# SIMPLE ITERATIVE METHODS FOR ROBUST POLE ASSIGNMENT

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## ABSTRACT

This paper presents a simple, inexpensive, iterative method for eigenstructure assignment. The aim is to get a robust solution in the sense that the sensitivities of the closed-loop eigenvalues, with respect to variations in the system matrices, are minimized. The algorithm is illustrated on four small size examples and compared to the existing methods. A matrix inversion method, specially suited for most of the occurring cases, is included.

## 1. INTRODUCTION

The problem of eigenstructure assignment by state feedback has been the object of various researches and has benefitted from different numerical techniques. The two major trends in the last decade and till now are: direct and iterative algorithms.

In the single input case, where the gain is unique and the poles sensitivities can not be controlled, a numerically stable direct method is advisable since it gives results which are as accurate and robust as the data permit. One of the most reliable ones is [14] and some inexpensive ones are [1] and [6].

In the multiinput case, where there are several degrees of freedom in the solution, even though a direct algorithm is numerically stable and guaranteed to compute an accurate gain, it is highly probable that the arbitrarily selected solution will not, in general, be robust and then, even small perturbations in the data or errors in the gain may give large errors in the prescribed poles (as shown in [11]).

Iterative methods, by contrast, aim to select from the possible set of solutions, a feedback which minimizes some measure of the sensitivity of the closed-loop poles. The most famous methods in this group are the ones in [9]. (More about direct versus iterative algorithms can be found in [12], [15] and [16]).

In the case of large scale systems, the standard technique is to reduce them to block Hessenberg form by means of orthogonal state coordinate transformations. The next step is either to decompose them into small size pole assignment problems as in [4], [5] and [18], or to project them into subspaces to obtain a reduced order problem as in [10] and [13].

In this paper we present a reliable, inexpensive method with different robustness measures. One of them is specially implemented for third order systems, and the others can be applied to systems of higher orders.

## 2. THE POLE ASSIGNMENT PROBLEM

Consider the linear, time invariant multiinput system with dynamic state equation

$$x(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$

where the state $x(t)$ and the input $u(t)$ are $n$ and $m$ dimensional vectors respectively, $n > m$, $\mathbf{A}$ and $\mathbf{B}$ are real constant matrices of compatible orders.

The poles of this system are the eigenvalues of $\mathbf{A}$. The pole assignment problem consists of choosing a state feedback control

$$u(t) = \mathbf{K}\,x(t) + v(t)$$

(i.e. choosing $\mathbf{K} \in \mathbf{R}^{m \times n}$) so that the closed-loop system
$x(t) = \mathbf{A}_{cl}\,x(t) + \mathbf{B}v(t), \quad \mathbf{A}_{cl} \equiv \mathbf{A} + \mathbf{BK}$

has a specified set of poles $\mathscr{L} = \{\lambda_i\}$.

When the pair $(\mathbf{A},\mathbf{B})$ is controllable, a matrix $\mathbf{K}$ exists for every choice of $\mathscr{L}$. Otherwise, all the uncontrollable modes of $(\mathbf{A},\mathbf{B})$ should belong to any choice of $\mathscr{L}$ [23]. Also, it is obvious that, since the system matrices are real, complex eigenvalues must appear in conjugate pairs in order to find a real $\mathbf{K}$. Multiplicities are allowed with condition that the closed-loop matrix is non defective (i.e. semi simple); we have

$$\mathbf{A}_{cl}\, \mathbf{X} = \mathbf{X}\,\Lambda \qquad (2.1)$$

where $\mathbf{X}$ is the matrix of right eigenvectors, $\Lambda$ is a diagonal matrix containing the required eigenvalues.

In the single input case ($m=1$) there is at most one solution, but if $1 < m < n$, multiple solutions may exist and extra conditions may be imposed to give $\mathbf{A}_{cl}$ other desirable properties.

## 3. ROBUSTNESS

Typically, the data, $\mathbf{A}$ and $\mathbf{B}$ are not known exactly and are subject to perturbations. Hence it is desirable that the chosen poles be as insensitive to changes in $\mathbf{A}_{cl}$ as possible. In [9] and [12], the conditioning of the closed-loop poles is taken as a measure of robustness, as follows: Let $\mathbf{x}_i$ and $\mathbf{y}_i$, $i = 1,2,\ldots m$, be the right and left eigenvectors of the closed-loop system matrix, corresponding to the eigenvalue $\lambda_i \in \mathscr{L}$. Then the magnitude of the condition number $c_i$,

$$c_i = \frac{\|\,\mathbf{y}_i\,\|_2 \, \|\,\mathbf{x}_i\,\|_2}{|\,\mathbf{y}_i^T \mathbf{x}_i\,|} \geq 1, \qquad (3.1)$$

should be kept as low as possible to ensure robustness. When $\mathbf{Y}^T$ is taken equal to $\mathbf{X}^{-1}$, $c_i = \|\mathbf{y}_i\|\, \|\mathbf{x}_i\|$. If a perturbation $O(\epsilon)$ is made in the coefficients of $\mathbf{A}_{cl}$, then the corresponding first order perturbation in the eigenvalue $\lambda_i$, is of $O(\epsilon\, n\, c_i)$, [9].

An upper bound on the sensitivities of the eigenvalues is given in [22]

$$\max_i c_i \leq \gamma_2(\mathbf{X}) \equiv \|\,\mathbf{X}\,\|_2 \, \|\,\mathbf{X}^{-1}\,\|_2 \equiv \frac{\sigma_1(\mathbf{X})}{\sigma_n(\mathbf{X})} \qquad (3.2)$$

where $\gamma(\mathbf{X})$ is the condition number of the matrix $\mathbf{X}$, $\sigma_1(\mathbf{X}) \geq \ldots \geq \sigma_n(\mathbf{X})$ are the singular values of $\mathbf{X}$.

So a necessary condition for a robust solution is that the condition numbers be minimized. They attain their minimum value $c_i = 1$ for all i, if and only if the closed-loop matrix is normal. That is to say, in the ideal case it should satisfy

$$(\mathbf{A}_{c1})^T\, (\mathbf{A}_{c1}) = (\mathbf{A}_{c1})\, (\mathbf{A}_{c1})^T \qquad (3.3)$$

from which we can deduce a bound on the "measure of normality" defined here as the norm of the difference between the l.h.s. and r.h.s. of (3.3), i.e.,

$$\|\mathbf{A}_{c1}^T\mathbf{A}_{c1} - \mathbf{A}_{c1}\mathbf{A}_{c1}^T\| \leq \|\mathbf{A}_{c1}^T\mathbf{A}_{c1}\| + \|\mathbf{A}_{c1}\mathbf{A}_{c1}^T\|$$

$$\leq 2\, \|\mathbf{A}_{c1}^T\|\, \|\mathbf{A}_{c1}\|$$

From (2.1), it is easy to see that

$$\|\mathbf{A}_{c1}\| \leq \|\mathbf{X}\|\, \|\Lambda\|\, \|\mathbf{X}^{-1}\| = \gamma(\mathbf{X})\|\Lambda\|,$$

and since for Frobenius or $\ell_2$ norms $\|\mathbf{M}^T\| = \|\mathbf{M}\|$, then an upper bound on the " relative measure of normality" is

$$\|\mathbf{A}_{c1}^T\, \mathbf{A}_{c1} - \mathbf{A}_{c1}\, \mathbf{A}_{c1}^T\| \,/\, \|\Lambda\|^2 \leq 2\,\gamma^2(\mathbf{X}).$$

For $\ell_1$ and $\ell_\infty$ norms, we have $\|\mathbf{M}^T\|_1 = \|\mathbf{M}\|_\infty$, then

$$\|\mathbf{A}_{c1}^T\, \mathbf{A}_{c1} - \mathbf{A}_{c1}\, \mathbf{A}_{c1}^T\| \,/\, |\lambda_{max}|^2 \leq 2\,\gamma_1(\mathbf{X})\,\gamma_\infty(\mathbf{X})$$

which means that, for a specified set of eigenvalues, any increase in $\gamma$ worsens greatly the measure of normality. This same conclusion is also drawn from the definition of the "departure from normality" [8]

$$\|\mathbf{N}\|_F^2 = \|\mathbf{A}_{cl}\|_F^2 - \sum_{i=1}^{n} |\lambda_i|^2 \equiv \Delta^2\,(\mathbf{A}_{cl})$$

$$\leq (\gamma_F^2(\mathbf{X}) - 1)\sum_{i=1}^{n} |\lambda_i|^2$$

i.e. to minimize $\Delta(\mathbf{A}_{cl})$, we should minimize $\gamma_F(\mathbf{X})$

According to [21], the $\ell_2$ condition number of a

real non-singular matrix, $\gamma_2(X)$ defined in (3.2) can be minimized by choosing an appropriate scaling matrix. For a matrix $M$ and a scaling matrix $S$, let $M_o$ denote the optimally scaled matrix $M_o = M S_o$, and $M_1$ denotes the matrix $M S_1$ with $S_1$ chosen such that the diagonal elements of $M_1^T M_1$ are equal to one.

Then $M_1$ is an "approximate" optimally scaled matrix, and the ratio

$$\frac{\gamma_2(M_1)}{\gamma_2(M_o)} \ll \sqrt{n}.$$

That is why in this paper, we impose the column normalization condition

$$\| x_i \|_2 = 1.$$

for each eigenvector such that the matrix $(X^T X)$ has ones as diagonal elements.

In the perfectly conditioned case, $X^T X$ is a unit matrix. The minimization of $\gamma_2(X)$ is now equivalent to the minimization of the residue matrix $R = X^T X - I$. This matrix is symmetric and has the very simple form

$$R = \begin{bmatrix} 0 & \cos\theta_{12} & \cdots \\ \cos\theta_{12} & 0 & \\ \cdot & & \\ \cdot & & \\ \cdot\cdot & & 0 \end{bmatrix} \quad (3.4)$$

where $\theta_{ij}$ is the angle between $x_i$ and $x_j$.
Another advantage is that the condition numbers defined in (3.1), become directly

$$c_i = \| y_i \|_2 \quad \text{and then} \quad \| c \|_2 = \| Y \|_F$$

The problem now can be stated as

$$\text{minimize } \| R \|$$

subject to

$$(A + BK) X = X\Lambda \quad (3.5)$$

Here there are $nm$ variables over which to optimize. This is a large number even for modest values of $n$ and

$m$ and the constraint is highly non-linear. methods have proved to be effective. The minimization of $\| R \|_2$ gives always the best results with respect to $\gamma$ (X) and also the $\nu$ measure $= \max_i | c_i |$. This makes it the best choice for rapid results in all general cases.

## 4. PARAMETERIZATION OF THE PROBLEM

Instead of solving (3.5), we parameterize the problem as in [9].

Step 1. Get the singular value decomposition (SVD) of B

$$B = [U_o \mid U_1] \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \\ & 0 & \sigma_m \end{bmatrix} V^T \quad (4.1)$$

where the dimensions are respectively. $U = [U_0 \mid U_1]$ is the orthogonal matrix containing the normalized eigenvectors of $BB^T$, $V$ is the orthogonal matrix containing the normalized eigenvectors of $B^TB$, and $\sigma_i$, $i = 1, 2, \ldots, m$ are the $m$ singular values of $B$. $B$ is assumed of full column rank $(m)$. This condition is always satisfied since, if rank $(B) < m$, it means that the inputs are not independent and thus can be reduced in number.
The above factorization (4-1) can be written

$$B = [U_0 \mid U_1] \begin{bmatrix} Z \\ 0 \end{bmatrix}$$

where $Z$ is a non singular matrix, and may also be obtained by $QR$ decomposition of $B$.

Step 2. Premultiplying (2.1) by $U_1^T$ gives.

$$U_1^T (A X - X \Lambda) = 0 \quad (4.2)$$

Then a matrix $K$ satisfying (2-1) exists if and only if $X$ satisfies (4-2), i.e. each eigenvector $x_i$ of $A_{cl}$, must belong to the space

$$S_i = \text{Ker}\{U_1^T (A - \lambda_i I)\}$$

where Ker$\{M\}$ denotes the null space of $M$. This

matrix is of order $(n-m) \times n$.

Let $S_i$ be an $n \times m$ matrix with columns spanning the space $S_i$, let $w_i$ be an $m \times 1$ vector containing the coordinates of the eigenvector $x_i$ with respect to this basis, such that

$$x_i = S_i \, w_i$$

and the eigenvector matrix $X$ is now expressible as a function of the $w$'s

$$X = S \, W = [S_1 \mid S_2 \mid \dots S_n] \; \text{diag} \, (w_1, w_2, \dots w_n)$$

where $S$ is a matrix of order $(nxnm)$ and diag $(w_1, w_2, \dots w_n)$ is an $(nmxn)$ block diagonal matrix with $mx1$ blocks.

In the case of controllability, dim $(S_i) = m$. Then the number of $w$'s is $nm$. This number may be reduced to $n(m-1)$ by expressing $(m-1)$ of the coefficients in each $w_i$, in terms of the remaining one.

The computational problem then becomes the unconstrained minimization of the specified measure of robustness, as a function of the $w$'s.

Step 3. The feedback matrix can be easily obtained by premultiplying (2.1) by $U_o^T$ to get

$$ZK = U_o^T (X \Lambda X^{-1} - A)$$

If $Z$ was obtained by QR decomposition, $K$ will be obtained by back substitution, or by isolating $K$ if SVD was used

$$K = Z^{-1} U_o^T (X \Lambda X^{-1} - A)$$

$Z^{-1}$ in this case is easily obtained from

$$Z^{-1} = V \begin{bmatrix} 1/\sigma_1 & & 0 \\ & & \\ 0 & & 1/\sigma_m \end{bmatrix}$$

As for the inverse of $X$, since the $x_i$'s must be maximally orthogonal to each other in any acceptable solution, and also normalized, then the process is satisfactorily stable. For a commonly occurring case (see section 5) of an eigenvector orthogonal to all the

others, a very simple algorithm is presented in section 6. The minimization technique used in step 2 is a modified version of the "simplex" optimization technique for non-linear, unconstrained objectives [17]. Its main step is a comparison of function values at the corners of the simplex. Here we change the size of the simplex (delta) to ensure that the obtained minimum is not a local one.

### 4.1. *Inverse of a Quasi-Orthogonal Matrix*

We have noticed from the results in section 5, that a commonly occurring case is the case where one of the eigenvectors is orthogonal to the other two. Since the order in which the eigenvectors are cited in the matrix is immaterial as long as it is in accordance with the order of the $\lambda$'s in $\Lambda$, we will describe here the case where $x_1$ is orthogonal to both $x_2$ and $x_3$

Let $$X = [x_1 \mid x_2 \mid x_3 ]$$

with the inner (or scalar) products

$$x_1^T x_2 = x_1^T x_3 = 0.$$

The $x_i$'s are normalized such that $\| x_i \| = 1 \; \forall i$.

i.e. $\cos \theta_{23} = x_2^T x_3$, and it is required to

find $Y = [y_1 \mid y_2 \mid y_3]$

such that $Y^T$ is the inverse of $X$: $XY^T = I$. Premultiplying by $X^T$ and substituting, we get

$$X^T X \, Y^T = X^T$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \cos\theta_{23} \\ 0 & \cos\theta_{23} & 1 \end{bmatrix} \begin{bmatrix} y_1^T \\ \overline{y_2^T} \\ \overline{y_3^T} \end{bmatrix} = \begin{bmatrix} x_1^T \\ x_2^T \\ x_3^T \end{bmatrix}$$

Then it is easy to see that $y_1^T = x_1^T$ i.e. the vector orthogonal to the other two is unchanged, and

$$\begin{bmatrix} 1 & \cos\theta_{23} \\ \cos\theta_{23} & 1 \end{bmatrix} \begin{bmatrix} y_2^T \\ y_3^T \end{bmatrix} = \begin{bmatrix} x_2^T \\ x_3^T \end{bmatrix}$$

from which

$$\begin{bmatrix} y_2^T \\ y_3^T \end{bmatrix} = \frac{1}{\sin^2\theta_{23}} \begin{bmatrix} 1 & -\cos\theta_{23} \\ -\cos\theta_{23} & 1 \end{bmatrix} \begin{bmatrix} x_2^T \\ x_3^T \end{bmatrix}$$

Thus the inverse is reduced to a very stable process. Even in the neighborhood of singularity, when $\theta_{23}$ is very small, the only condition is that $\sin^2\theta_{23}$ and $1/\sin^2\theta_{23}$, be within the machine lower bound and upper bound respectively.

## 5. MEASURES OF ROBUSTNESS

The measures of robustness considered here are
1. The $\ell_1$ norm of $\mathbf{R}$ (which is equal to the $\ell_\infty$ norm since $\mathbf{R}$ is symmetric)

$$\|\mathbf{R}\|_1 = \max_j \sum_{i=1}^n |r_{ij}|. \qquad (5\text{-}1)$$

Minimizing $\|\mathbf{R}\|_1$ is equivalent to mininizing

$$f_o = \sum_{i=1}^{n-1} \sum_{j=i+1}^n |r_{ij}|.$$

From (5-1)

$$\|\mathbf{R}\|_1 \leq n \max_{i,j} |r_{ij}|$$

and since all the diagonal elements are zeroes and the $r_{ij}$ are cosine functions, we get

$$\|\mathbf{R}\|_1 \leq n\text{-}1.$$

$f_o$ represents the sum of the magnitudes of the cosines of all angles between the eigenvectors, i.e. $f_o$ is bounded by their number:

$$f_o \leq n(n\text{-}1)/2$$

2. The Frobenius norm of $\mathbf{R}$

$$\|\mathbf{R}\|_F = (\sum_{i=1}^n \sum_{j=1}^n (r_{ij})^2)^{1/2} = (2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n (r_{ij})^2)^{1/2}$$

which is equivalent to minimizing

$$f = \sum_{i=1}^{n-1} \sum_{j=1+1}^n (r_{ij})^2$$

the upper bounds are:

$$\|\mathbf{R}\|_F \leq \sqrt{n(n\text{-}1)} \text{ and } f \leq n(n\text{-}1)/2.$$

3. The spectral norm of $\mathbf{R}$, since $\mathbf{R}$ is symmetric, is

$$\|\mathbf{R}\|_2 = \max_i |\lambda_i(\mathbf{R})|$$

In the case $n = 3$, this quantity is easy to get by solving the characteristic polynomial of R which readily has the form of the reduced cubic equation

$$\lambda^3 + p\lambda + q = 0$$

where $p = -f$

$$q = -2 \prod_{\forall i,j} \cos\theta_{ij} \equiv -2f_2,$$

for which the three real roots are explicitly

$$t_s = \frac{2}{\sqrt{3}} \sqrt{f} \cos(\frac{\beta + 2\pi(s-1)}{3}), \quad s = 1, 2, 3$$

where $\beta = \cos^{-1}(-\sqrt{27f_2}/\sqrt{(f)^3})$ [2].

From matrix norm relations [8], and since $\mathbf{R}$ is symmetric,

$$\|\mathbf{R}\|_2 \leq \sqrt{\|\mathbf{R}\|_1 \|\mathbf{R}\|_\infty} = \|\mathbf{R}\|_1$$

$$\|\mathbf{R}\|_2 \leq (n\text{-}1)$$

which is a tighter bound than the standard

$$\|R\|_2 < \sqrt{n} \, \|R\|_1 \quad \text{and} \quad \|R\|_2 \leq \|R\|_F.$$

## 6. NUMERICAL EXAMPLES AND DISCUSSION

For each example, different initial guess points were tried and gave the same minimal point. The difference was in the number of changes of delta required. This maximum number was 4.

The first three columns of the tables show the results of the minimization of $f_o$, $\|R\|_2$ and $f$ respectively

**Example 1.** $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

The required closed-loop eigenvalues are: $\mathscr{L} = \{-0.2, -0.2, -10\}$. The results are shown in Table I. The fourth and fifth columns contain the best results of [19] and [20] respectively. The three last columns contain the results of methods 0,1,2/3 of Kautsky et al. 1983, 1984, as reported to us by [19]. (The eigenvector and feedback matrices are not available).

Their measures of robustness are

$$\nu_1 = \|c\|_\infty = \max_i |c_i|$$

$$\nu_2 = \gamma_2(X)$$

and $\nu_3 = \gamma_F(X)$

($\|c\|_2$ equals $\|Y\|_F$ so long as $X$ is normalized. In [20] the obtained $X$ is not normalized and hence; $\|c\|_2 = 2.616$ not 2.4).

We notice that all methods give acceptable good results for this example. Method 0 of [9] was criticized by [3] as being heuristic and typically it does not converge. Method 1 alters $X$ one column at a time performing inexpensive local optimization at each step but is generally slow. Method 2/3 does not attempt to minimize $\gamma(X)$ as claimed, but instead it optimizes a related heuristic measure of condition. Also it does not perform well on ill-conditioned problems. The method

in [19] is slow and difficult to implement and that in [20] requires gradients and thus is expensive.

**Example 2.** Same $A$ matrix as example 1, but

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The required closed-loop eigenvalues are $\mathscr{L} = \{-1, -2, -3\}$. The results are shown in Table II.

**Example 3.** This example is given in [3] after Atkinson (1985).

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The required closed-loop eigenvalues are: $\mathscr{L} = \{-1, -2, -3\}$. The results are shown in Table III.

The fourth and the fifth columns expose the results of [3] after full convergence and half convergence respectively. The measure of sensitivity is the Frobenius norm, but the minimization needs explicit formulae for the gradient and Hessian of the objective function. The numerical stability of the straight forward procedure for derivatives, is in doubt. The more circuitous method via the SVD yields easy to compute factors, but the cost is considerable. The method delivers highly accurate minimizers but these are not often required in practical applications. (Here also $X$ is not normalized and hence $\|c\|_2 = 7.75$ not 3.445).

**Example 4.** This example is given in [7], p 622.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The required closed-loop eigenvalues are: $\mathscr{L} = \{-2, -3, -4\}$. The results are shown in Table IV.

In [7] the freedom in eigenvectors selection is used to produce designs such that only certain modes appear in each state (and hence in each output). In our example, it is a coincidence that the required combination of modes corresponds to the maximally orthogonal set of eigenvectors. We remark that the first and fourth columns are almost identical, except that in [7] the $X_i$'s are not normalized and hence $\gamma(X)$ is increased and also $\|c\|_2 = 6.595$ not 2.549. The disadvantage of D'Azzo's method is that the size of the problem is increased to $(n+m) \times (n+m)$ since he gets the null space of $(A-\lambda_i I)$ augmented by $B$. This, compared to our method, is very expensive both in time and in storage. Our method gets the null space of the reduced matrix $U_1^T (A- \lambda_i I)$ which is of size $(n-m) \times n$.

In all examples, it is clear that the introduced methods have proved to be effective. The minimization of $\|R\|_2$ gives always the best results with respect to $\gamma(X)$ and also the $\nu$ measure $= \max_i |c_i|$. This makes it the best choice for rapid results in all general cases.

Table I: Results of Example 1.

| | $f_o$ | $\|R\|_2$ | $f$ | [19] | [20] | 0 | 1 | 2/3 |
|---|---|---|---|---|---|---|---|---|
| $\|R\|_1$ | 0.82943 | 1.1503 | 0.949 | 0.8292 | 1.6 | | | |
| $\|R\|_2$ | 0.82943 | 0.82943 | 0.86545 | 0.8293 | 1.5148 | | | |
| $\|R\|_F$ | 1.1724 | 1.17192 | 1.0672 | 1.1727 | 1.1668 | | | |
| $\theta_{12}$ | 90° | 90° | 70° | 90° | 106°.55 | | | |
| $\theta_{13}$ | 146° | 133° | 61.6° | 90° | 142°.62 | | | |
| $\theta_{23}$ | 90° | 62°.4 | 118° | 33°.9 | 36°.9 | | | |
| $c_1$ | 1.79 | 1.585 | 1.67 | 1 | 1.36 | 1.43 | 1.473 | 1.59 |
| $c_2$ | 1 | 1.3 | 1.67 | 1.789 | 1.38 | 1.47 | 1.425 | 1.41 |
| $c_3$ | 1.79 | 1.78 | 1.78 | 1.789 | 1.749 | 1.49 | 1.789 | 1.79 |
| $\|Y\|_F$ | 2.722 | 2.72 | 2.97 | 2.53 | 2.4 | | | |
| $\gamma_F(X)$ | 4.7146 | 4.714 | 5.147 | 4.383 | 4.506 | | | |
| $\gamma_2(X)$ | 3.273 | 3.273 | 3.36 | 3.273 | 3.292 | 3.273 | 3.273 | 3.28 |
| $\|K\|_F$ | 16.42 | 16.47 | 16.49 | 16.5 | 16.19 | | | |
| $\|K\|_2$ | 16.38 | 16.43 | 16.46 | 16.46 | 16.16 | 16.49 | 16.49 | 16.54 |

**Table II**. Results of Example 2.

|  | $f_0$ | $\|\mathbf{R}\|_2$ | $f$ |
|---|---|---|---|
| $\|\mathbf{R}\|_1$ | 0.98298 | 0.99101 | 0.9965 |
| $\|\mathbf{R}\|_2$ | 0.98298 | 0.98299 | 0.9917 |
| $\|\mathbf{R}\|_F$ | 1.39014 | 1.39 | 1.2147 |
| $\theta_{12}$ | 90° | 89°.05 | 120° |
| $\theta_{13}$ | 10:58 | 10°.6 | 60° |
| $\theta_{23}$ | 90° | 90° | 60° |
| $c_1$ | 5.443 | 5.445 | 6.416 |
| $c_2$ | 1 | 1.0014 | 6.364 |
| $c_3$ | 5.443 | 5.445 | 6.4209 |
| $\|Y\|_F$ | 7.7635 | 7.766 | 11.08 |
| $\gamma_F(X)$ | 13.446 | 13.45 | 19.20 |
| $\|K\|_F$ | 11.446 | 11.44 | 11.06 |
| $\|K\|_2$ | 11.31 | 11.30 | 10.913 |

**Table III**: Results of example 3.

|  | $f_0$ | $\|\mathbf{R}\|_2$ | $f$ | [3] full | [3] half |
|---|---|---|---|---|---|
| $\|\mathbf{R}\|_1$ | 0.985 | 0.98926 | 0.99748 | 0.9829 | |
| $\|\mathbf{R}\|_2$ | 0.985 | 0.98293 | 0.99155 | 0.9829 | |
| $\|\mathbf{R}\|_F$ | 1.393 | 1.39005 | 1.21448 | 1.3900 | |
| $\theta_{12}$ | 90° | 90.17 | 90.67 | 90° | |
| $\theta_{13}$ | 10°.3 | 10°.61 | 59.63 | 10°.6 | |
| $\theta_{23}$ | 90° | 89°.63 | 119.47 | 90° | |
| $c_1$ | 5.433 | 5.435 | 6.33 | 5.432 | |
| $c_2$ | 1 | 1 | 6.27 | 1 | |
| $c_3$ | 5.433 | 5.535 | 6.34 | 5.432 | |
| $\|Y\|_f$ | 7.748 | 7.7519 | 10.94 | 3.4458 | |
| $\gamma(X)$ | 13.42 | 13.426 | 18.95 | 11.87 | 14.43 |
| $\gamma_2(X)$ | 10.8 | 10.7811 | 13.354 | 10.77 | 13.15 |
| $\|K\|_F$ | 9.07 | 9.068 | 7.5037 | 9.446 | |
| $\|K\|_2$ | 9.068 | 9.0655 | 7.49878 | 9.44 | 12.6 |

**Table IV**: Results of example 4.

|  | $f_0$ | $\|\mathbf{R}\|_2$ | $f$ | [7] |
|---|---|---|---|---|
| $\|R\|_1$ | 0.976 | 1.03 | 0.997 | 0.976 |
| $\|R\|_2$ | 0.976 | 0.976 | 0.987 | 0.976 |
| $\|R\|_F$ | 1.38 | 1.34 | 1.03 | 1.38 |
| $\theta_{12}$ | 90° | 87°.6 | 60°.6 | 90° |
| $\theta_{13}$ | 12°.528 | 13°.77 | 59°.5 | 12°.528 |
| $\theta_{23}$ | 90° | 93°.4 | 118°.9 | 90° |
| $c_1$ | 4.609 | 4.613 | 5.2157 | 4.609 |
| $c_2$ | 4.609 | 1.1 | 5.13912 | 4.609 |
| $c_3$ | 1 | 4.617 | 5.13912 | 1 |
| $\|Y\|_F$ | 6.595 | 6.619 | 8.9768 | 2.549 |
| $\gamma_F(X)$ | 11.423 | 11.465 | 15.548 | 12.227 |
| $\gamma_2(X)$ | 9.1173 | 9.134 | 9.5614 | 10.908 |
| $\|K\|_F$ | 13.96 | 13.978 | 10.499 | 13.964 |
| $\|K\|_2$ | 13.72 | 13.74 | 9.9848 | 13.7197 |

## 7. CONCLUSION

In this paper we discussed the case of third order systems with respect to different measures of robustness. The method proved to be efficient and inexpensive as compared to other methods. A measure of normality is introduced and its bounds calculated. A straight forward expression for the calculation of $\|\mathbf{R}\|_2$ is deduced and found to yield the best results. A matrix inversion scheme is proposed which exploits

the special quasi-orthogonal eigenvector matrix obtained. The same iterative minimization for both $f_o$ and $f$ gave good results for higher order systems and will be reported elsewhere. A special implementation for $\|R\|_2$ is being studied. Also the case of non semisimple matrices, or eigenvalues having more than m multiples must be further investigated.

## REFERENCES

[1] Arnold M. and Datta B.N., "An algorithm for the multiinput eigenvalue assignment problem". IEEE TAC vol 35, pp. 1148-1152, Oct. 1990.

[2] Bakhoom N.G., "A course in Mathematics", Part II, 2nd edition, pp. 568-578, 1955.

[3] Byers R. and Nash S.G., "Approaches to robust pole assignment" Int. J. Control, vol. 49 No. 1 pp. 97-117, 1989.

[4] Chu K-W.E., "A pole assignment algorithm for linear state feedback". Systems and Control letters (North-Holland) pp. 289-299, July 1986.

[5] Chu K-W.E., "A controllability condensed form and a state feedback pole assignment algorithm for descriptor systems" IEEE TAC vol. 33, pp. 366-370, Apr. 1988.

[6] Datta B.N., "An algorithm to assign eigenvalues in a Hessenberg matrix: Single-input case". IEEE TAC vol. 32, pp. 414-417, May 1987.

[7] D'Azzo J.J. and Houpis C.H. "Linear control system analysis and design. Conventional and modern", 3rd edition. McGraw-Hill, 1988.

[8] Golub G. and Van Loan C.F., "Matrix Computations" 2nd edition the John Hopkins Univ. Press. Baltimore, 1989.

[9] Kautsky J., Nichols.N.K., and Van Dooren P., "Robust pole assignment in linear state feedback". Int. J. Control, vol. 41, pp. 1129-1155, 1985.

[10] Lu J., Chiang H-D. and Thorp J.S., "Partial eigenstructure assignment and its application to large scale systems", IEEE TAC vol. 36, March 1994.

[11] Nichols N.K., "On Computational algorithms for pole assignment". IEEE TAC vol 31 pp 643-645, July 1986.

[12] Nichols N.K., "Authors reply" about Comments on "On computational algorithms for pole assignment" by Petkov et al. IEEE TAC vol. 33, pp. 893-894, Sept. 1988.

[13] Nichols N.K., "Robustness in partial pole placement." IEEE TAC vol. 32 pp. 728-732, Aug. 1987.

[14] Petkov P. Hr., Christov N.D. and Konstantinov M.M, "A computational algorithm for pole assignment of linear single input systems". IEEE TAC vol. 29 pp 1045-1048, Nov. 1984.

[15] Petkov P.Hr., "A computational algorithm for pole assignment of linear multiinput-systems". IEEE TAC vol. 31, pp. 1044-1047, Nov. 1986.

[16] Petkov P. Hr., Comments on "On Computational algorithms for pole assignment" IEEE TAC vol. 33, pp. 892-893, Sept. 1988.

[17] Press W.H. et al., "Numerical Recipes. The art of scientific computing". Cambridge Univ. Press, 1986.

[18] Shafai B. and Bhattacharyya S.P. "An algorithm for pole assignment in high order multivariable systems". IEEE TAC vol. 33 pp 870-876, Sept. 1988.

[19] Shalaby M.A. and El-Attar R.A., "A computational procedure for computing a robust solution to the state feedback pole assignment problem" Proc. 3rd Conf. on AME, MTC Cairo, Egypt (1988), pp. 41-53.

[20] Shalaby M.A., "An algorithm for robust pole assignment in linear state feedback", Proc. 5th Int. Conf. on Computer-aided P.E, Univ. of Edinburgh (1989), pp. 305-310.

[21] Watson G.A. "An algorithm for optimal $\ell_2$ scaling of matrices". IMA Journal of Numerical Analysis II, pp. 481-492, 1991.

[22] Wilkinson J.H. "The Algebraic Eigenvalue Problem". Oxford Univ. Press 1965.

[23] Wonham. W.M., "Linear Multivariable Control: A geometric approach", Springer-Verlag, 1979.