# FIRST AND SECOND FUNDAMENTAL PROBLEMS FOR AN ELASTIC INFINITE PLATE WITH A CURVILINEAR HOLE

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ABSTRACT

Complex variable methods are used to solve the first and second fundamental problems for the infinite plate with a curvilinear hole conformally mapped on the domain outside a unit circle by means of a rational mapping function. Some applications are investigated and some special cases are derived.

*Keywords*: Complex Variable methods, First and Second fundamental problems An infinite plate, Transformation mapping.

## INTRODUCTION

Problems dealing with the isotropic homogeneous infinite plates have been investigated by many authors [1,2,3,6]. Some authors [2,3] used Laurent's theorem to express each complex potential as a power series, others [1,6] used complex variable method of Cauchy type integrals.

It is known that [4] the first and second fundamental problems in the plane theory of elasticity are equivalent to finding two analytic functions, $\Phi(z)$ and $\Psi(z)$ of one complex argument $z = x + iy$ satisfying the boundary conditions.

$$K\,\Phi_1(t) - t\,\overline{\Phi_1'(t)} - \overline{\Psi_1(t)} = f(t) \qquad (1.1)$$

where for the first fundamental problem $k = -1$, $f(t)$ is a given function of stresses; while for the second fundamental problem $k = æ = \dfrac{\lambda + 3\mu}{\lambda + \mu} > 1$, $f = 2\mu g$ is a given function of the displacement, $\lambda, \mu$ are called Lame's constants, æ is called Muskhelishvili's constant and with t denoting the affix of a point on the boundary.

The transformation mapping

$$z = c\,w(\zeta) = c(\zeta + m\zeta^{-1}) \quad (c > o, w' \neq 0 \text{ or } \infty \text{ for } |\zeta| > 1) \qquad (1.2)$$

is used by Muskhelishvili [4] to solve the first and second fundamental problems of the infinite plate with an elliptic hole. Solkolnikoff [3] used the same rational mapping of (1.2) to solve the problem of elliptical ring, where he used Laurant's theorem. England [2] considered an infinite plate which is weakened by a hypotrochoid hole, conformally mapped onto a unit circle $\gamma$ by the transformation mapping.

$$z = c(\zeta + m\zeta^{-n}) \quad (c > o, z' \neq 0 \text{ or } \infty \text{ for } |\zeta| > 1)$$

$(c < m < 1/n)$ and he solved the boundary value problem.

The main reason for interest in this mapping is that the general shapes of the hypotrochoids are curvilinear polygons; for $n = 1$ an ellipse, for $n = 2$ a curvilinear triangle, for $n = 3$ a curvilinear square and hence approximate regions of physical interest. El-Sirafy and Abdou [5] used complex variable methods to solve the first and second fundamental problems of the infinite plate with a curvilinear hole C conformally mapped on the domain outside a unit circle $\gamma$ by the rational function

$$z = \frac{c(\zeta + m\zeta^{-1})}{1 - n\zeta^{-1}} \quad (c > 0, |n| < 1, z'(\zeta) \neq 0 \text{ or } \infty \text{ for } |\zeta| > 1).$$

In this paper complex variable methods are used to solve the first and second fundamental problems of the same previous domain of the infinite plate with a curvilinear hole C conformally mapped on the domain outside a unit circle $\gamma$ by the mapping.

$$z = c\frac{\zeta + m\zeta^{-1} + l\zeta^{-2}}{1 - n\zeta^{-1}} \quad (c>0, z(\zeta) \neq 0 \text{ or } \infty \text{ for } |\zeta| > 1).$$

Some applications of the first and second fundamental problems on these domains are investigated, the interesting cases when the shape of the hole is an ellipse, triangle, a crescent or a cut having the shape of a circular arc are included as special cases. Several previously known solutions appear as special cases of the complex potential given here. Holes corresponding to certain combinations of the parameters m, n, l are sketched (See Figure (1,2,3)).

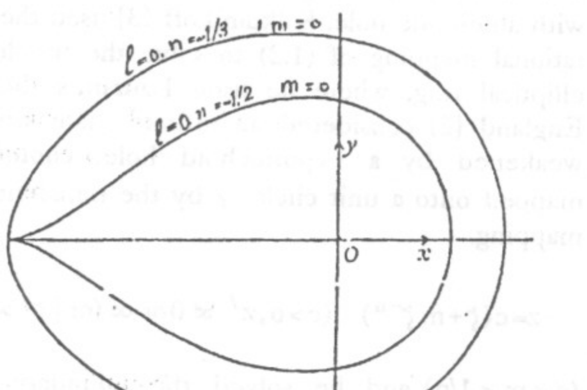

Figure 1.

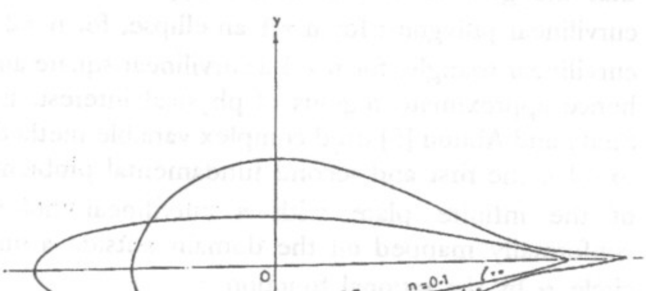

Figure 2.

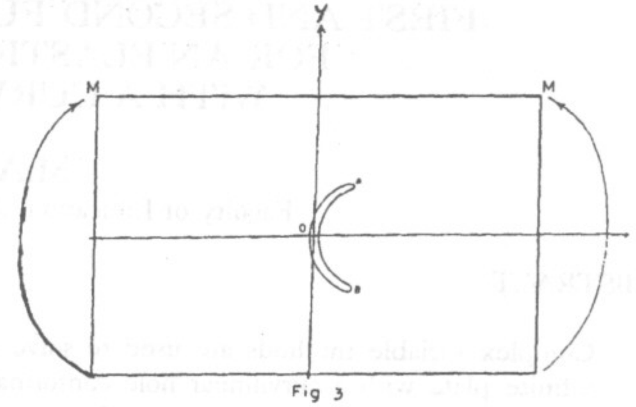

Figure 3.

## 2. BASIC EQUATIONS

Consider a region of an elastic media of infinite plate denoted by S bounded by a single contour L, with a curvilinear hole C, and assume that the origin lies inside the hole. Let $\overline{xx}, \overline{yy}, \overline{xy}$ be the components of stress, and u, v be the components of displacement. In the absence of body forces, it is known from Kolsov-Muskhelishvili [4] formulae

$$\overline{xx} + \overline{yy} = 4\,\mathrm{Re}\{\Phi_1'(z)\} \tag{2.1}$$

$$\overline{xx} - \overline{yy} + 2i\,\overline{xy} = 2[z\Phi_1''(z) + \Psi_1'(z)] \tag{2.2}$$

and

$$2\mu(u + iv) = æ\,\Phi(z) - z\overline{\Phi_1'(z)} - \overline{\Psi_1(z)}, \tag{2.3}$$

where the complex functions $\Phi_1(z)$ and $\Psi(z)$ take the form

$$\Phi_1(z) = -\frac{x + iy}{2\pi(i + æ)}\ln\zeta + c\,\Gamma\,\zeta + \Phi_o(\zeta) \tag{2.4}$$

$$\Psi_1(z) = \frac{æ(x - iy)}{2\pi(i + æ)}\ln\zeta + c\,\Gamma\,\zeta + \Psi_o(\zeta), \tag{2.5}$$

x,y being the components of the resultant vector of all external forces acting on L; $\Gamma, \Gamma'$ are constants and $\Phi_o(z), \Psi_o(z)$ are homomorphic functions at infinity, using (2.4)-(2.5) in (2.3), in general, we have

$$K\Phi(\sigma)-\frac{w(\sigma)}{w'(\sigma)}\overline{\Phi'(\sigma)}-\overline{\Psi(\sigma)}=f(\sigma) \text{ on } \gamma \quad (2.6)$$

Here $K=-1, f(\sigma)=-f^*(\sigma)$ for the first displacement problem while $K=\text{æ}, f(\sigma)=2\mu g(\sigma)$ for the second fundamental problem and $w(\zeta)$ is the rational mapping

$$z(\zeta)=cw(\zeta)=c\frac{\zeta+m\zeta^{-1}+l\zeta^{-2}}{1-n\zeta^{-1}}, \quad (c>0, |n|<1), \quad (2.7)$$

where m, n, l are real parameters subject to the condition $w'(\zeta)$ does not vanish or become infinite outside $\gamma$.

## 3. METHODS OF SOLUTION

The expression $\frac{w(\zeta^{-1})}{w'(\zeta)}$ may be written in the form

$$\frac{w(\zeta^{-1})}{w'(\zeta)}=\alpha(\zeta^{-1})+\beta(\zeta), \quad (3.1)$$

where

$$\alpha(\zeta)=\frac{l+h\zeta}{\zeta(\zeta-n)} \quad (3.2)$$

$$\beta(\zeta)=\frac{\zeta}{1-n\zeta}\left[\frac{(\zeta-n)^2(l\zeta^3+m\zeta^2+1)}{\zeta^4-2n\zeta^3-m\zeta^2-2l\zeta+nl}-(l\zeta+h)\right], \quad (3.3)$$

$$h=\frac{(1-n^2)^2(m+n^2)+ln(m+2ln+n^2-ln^3)}{[l-2n^2-mn^2-2ln^3+ln^5]}, \quad (3.4)$$

and $\beta(\zeta)$ is regular function for $|\zeta|>1$.

Using (2.7) and (3.1) the boundary condition (2.6) can be written in the form

$$K\Phi(\sigma)-\alpha(\sigma)\overline{\Phi'(\sigma)}-\overline{\Psi_{\star}(\sigma)}=f_{\star}(\sigma), \quad (3.5)$$

where

$$\Psi_{\star}(z)=\Psi(\zeta)+\beta(\zeta)\,\Phi'(\zeta), \quad (3.6)$$

$$f_{\star}(z)=F(\zeta)-cK\Gamma\zeta+c\overline{\Gamma^{\star}}\zeta^{-1}+N(\zeta)(\alpha(\zeta)+\overline{\beta(\zeta)}) \quad (3.7)$$

$$N(\zeta)=c\Gamma-\frac{x-iy}{2\pi(1+\text{æ})}\zeta \quad (3.8)$$

and

$$F(\zeta)=f(cw(\zeta)). \quad (3.9)$$

It will be assumed that $F(\zeta)$ has a derivative satisfying the HÖlder condition. Multiplying both sides of (3.5) by $\frac{1}{2\pi i(\sigma-\zeta)}$ and integrating with respect to $\sigma$ on $\gamma$, we have

$$K\Phi(\zeta)+\frac{1}{2\pi i}\int_\gamma\frac{\alpha(\sigma)\overline{\Phi'(\sigma)}}{\sigma-\zeta}d\sigma=$$

$$\frac{c\overline{\Gamma^*}}{\zeta}-A(\zeta)+\frac{h}{\zeta-n}N(n)+\frac{l}{\zeta(\zeta-n)}N(\zeta) \quad (3.10)$$

where

$$A\zeta=\frac{1}{2\pi i}\sum_{i=0}^{\infty}\zeta^{-\eta-1}\int_\gamma\sigma^{\eta}F(\sigma)d\sigma, \quad |\zeta|>1 \quad (3.11)$$

Using (3.2) in (3.10), one gets

$$\frac{1}{2\pi i}\int_\gamma\frac{\alpha(\sigma)\Phi'(\sigma)}{\sigma-\zeta}d\sigma=\frac{cb}{n-\zeta}, \quad (3.12)$$

where b is a complex constant to be determined. Hence

$$-K\Phi(\zeta)=A(\zeta)-\frac{c\overline{\Gamma^*}}{\zeta}+\frac{cl\Gamma}{\zeta(n-\zeta)}+\frac{cb_o}{n-\zeta}, \quad (3.13)$$

where

$$b_o=b+h\Gamma-\frac{(l+nh)(x-iy)}{2\pi c(1+\text{æ})}. \quad (3.14)$$

Inserting $\overline{\Phi'(\sigma)}$ from (3.12) in (3.10) we get

$$cbK+c\overline{b_o}\nu+nc(l+nh)\Gamma+ncl(2-n^2)\nu\Gamma+\frac{1+nh}{n}\overline{A'(n^{-1})}=0$$

Hence

$$b=\frac{KE-\nu\bar{E}}{c(K^2-\nu^2)}, \quad (3.15)$$

where

$$E=(l+nh)\left[-\frac{\overline{A'(n^{-1})}}{n}nc\Gamma+\frac{\nu(x+iy)}{2\pi(1+\text{æ})}\right. \quad (3.16.1)$$

$$\left.-ch\nu\Gamma+nc\nu\Gamma(n^2-2)l\right]$$

and

$$\nu=\frac{n(l+nh)}{(1-n^2)^2}. \quad (3.16.2)$$

Hence $\Phi(\zeta)$ is completely determined in the form

$$k\phi = \zeta\frac{-c\overline{\Gamma^*}}{\zeta} + \frac{cl\Gamma}{\zeta(n-\zeta)} + A(\zeta) + \frac{H}{n-\zeta}, \quad (3.17)$$

where

$$h = h\,N(n) + cb - \frac{l(x-iy)}{2\pi(1+æ)} \quad (3.18)$$

Also from the boundary conditions (3.5) and (3.6):

$$\psi = (\zeta) = \frac{cK\Gamma}{\zeta} - \frac{w(\zeta^{-1})}{w'(\zeta)}\Phi_{☼}(\zeta) + \frac{h\zeta}{1-n\zeta}\Phi_{☼}(n^{-1})$$

$$+ \frac{l\zeta}{1-n\zeta}B_{☼}(\zeta) + B(\zeta) - B, \quad (3.19)$$

where

$$\Phi_{☼}(\zeta) = \Phi'(\zeta) + \overline{N(\zeta)} \quad (3.20.1)$$

$$B(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\overline{F(\sigma)}}{\sigma-\zeta}d\sigma \quad (3.20.2)$$

$$B_{☼}(\zeta) = c\Gamma\zeta + n^{-1}\Phi'(n^{-1}) - \frac{x+iy}{2\pi(1+æ)} \quad (3.20.3)$$

and

$$B = \frac{1}{2\pi i}\int_\gamma \frac{\overline{F(\sigma)}}{\sigma}d\sigma. \quad (3.20.4)$$

## 4. SPECIAL CASES

a. For m = 0, the mapping function is $z = c\frac{\zeta + l\zeta^{-2}}{1-n\zeta^{-1}}$ and the two complex functions $\Phi(\zeta)$, $\Psi(\zeta)$ can be directly determined from (3.17)-(3.19).

b. For $m=n=0$, the rational mapping function is $z = c\zeta + l\zeta^{-2}$, where the hole takes the triangular form and the two complex functions take the form

$$K = \phi(\zeta) = c\overline{\Gamma^*}\zeta^{-1} - cl\overline{\Gamma}\zeta^{-2} + A(Z) + \frac{x-iy}{2\pi(1+æ)}l\zeta^{-1} \quad (4.1)$$

and

$$\psi(\zeta) = cK\overline{\Gamma}\zeta^{-1} - \frac{w(\zeta^{-1})}{w'(\zeta)}\Phi_{☼}(\zeta) + l\zeta^2 N(\zeta^{-1}) + B(\zeta) - B \quad (4.2)$$

c. For $m=0, l=-n^2$, the mapping function is $z = c(\zeta + n^2\zeta^{-1} + n)$, and the hole takes an elliptic form.

d. For n = 0, the mapping is $z = c(\zeta + m\zeta^{-1} + l\zeta^{-2})$, and the two complex functions can be obtained from (3.17)-(3.19).

e. For l = 0, the mapping function is $z = c\frac{\zeta + m\zeta^{-1}}{1-n\zeta^{-1}}$, and (3.17)-(3.19), in this case, are in the agreement with [5].

## 5. EXAMPLES

Curvilinear hole for an infinite plate subjected to a uniform tensile stress:

5.1 For $K=-1, \Gamma = \frac{p}{4}, \Gamma^* = -\frac{1}{2}P\exp((-2i\theta)$ and x=y=f=0, we have an infinite plate stretched at infinity by the application of a uniform tensile stress of intensity p, making an angle $\theta$ with the X-axis. The plate is weakened by a curvilinear hole C which is free from stress.

The functions (3.17) - (3.19) take the form

$$\phi(\zeta) = \frac{cP}{2\zeta}e^{2i\theta} + \frac{cPl}{\zeta(n-\zeta)} + \frac{cP}{n-\zeta}\left[\frac{h}{4} + \frac{h\nu}{4(1-\nu)} - \right.$$

$$\left. -\frac{n(l+nh)}{2(i-\nu)}\cos 2\theta + \frac{n(l+nh)}{2(1+\nu)}i\sin 2\theta + \frac{nl(2-n^2)\nu}{4(1-\nu)}\right] \quad (5.1)$$

and

$$\Psi(\zeta) = \frac{-cP}{4}\zeta^{-1} - \frac{w(\zeta^{-1})}{w'(\zeta)}\Phi_{☼}(\zeta) + \frac{\zeta}{1-n\zeta}(h\Phi_{☼}(n^{-1}) + lB_{☼}(\zeta)) \quad (5.2)$$

5.2 *Curvilinear hole the edge of which is subject to a uniform pressure*:

For $K=-1, x=y=\Gamma=\Gamma^*=0$ and f=Pt; where P is a real constant, the formulae (3.17) and (3.19) become

$$\Phi(\zeta) = \frac{cP}{(1-\nu)(n-\zeta)}[m + n^2 + nl(2-n^2)\nu + l(1-\nu)\zeta^{-1}] \quad (5.3)$$

and

$$\Psi(\zeta)=\frac{-w(\zeta^{-1})}{w'(\zeta)}\Phi'(\zeta)-cP\zeta^{-1}-ncP + \frac{cP\nu[m+n^2+2nl-n^3l]\zeta}{(1-n\zeta)(1-\nu)} \quad (5.4)$$

Hence (5.3) - (5.4) give the solution of the first fundamental problem when the edge of the hole is subject to a uniform pressure P. Putting in (5.3) and (5.4). -iT instead of P, we have the first fundamental problem when the edge of the hole is subject to a uniform tangential stress T.

### 5.3 *Uni-Directional Tension of an Infinite plate with a rigid curvilinear center*:

a. For $K=æ,\Gamma=\frac{P}{4},\Gamma^*=-\frac{P}{2}e^{-2i\theta},x=y=0,f=2ip\in^z$, we have the two complex functions

$$-æ\Phi(\zeta)=\frac{cPe^{2i\theta}}{2}\zeta^{-1}+c(n-\zeta)^{-1}\left[b_o^*+l(\frac{P}{4}+2ip\in)\zeta^{-1}\right] \quad (5.5)$$

$$\Psi(\zeta)=\frac{cPæ}{4}\zeta^{-1}-\frac{w(\zeta^{-1})}{w'(\zeta)}\Phi_{☼}(\zeta)+\frac{h\zeta}{1-n\zeta}\Phi_{☼}(n^{-1})+$$

$$2icp\in(n+\zeta^{-1})+\frac{l\zeta}{1-n\zeta}\left\{n^{-1}\Phi'(n^{-1})+\frac{cP\zeta}{4}\right\} \quad (5.6)$$

where

$$b_o^*=\frac{P}{4(æ+\nu)}[hæ+nl\nu(n^2-2)+2n(l+nh)\cos2\theta]+$$

$$+\frac{i}{2(æ-\nu)}[4\mu\in\{(m+n)^2æ+nl\nu(2-n^2)\}-nP(l+nh)\sin2\theta]$$

Therefore we have the case of uni-directional tension of an infinite plate with a rigid curvilinear center.

The constant $\in$ can be determined from the condition that the resultant moment of the forces acting on the curvilinear center from the surrounding material must vanish. i.e.

$$M=\mathrm{Re}\left\{\int\left[\Psi(\zeta)-\frac{cP}{2}e^{-2i\theta}\zeta\right]w'(\zeta d\zeta\right\}=0 \quad (5.7)$$

Hence,

$$\in=\frac{P(1-n^2)^2[æ(m+n^2)+nl(2-n^2)\nu](æ+1)\sin2\theta}{4p[æ(1+\nu)(m+n^2)(m+n^2+4nl-2n^3l)+D]}, \quad (5.8.1)$$

where

$$D=æ(æ-\nu)(1-n^2)^2+\nu n^2l^2(2-n^2)^2(æ+1) +l^2(2-n^2)(æ-\nu) \quad (5.8.2)$$

b. For $K=æ,x=y=\Gamma^*=0,\Gamma=\bar{\Gamma}=\frac{P}{2}$ and $f=2\mu g$, under the same condition of the previous example, one obviously has $\in=0$, the two complex functions are

$$\Phi(\zeta)=\frac{cP\zeta^{-1}}{2æ(æ+\nu)(\zeta-n)}[l(æ+\nu)+hæ+nl(n^2-2)\nu\zeta] \quad (5.9)$$

and

$$\Psi(\zeta)=\frac{w(\zeta^{-1})}{w'(\zeta)}\Psi_*(\zeta)+\frac{\zeta}{1-n\zeta}\left[(h+n^{-1}l)\Phi'(n^{-1})+\frac{cP}{2}(h+l\zeta)\right]+ + \frac{1}{2}æP\zeta^{-1} \quad (5.10)$$

The previous results give the solution of the second fundamental problem in the case of bi-axial tension.

c. Under the condition of (iii.1) when the rigid curvilinear kernel is restrained in the original position by a couple which is not sufficient to rotate the kernel then $\in=0$. The complex functions are directly obtained in the form:

$$-æ\Phi(\zeta)=\frac{cP}{2}e^{2i\theta}\zeta^{-1}+c(n-\zeta)^{-1}\left[\frac{P}{4}l\zeta^{-1}+b_o^*\right] \quad (5.11)$$

and

$$\Psi(\zeta)=\frac{cPæ}{4}\zeta^{-1}-\frac{w(\zeta^{-1})}{w'(\zeta)}\Phi_*(\zeta)+\frac{h\zeta}{1-n\zeta}\Phi_*(n^{-1})+$$

$$\frac{l\zeta}{1-n\zeta}(n^{-1}\Phi'(n^{-1}+\frac{cP}{4}\zeta), \quad (5.12)$$

where

$$b_o^*=\frac{P}{4(æ+\nu)}[hæ+nl(n^2-2)\nu+2n(l+nh)\cos2\theta] - \frac{in(nh+l)P}{2(æ-\nu)}\sin2\theta \quad (5.13)$$

The resultant moment is given by

$$M = \frac{c\pi P(æ+1)[(m+n^2)æ+nl(2-n^2)\nu]}{æ(æ-\nu)}\sin 2\theta \quad (5.14)$$

5.4 *When the external force acts on the center of the curvilinear:*

For $\Gamma = \Gamma^* = f = 0$ and $k = æ$, we have the second fundamental problem when the force acts on the curvilinear kernel. It will be assumed that the stresses vanish at infinity and it is easily seen that the kernel does not rotate.

In general, the kernel remains in the original position. The Goursat's functions are

$$\Phi(\zeta) = \frac{1+nh}{2\pi æ(1+æ)(\zeta-n)}\left[\frac{\nu æ(x+iy)}{æ^2-\nu^2} - (1+\frac{\nu^2}{æ^2-\nu^2})(x-iy)\right] \quad (5.15)$$

and

$$\Psi(\zeta) = \frac{h\zeta}{1-n\zeta}\Phi_*(n^{-1} - \frac{w(\zeta^{-1})}{w'(\zeta)}\Phi_*(\zeta) + \frac{l\zeta}{1-n\zeta}\left[n^{-1}\Phi'(n^{-1}) - \frac{x+iy}{2\pi(1+æ)}\right] \quad (5.16)$$

## CONCLUSIONS

From the above results and discussions the following may be concluded:

1) In the theory of two dimensional linear elasticity one of the most useful techniques for the solutions of boundary value problems for a wkwardly shaped region is to transformation the region into one simpler shape.
2) The mapping function (2.7) maps the curvilinear hole C in Z-plane onto the domain outside a unit circle $\xi$-plane under the condition $w(\xi) \neq 0$ or $\infty$ for $|\xi|>|$.
3) The physical interest of the mapping (2.7) comes from its special cases for example $\ell$=n=0, the hole takes an elliptic form. For m=m=0, the hole takes a triangle form etc. More information and applications on technology for the mapping function are found in [1,4].
4) The complex variable method (Cauchy method) is considered as one of the best method for solving the integro-differential equations (1.1) and obtaining the two complex potential functions $\phi(z)$ and $\psi(z)$ directly
5) This paper can be considered as a generalization of the work of the infinite plate with a eurvilineor hole under certain conditions [1-6].

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