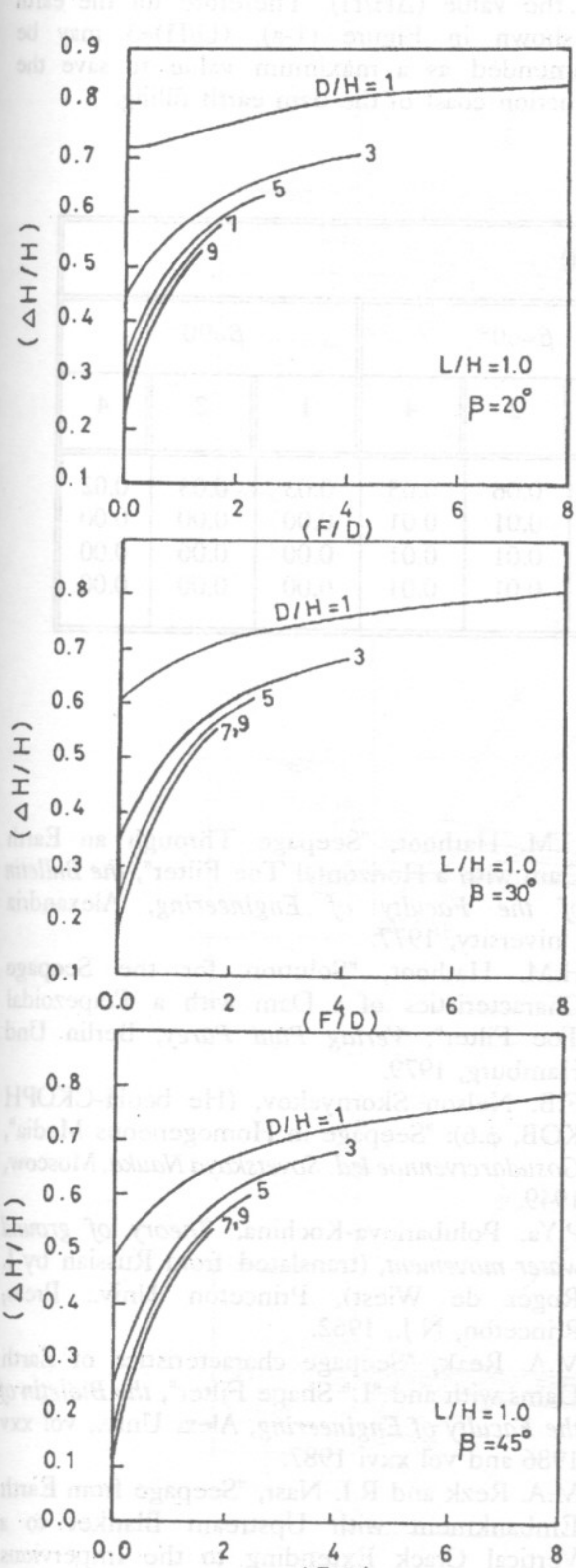
Effect of the relative depth of the impervious Stratum (L/H).

Figure (3) shows that, For (F/D) = 0.0, the effect of (L/H) on the relative head loss (ΔH/H) is pronounced. The head loss increases with increasing (L/H). The head loss varies from 0.44 to 0.72 for β=20°, and ranges from 0.04 to 0.67 for β = 90°. The percentage (ΔH/H) increases relative to (L/H) from 0.28 to 0.63 for β = 20° and 90° respectively. A major difference in the above mentioned percentage is evident.

For (F/D) = 1.0, the percentage of increasing (ΔH/H) with increasing (L/H) ranges from 0.15 to 0.24 for β = 20° and 90° respectively, corresponds to a head loss varies from 0.58 to 0.73 for β = 20°, and a head loss ranges from 0.44 to 0.68 for β = 90°. In this case study, the relative length of the upstream blanket (F/D) is not long enough, so as to a pronounced head loss is achieved. At the same time, the value of (ΔH/H) is clearly changing as (L/H) changes.

For (F/D) = 2.0, the percentage of increasing (ΔH/H) as (L/H) increases, decreases to 0.08, and 0.14 for β=20° and 90° respectively. It is shown that, as (F/D) reaches a value of 2 the effect of (L/H) on the head loss reduces, while the head loss, in this case, raised from 0.65 to 0.73 for β = 20°, and from 0.55 to 0.69 for β = 90°.

For (F/D)=4, the effect of (L/H), in this case could be ignored. The increase in the percentage (ΔH/H) with (L/H), for β = 20° and 90° equals to 0.05 and 0.02 respectively, corresponds to a head loss 0.72 and 0.66 which is not noticeable, if compared with the case of (F/D) = 2. From Table (I-Appendix), the total average relative head loss (ΔH/H) was found equals to 0.51, 0.61, 0.66 and 0.69 for the relative length of the upstream blanket (F/D) = 0.0,1,2 and 4 respectively. The results showed above recommends an optimum value of "2" for ratio (F/D), which will develop an average head loss equals to 0.66. Increasing (F/D) beyond this value produces a negligible effect on (ΔH/H). In other words (F/D) = 2 may be recommended if an earth dam is required to be constructed downstream the impervious stratum, Figure (1-a).



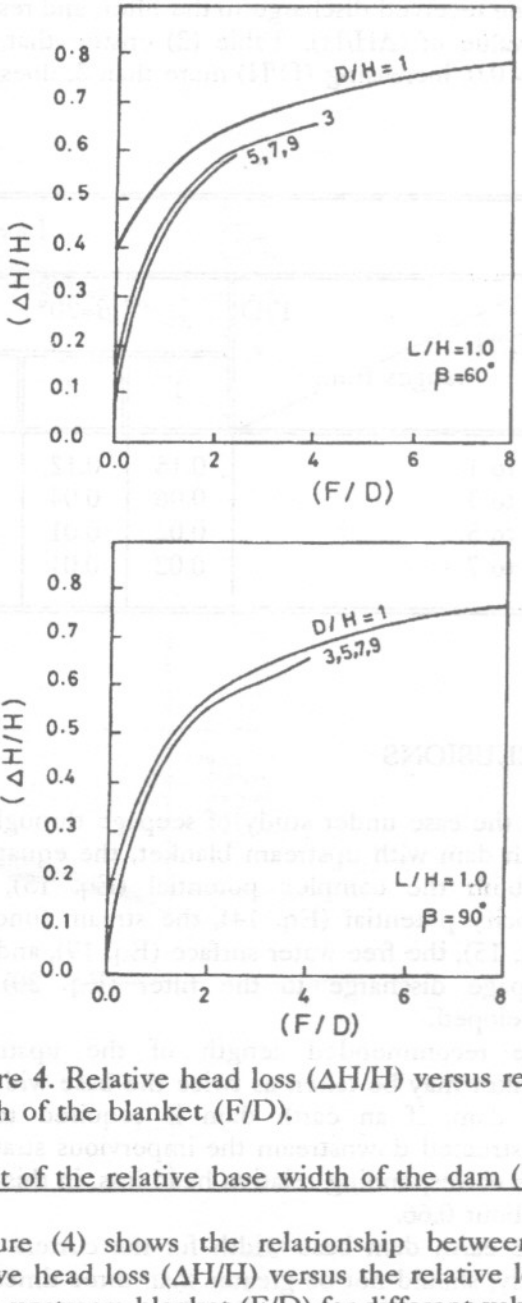


Figure 4. Relative head loss (ΔH/H) versus relative length of the blanket (F/D).

Effect of the relative base width of the dam (D/H)

Figure (4) shows the relationship between the relative head loss (ΔH/H) versus the relative length of the upstream blanket (F/D) for different values of (D/H), and β ranges from 20° to 90°. For a constant value of (D/H), it is clear that, the head loss increases rapidly for (F/D) ranges from 0.0 to 2, after which a slight increase occurs. Also it is noticed from Figure (4) that, for a constant value of (F/D), the head loss increases with the decreasing of (D/H). This related to the filter action which drops quickly the free water surface for small value of (D/H), and large value of the discharge seepaged to the filter, as shown from Table (1). However increasing (D/H) limits the received discharge to the filter, and results a less value of (ΔH/H). Table (2) ensure that, for (F/D) > 0.0, increasing (D/H) more than 3, does not affect the value (ΔH/H). Therefore for the earth dam shown in Figure (1-a), (D/H)=3 may be recommended as a maximum value to save the construction coast of the dam earth filling.

Table 2.

Increase in (ΔH/H)									
D/H ranges from \ F/D	β=20°			β=60°			β=90°		
	1	2	4	1	2	4	1	2	4
3 to 1	0.16	0.12	0.11	0.08	0.06	0.05	0.03	0.03	0.02
5 to 3	0.06	0.04	0.04	0.02	0.01	0.01	0.00	0.00	0.00
7 to 5	0.02	0.01	0.01	0.02	0.01	0.01	0.00	0.00	0.00
9 to 7	0.02	0.01	0.01	0.02	0.01	0.01	0.00	0.00	0.00

CONCLUSIONS

1- For the case under study of seepage through an earth dam with upstream blanket, the equations of both the complex potential (Eq. 13), the velocity potential (Eq. 14), the stream function (Eq. 15), the free water surface (Eq. 19), and the seepage discharge to the filter (Eq. 20) are developed.

2- The recommended length of the upstream blanket may be taken as twice the base width of the dam, if an earth dam is required to be constructed downstream the impervious stratum. The corresponding relative head loss, in this case is about 0.66.

3- The earth dam base width for the current case study, should not be greater than three times the upstream retained water head.

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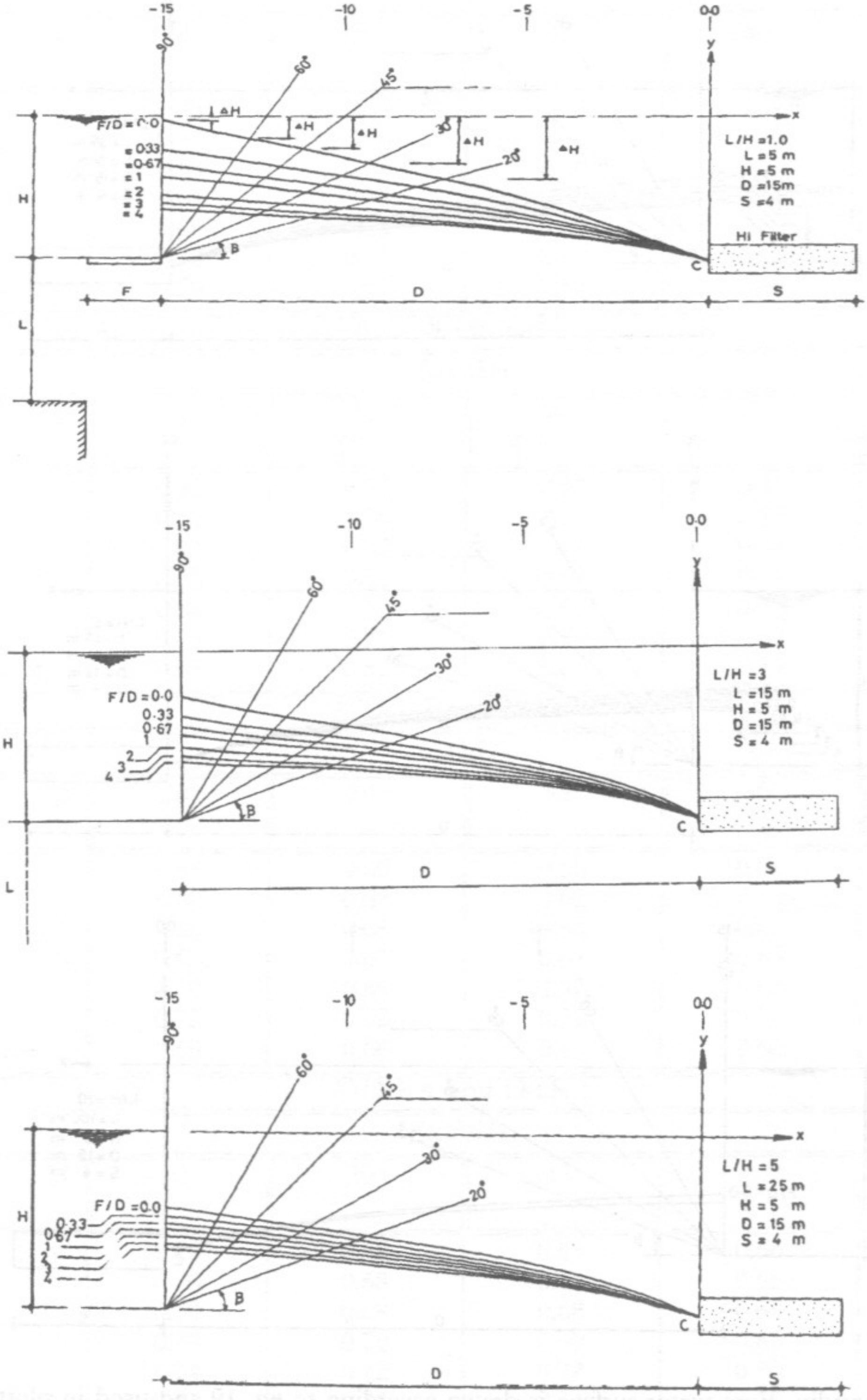
Appendix I.



-15
-10
-5
0.0
y
x
90°
60°
45°
30°
20°
ΔH
F/D = 0.0
= 0.33
= 0.67
= 1
= 2
= 3
= 4
H
B
L
F
D
S
C
Hi Filter
L/H = 1.0
L = 5 m
H = 5 m
D = 15m
S = 4 m
L/H = 3
L = 15 m
H = 5 m
D = 15 m
S = 4 m
L/H = 5
L = 25 m
H = 5 m
D = 15 m
S = 4 m

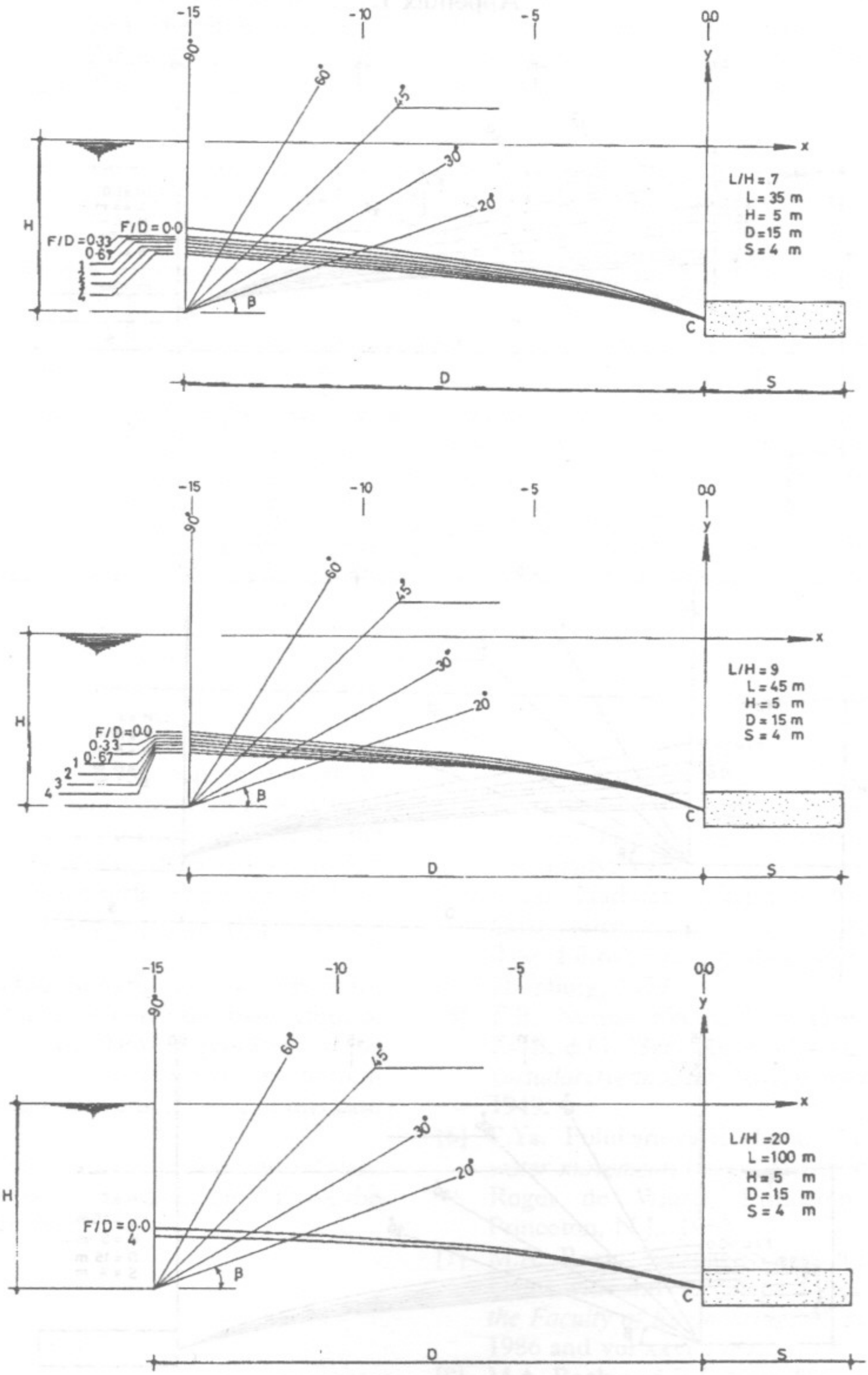


Figure I-Appendix. Free water surface is drawn according to eq. 19 and used in plotting Figure 3.

Table (I-Appendix) F/D = 0.0 and D/H=3

(ΔH/H)					
β° / L/H	20°	30°	45°	60°	90°
1	0.44	0.34	0.24	0.14	0.04
3	0.52	0.46	0.40	0.34	0.28
5	0.58	0.54	0.50	0.47	0.43
7	0.61	0.58	0.56	0.54	0.52
9	0.66	0.64	0.60	0.60	0.56
20	0.72	0.71	0.69	0.68	0.67
average value	0.59	0.55	0.50	0.46	0.42

F/D = 1 and D/H=3

(ΔH/H)					
β° / L/H	20°	30°	45°	60°	90°
1	0.58	0.52	0.50	0.46	0.44
3	0.62	0.58	0.55	0.53	0.50
5	0.64	0.62	0.59	0.58	0.55
7	0.66	0.64	0.63	0.61	0.60
9	0.69	0.67	0.65	0.65	0.62
20	0.73	0.71	0.70	0.69	0.68
average value	0.65	0.62	0.60	0.59	0.57

F/D = 2 and D/H=3

(ΔH/H)					
β° / L/H	20°	30°	45°	60°	90°
1	0.65	0.60	0.60	0.58	0.55
3	0.65	0.62	0.62	0.60	0.58
5	0.66	0.66	0.65	0.61	0.60
7	0.68	0.67	0.67	0.63	0.62
9	0.70	0.69	0.67	0.66	0.65
20	0.73	0.72	0.71	0.70	0.69
average value	0.69	0.66	0.65	0.63	0.62

F/D = 4 and D/H=3

(ΔH/H)					
β° / L/H	20°	30°	45°	60°	90°
1	0.70	0.68	0.68	0.66	0.66
3	0.70	0.68	0.68	0.66	0.66
5	0.70	0.68	0.68	0.68	0.66
7	0.72	0.70	0.69	0.68	0.66
9	0.74	0.72	0.72	0.70	0.66
20	0.75	0.74	0.74	0.72	0.68
average value	0.72	0.70	0.70	0.68	0.66