A FOURIER SERIES METHOD FOR LINEARIZED SOLUTION OF A FLOW OVER A RAMP

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ABSTRACT

An approximate method is developed for the analytical solution of singular integral equations related to the problem of steady free-surface flow of an ideal fluid obstructed by a ramp in the bottom. Schawrtz-Christoffel transformation is used to map the region of flow,in the complex potential-plane, onto the upper half-plane. The Hilbert transformation as well as the Fourier series method are used as a basis for the approximate solution of the problem for small inclination angle of the ramp. Evaluation of the free-surface shape for different ramp heights and different Froude numbers, for the case of supercritical flow, are plotted. A comparison with the solution of other authors shows good agreement.

1. INTRODUCTION

Problem of steady free-surface of a running two-dimensional, irrotational, inviscid and incompressible flow over irregular bottom has received considerable attention throughout the history of fluid mechanics, due to its wide application in the fields of coastal engineering as waves approaching the seashore or propagation of bores in rivers and hydraulics.

Early work in this area, characterized by the use of a linearized free-surface condition, is reviewed in [10] by Wehausen and Laitone in 1960. More recently in 1981, Forbes[7] investigated the flow over a submerged semi-elliptical body, by allowing the velocity potential and stream function to be independent variables and the solution is obtained by a boundary integral technique. In 1981, Abd-el-Malek [1] considered the nonlinear problem of a flow over a ramp by applying the Hilbert transformation. In 1982, Forbes and Schwartz [8] considered the case of a flow over a semi-circular bump.

Fourier double integral formula has been applied extensively to find the linearized solution of a flow over a crump weir by Boutros, et al [5] in 1986, a flow over a ramp by Abd-el-Malek and Masoud [2] in 1988 and a flow over a trapezoidal obstacle by Faltas, et al [6] in 1989.

Approximate solutions of gravity flow from a uniform channel over various topographies for large Froude number has been applied successfully by Abd-el-Malek and Hanna [3] in 1989 for the case of a ramp, Abd-el-Malek, et al [4] in 1991 for the case of a crump weir.

In this paper, the Hilbert method as well as the Fourier series procedure are used as a basis for the approximate solution of a two-dimensional problem. First, as customary,the physical and complex potential-planes are mapped onto an auxiliary upper half-plane. The nonlinear integral equations are found giving the solution of a mixed boundary-value problem. Lastly the Fourier series procedure is applied, based upon small inclination angle of the ramp, to find an approximate shape for the free surface of the flow.

2. FORMULATION OF THE PROBLEM

An inviscid, incompressible fluid flows over a bottom which consists of a horizontal plane *AB*, inclined plane *BC* with an inclination angle α and length L, and a horizontal plane *CD*, where the bottom extends from - ∞ (point *A*) to ∞ (point *D*), as shown in Fig.1.

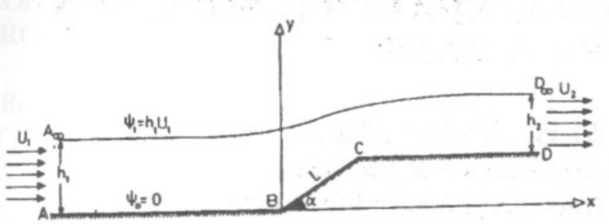

Figure 1. Physical plane of a flow over a ramp of length L and inclination angle α.

The flow is considered to be two-dimensional, steady, irrotational, and subject to the action of gravitational force. Far upstream the fluid is of depth h_1 and has a uniform horizontal velocity U_1. Far downstream the fluid is of depth h_2 and has a uniform horizontal velocity U_2.

For convenience, we choose point *B* to be the origin in the z-plane, the x-axis from left to right and the y-axis upwards.

The complex potential

$$W(z) = \phi(x,y) + i\,\psi(x,y),$$

is an analytic function of z, (z=x+iy), within the flow region, with complex conjugate velocity

$$\frac{dW(z)}{dz} = u(x,y) - i\,v(x,y) = q\,e^{-i\theta}, \qquad (2.1)$$

where (u,v) are the components of the velocity of the fluid, q is the speed of the fluid, and θ is the angle of inclination of the velocity with the horizontal.

Let the dimensionless variables z', q', and W' be

$$z' = \frac{z}{h_1}, \quad q' = \frac{q}{U_1}, \quad W' = \frac{W}{\psi_1}, \qquad (2.2)$$

where $\psi_1 = U_1 h_1$.

In dimensionless form, the Bernoulli condition on the free-surface is

$$q'^2 + \frac{2}{F^2}(y' - 1) = 1, \qquad (2.3)$$

where F is the Froude number defined by

$$F = \frac{U_1}{\sqrt{g\,h_1}}. \qquad (2.4)$$

Denote the dimensionless ramp height by ε, where

$$\varepsilon = \frac{L\,\sin\alpha}{h_1} = L'\sin\alpha, \qquad (2.5)$$

In dimensionless form, (2.1) becomes

$$\xi = \frac{dW'}{dz'} = q'\,e^{-i\theta}. \qquad (2.6)$$

Let

$$\omega = \ln\xi = \ln q' - i\theta, \qquad (2.7)$$

where ω is called the logarithmic hodograph variable.

Then, from (2.6) and (2.7) we get

$$z' = \int e^{-\omega}\, dW'. \qquad (2.8)$$

Using the Schwartz-Christoffel transformation, we map the flow region in the W'-plane onto the upper half of the auxiliary t-plane, so that the following points correspond (see Figures 2 and 3):

$$\left.\begin{array}{ll} B & :W' = 0,\ t = 0, \\ D, D_\infty & :W' \to +\infty,\ t = 1, \\ A, A_\infty & :W' \to -\infty,\ t \to +\infty. \end{array}\right\} \qquad (2.9)$$

The mapping function is

$$W'(t) = -\frac{1}{\pi}\ln(1-t), \qquad 0 \le \arg(1-t) \le \pi. \qquad (2.10)$$

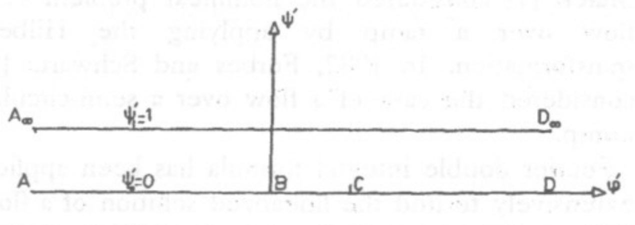

Figure 2.Normalized complex potential (or W'-) plane .

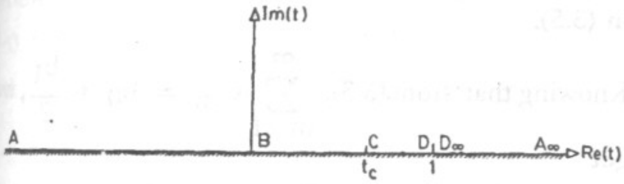

Figure 3. The auxiliary upper half (or t-) plane.

From Abd-el-Malek [1], using the transformation $t = \operatorname{cosec}^2\left(\frac{s}{2}\right)$ after dropping the primes, we have the following system of equations:

(i) Along the free surface $(0 < s < \pi)$

$$q^3(s) = 1 - \lambda \int_0^s \frac{\sin\theta(r)}{\sin r}\, dr \qquad (2.11)$$

$$\theta(s) = \frac{\sin s}{\pi}\int_0^\pi \frac{\ln q(r) - \ln q(s)}{\cos r - \cos s}\, dr + \frac{2\alpha}{\pi}\tan^{-1}\left[\frac{\sin s}{\mu + \cos s}\right] \qquad (2.12)$$

$$y(s) = 1 + \frac{2}{\pi}\int_0^s \frac{\sin\theta(r)}{\sin r\, q(r)}\, dr \qquad (2.13)$$

$$x(s) = -\frac{2}{\pi}\ln\left(\cos\frac{s}{2}\right) + \frac{1}{\pi}\int_0^s \frac{\cos\theta(r)}{\sin r\, q(r)}\, dr \qquad (2.14)$$

(ii) Along the boundary of the ramp

$$\ln q_b(t) = \frac{2\sqrt{1-t}}{\pi}\int_0^\pi \frac{\ln q(r)}{2 - t(1-\cos r)}\, dr + \frac{2\alpha}{\pi}\tanh^{-1}\left(\frac{k\sqrt{1-t}}{t-k}\right), \quad 0 < t < t_c \qquad (2.15)$$

$$L = \frac{1}{\pi}\int_0^{t_c} \frac{1}{(1-r)\, q_b(r)}\, dr \qquad (2.16)$$

where

$$\lambda = \frac{6}{\pi F^2},\ k = 1 - \sqrt{1 - t_c},\ \mu = \frac{2}{k} - 1 \qquad (2.17)$$

3. SOLUTION OF THE PROBLEM

For small inclination angle α, the change in $\theta(s)$ will be very small, then we can approximate $\sin\theta(s)$ by $\theta(s)$ and $\cos\theta(s)$ by one.

Now express $\theta(s)$ in Fourier sine series

$$\theta(s) = \sum_{m=1}^{\infty} C_m \sin ms, \qquad (3.1)$$

then

$$\frac{\sin\theta(s)}{\sin s} = \sum_{m=1}^{\infty} C_m \frac{\sin ms}{\sin s} = \sum_{m=0}^{\infty} b_m \cos ms \qquad (3.2)$$

where

$$\left.\begin{aligned} C_1 &= 0.5\,(2b_0 - b_2) \\ C_m &= 0.5\,(b_{m-1} - b_{m+1}),\quad m = 2,3,4,\ldots \end{aligned}\right\} \qquad (3.3)$$

Upon employing the expression (3.2) into the equation (2.11), we get

$$q^3(s) = 1 - \lambda b_0 s - \lambda \sum_{m=1}^{\infty} \frac{b_m}{m}\sin ms \qquad (3.4)$$

from the condition at far downstream, where $s = \pi$, $q = q_2$ and from which it follows that:

$$b_0 = \frac{1 - q_2^{\,3}}{\lambda\pi} \qquad (3.5)$$

For q_2 very close to one and $\lambda < 1$, we can approximate (3.4) by

$$\ln q(s) \simeq -\frac{\lambda}{3}\left[b_0 s + \sum_{m=1}^{\infty} \frac{Zb_m}{m}\sin ms\right] \qquad (3.6)$$

Now, upon substituting (3.1) and (3.6) in (2.12), taking into consideration:

$$\text{(i)}\ \tan^{-1}\left(\frac{\sin s}{\mu + \cos s}\right) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m\,\mu^m}\sin ms,$$

$$\text{(ii) at } s=0:\ \sum_{m=1}^{\infty} \frac{b_m}{m} = 0,$$

carrying out the integrations, equating the corresponding coefficients of sin(ms) ; m = 1,2,3,..., and using Tolstov [9],we get

$$C_m = \begin{cases} \dfrac{4\lambda b_0}{3\pi m^2} - \dfrac{2\alpha}{m\pi\mu^m} ; & m=1,3,5,... \\ \dfrac{2\alpha}{m\pi\mu^m} ; & m=2,4,6,... \end{cases} \quad (3.7)$$

Hence we have

$$\theta(s) = \frac{4\lambda b_0}{3\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)s}{(2n-1)^2} + \frac{2\alpha}{\pi}\tan^{-1}\left[\frac{\sin s}{\mu+\cos s}\right] \quad (3.8)$$

where λ and μ are defined in (2.17), and b_0 is given in (3.5).

Knowing that, from (3.3), $\sum_{m=1}^{\infty} C_m \simeq b_0 + \frac{b_1}{2}$, we get

$$b_1 = 2\left(\frac{1}{F^2} - 1\right) b_0 - \frac{4\alpha}{\pi}\ln\left(1+\frac{1}{\mu}\right) \quad (3.9)$$

Hence we have

$$b_m = \begin{cases} 2b_0\left(1 - \dfrac{4\lambda}{3\pi}\displaystyle\sum_{n=1}^{\frac{m}{2}}\frac{1}{(2n-1)^2}\right) + \dfrac{4\alpha}{\pi}\displaystyle\sum_{n=1}^{\frac{m}{2}}\frac{1}{(2n-1)\mu^{2n-1}} ; & m=2,4,... \\ b_1 - \dfrac{2\alpha}{\pi}\displaystyle\sum_{n=1}^{\frac{m-1}{2}}\frac{1}{\mu^{2n}} ; & m=3,5,... \end{cases} \quad (3.10)$$

Therefore, from (2.13), (2.14), (3.6) and (3.8), the shape of the free surface can be found from, $(0 < s < \pi)$:

$$\eta(s) = \frac{2}{\pi}\int_0^s \frac{1}{\sin r}\left[\frac{4\lambda b_0}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)r}{(2n-1)^2} + \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\mu^n}\sin nr\right] \quad (3.11)$$

$$\times\left[1 + \frac{\lambda r b_0}{3} + \frac{\lambda}{3}\sum_{n=1}^{\infty} b_n\frac{\sin nr}{n}\right] dr$$

$$x(s) = -\frac{2}{\pi}\ln\left(\cos\frac{s}{2}\right) + \frac{1}{\pi}\int_0^s\frac{1}{\sin r}\left[1 + \frac{\lambda r b_0}{3} + \frac{\lambda}{3}\sum_{n=1}^{\infty} b_n\frac{\sin nr}{r}\right] dr \quad (3.12)$$

where
$\eta(s) = y(s) - 1$, (3.13)
and b_n , $n = 0,1,2,...$ are given by (3.10).

4 NUMERICAL RESULTS AND DISCUSSION

Upon substituting (3.10) in (3.11), (3.12) and (3.13), for $n = 4$, and then carrying out the involved integrations,we get the following expressions for $\eta(s)$ and $x(s)$:

$$\eta(s) = (2/\pi)(L_0 + L_1 s + L_2 s^2 + L_3\sin(s) + L_4\cos(s) + L_5 s\sin(s) + L_6\sin(2s) + L_7\cos(2s) + L_8 s\sin(2s) + L_9\sin(3s) + L_{10}\cos(3s) + L_{11} s\sin(3s) + L_{12}\cos(4s) + ...) \quad (4.1)$$

where

$$L_0 = L_4 - (L_7 + L_{10} + L_{12}),$$

$$L_1 = A + C,$$

$$L_2 = \frac{\lambda b_0}{6} L_1,$$

$$L_3 = \frac{2}{3}(3 + \lambda b_0) B + 2D,$$

$$L_4 = -\frac{\lambda}{3}[b_1 A + \frac{b_2}{2} B + \frac{b_3}{3} C + (\frac{b_4}{4} - 2b_0) D],$$

$$L_5 = \frac{2\lambda}{3} b_0 D,$$

$$L_6 = C,$$

$$L_7 = -\frac{\lambda}{72}[6b_2 A + 8(3b_1 + b_3) B - 3(4b_0 - 2b_2 - b_4) C + 4b_3 D],$$

$$L_8 = \frac{\lambda}{3} b_0 C,$$

$$L_9 = \frac{2}{3} D, \tag{4.2}$$

$$L_{10} = -\frac{\lambda}{108}[4b_3 A + 3(2b_2 + b_4) B + 4(3b_1 + b_3) C - (8b_0 - 6b_2 - 3b_4) D]$$

$$L_{11} = \frac{2\lambda}{9} b_0 D,$$

$$L_{12} = -\frac{\lambda}{144}[3b_4 A + 4b_3 B + 3(2b_2 + b_4) C + 4(3b_1 + b_3) D]$$

where

$$A = \frac{4\lambda b_0}{3\pi} - 2\mu B,\ B = \frac{\alpha}{\pi \mu^2},\ C = \frac{4\lambda b_0}{27\pi} - \frac{2}{3} B,$$

$$D = \frac{B}{2\mu^2} \tag{4.3}$$

Also,

$$x(s) = -\frac{2}{\pi} \ln(\cos\frac{s}{2}) + \sum_{n=0}^{4} \frac{H_n}{n} \sin ns, \tag{4.4}$$

where

$$H_0 = \frac{\lambda}{9}(3b_1 + b_3) + \frac{23}{15} H_4,$$

$$H_1 = -\frac{\lambda}{6}(3b_0 - 2b_2 - b_4),$$

$$H_2 = \frac{22}{15} H_4 + \frac{2\lambda b_3}{9}, H_3 = \frac{\lambda}{6}(b_4 - b_0),$$

$$H_4 = (\frac{\lambda b_0}{3} + \frac{4}{\pi}) \tag{4.5}$$

where $b_0, b_1, b_2, ...$ are given by (3.5), (3.9) and (3.10) respectively.

It is clear from (2.17) that since $0 \leq t_c \leq 1$, then $0 \leq k \leq 1$ and consequently $\mu \geq 1$. For the validity of the approximation, $\lambda < 1$, which corresponds to $F^2 > 1.9$. Consequently the main effective constants in (4.3) are A and C, while the rest decrease most rapidly with the increase of μ, which corresponds to the decrease of ramp length. Also the constants A and C decrease monotonically with the increase of the Froude number F^2.

Therefore in the expression of $\eta(s)$ in (4.1), the main effective coefficients are L_0, L_1, L_2, and in the expression of x(s) in (4.4), the variation in H_n with the change of α and F^2 is very small and therefore the main contribution in x(s) is due to the first term.

Table (1) shows the values of the coefficients L_n and H_n ; n = 1,2,3,...for the case of $\alpha = \pi/6$ and $F^2 = 20$.

Results showing the free-surface elevation over a range of Froude number F^2, 2 to 20 for fixed $\varepsilon = 0.018$ and over a range of ramp height ε, 0.018 to 0.03 for fixed $F^2 = 10$ are given in Figs. 4 and 5.

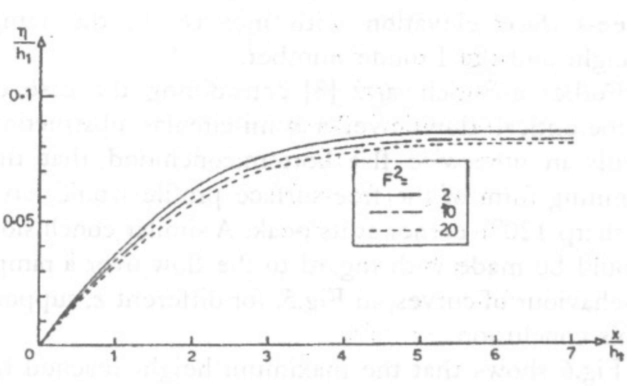

Figure 4. Effect of the Froude number on the free-surface profile for ε=0.018.

Table 1. Values of L_n and H_n for $\alpha = \pi/6$ and $F^2 = 20$

n	L_n	H_n
0	-1.68253×10^{-3}	1.92351
1	4.02088×10^{-2}	3.95502×10^{-2}
2	5.78082×10^{-4}	1.87315
3	3.89708×10^{-5}	1.30971×10^{-2}
4	2.04915×10^{-3}	5.20797×10^{-1}
5	3.52396×10^{-9}	------------
6	4.06684×10^{-3}	------------
7	-5.24070×10^{-4}	------------
8	1.16938×10^{-4}	------------
9	4.08518×10^{-8}	------------
10	3.18052×10^{-4}	------------
11	1.17465×10^{-9}	------------
12	-1.66065×10^{-4}	------------

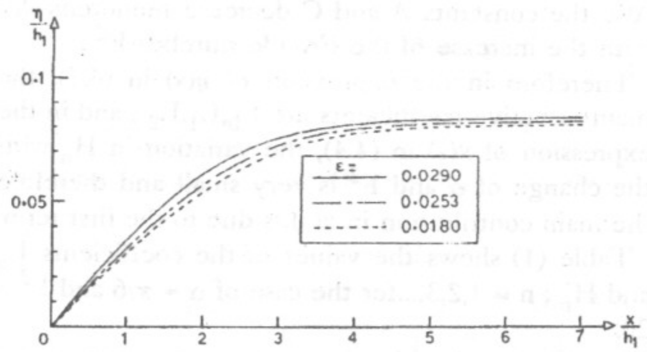

Figure 5. Effect of the ramp height ε on the free surface profile for $F^2 = 10.0$.

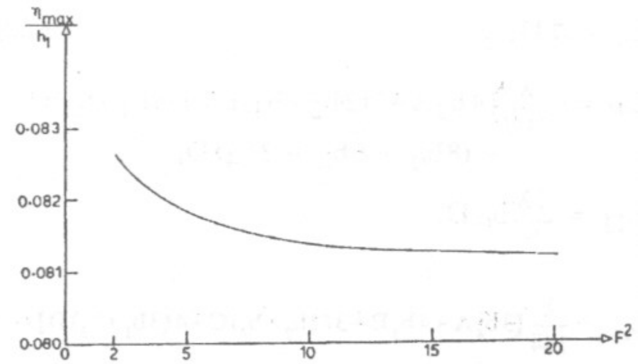

Figure 6. Normalized maximum free-surface height against the Froude number F^2 for $\varepsilon = 0.018$.

These results demonstrate the monotonic rise in free-surface elevation with increase of the ramp height and the Froude number.

Forbes and Schwartz [8] considering the case of supercritical flows over a semi-circular obstruction with an otherwise flat bottom,concluded that the limiting form of the free-surface profile would have a sharp 120^o - corner at its peak. A similar conclusion could be made with regard to the flow over a ramp. Behaviour of curves, in Fig.5, for different ε, support this conclusion.

Fig.6 shows that the maximum height reached by the free surface increases as the Froude number decreases while in Fig.7, the maximum height of the free-surface elevation increases with the increase of the ramp height.

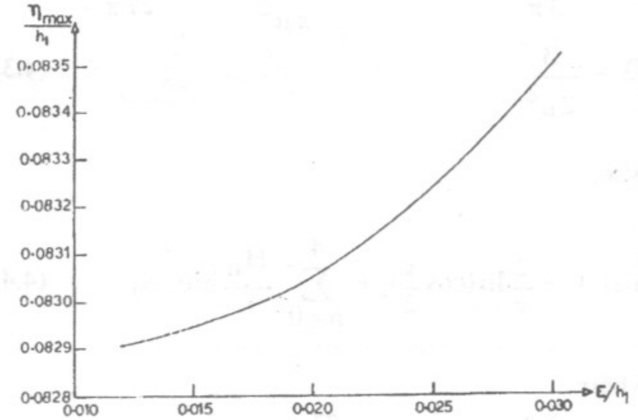

Figure 7. Normalized maximum free-surface height against the ramp height ε for $F^2 = 2.0$.

For the two cases of subcritical ($F^2 < 1$) and critical ($F^2 = 1$) flows, the solution is not valid due to the nature of the approximation used.

The results obtained by Abd-el-Malek and Masoud [2] using Fourier's double -integral theorem show a reasonable qualitative agreement in the form of the free-surface predicted above the ramp, as shown in Fig.8, with a difference of order 10^{-3}.

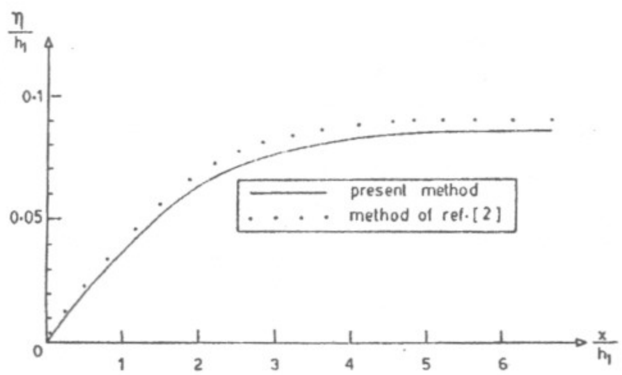

Figure 8. Comparison of current results with those of Abd-el-Malek and Masoud [2] for supercritical flow over a ramp obstruction, for $\varepsilon = 0.02$ and $F^2 = 2.0$.

5. CONCLUSIONS

The steady free-surface of a fluid over a ramp with gravity acting as the restoring force has been studied using the Schwartz - Christoffel transformation, Hilbert transformation and the Fourier series method. A linearized solution for such flows is developed for the case of the supercritical flow. Some computed examples of such flows are shown for a variety of bottom heights and Froude number. The obtained solution provides a reasonably accurate indication of the type of free surface formed.

REFERENCES

[1] M.B. Abd-el-Malek, "Boundary integral methods and free surface problems", *Ph.D. dissertation*, University of Windsor, Windsor, Ontario, Canada, 1981.

[2] M.B. Abd-el-Malek and S.Z. Masoud, "Linearized solution of a flow over a ramp", *Appl. Math. Modelling*, vol. 12, pp. 406-410, 1988.

[3] M.B. Abd-el-Malek and S.N. Hanna, "Approximate solution for a flow over a ramp for large Froude number", *J. Comput. Appl. Math.*, vol. 28, pp. 105-117, 1989.

[4] M.B. Abd-el-Malek, S.N. Hanna and M.T. Kamel, "Approximate solutions of gravity flow from a uniform channel over triangular bottom for large Froude number", *Appl. Math. Modelling*, 15, Vol.1, pp. 25-32, 1991.

[5] Y.Z. Boutros, M.B. Abd-el-Malek,M.B. and S.Z. Masoud, "Linearized solution of a flow over a nonuniform bottom", *J. Comput. Appl. Math.*, vol. 16, pp. 105-116, 1986.

[6] M.S. Faltas, S.N. Hanna and M.B. Abd-el-Malek, "Linearized solution of a free surface flow over a trapezoidal obstacle", *Acta Mech.*, vol. 78, no. 3-4, pp. 219-233, 1989.

[7] L.K. Forbes, "On the wave resistance of submerged semi-elliptical body", *J. Engrg. Math.*, vol. 15, pp. 287-198, 1981.

[8] L.K. Forbes and L.M. Schwartz, "Free-surface flow over a semi-circular obstruction", *Fluid Mech.*, vol. 114, pp. 299-314, 1982.

[9] G.P. Tolstov, *Fourier series*, Dover Pub., Inc., New York, 1962.

[10] J.V. Wehausen and E.V. Laitone, *Surface waves, Handbuch der Physik*, Vol.9, Springer, Berlin, 1960.