FRACTURE OF SHELLS AND PLATES ON AN ELASTIC FOUNDATION UNDER CONVECTIVE COOLING

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ABSTRACT

In this paper, the transient thermal stress problem for circumferential crack in a cylindrical shell and a plate on an elastic foundation are investigated. The shell is approximated by a plate on an elastic foundation. Using the principle of superposition, the formulation results in a singular integral equation which is solved for an edge crack geometry. The numerical results presented include the transient temperature and thermal stress distributions in the uncracked plate, the stress intensity factors as a function of nondimensional time (Fourier number), the crack length,and some values of Biot number. Also, the influence of the stiffness of the elastic foundation on the stress intensity factors is presented. The results are compared with Nied's work for circumferentially cracked hollow cylinder problem and showed satisfactory approximation.

INTRODUCTION

The important failure mode in many structural components is the cracking of the material under transient thermal stress. In the literature, there are many studies of crack problem subjected to thermal shock. The problem of a cracked plate subjected to sudden cooling with free boundaries has been discussed in [1,2]. Rizk [3] considered an edge cracked plate with one free and one fully constrained boundaries subjected to convective cooling. Nied and Erdogan [4] analyzed the stress intensity factor for a circumferentially cracked hollow cylinder under transient thermal stresses.

In [4] the problem was solved by using a general technique for the solution of three dimensional elasticity problem in cylindrical coordinate with mixed boundary conditions imposed along a plane perpendicular to the cylindrical axis which is very complicated and intractable. In the solution the displacements and stresses are expressed in terms of a single potential function and a set of four harmonic functions equivalent to Papkovich-Neuber potentials in cylindrical coordinates. Fourier and Hankle transforms are used to formulate the problem with proper boundary conditions resulting in, after lengthy analysis, a singular integral equation with two Fredholm kernels L_1, L_2. The numerical solution of the singular integral equation was obtained by using a Gaussian integration procedure which had some difficulties in evaluating the Fredholm kernels due to unbounded terms at the end points, slow convergent, and computational of Lommmel's functions. It is well known that the cylindrical shell may be modeled by a plate on an elastic foundation [5]. This model will make the problem of circumferentially edge cracked cylindrical shell under transient thermal stresses analytically tractable. The Fourier transform is used to formulate the mathematical model with suitable boundary conditions to obtain the singular integral equation which is solved numerically in straight forward manner by using expansion method. this method is easy to use and gives faster convergence when compared with the other method without any difficulties. So, the problem of interest is to study the edge cracked plate on an elastic foundation subjected to convective cooling on the surface containing the edge crack which is shown in Figure (1-a). The results are compared with Nied's results [4] for $R_i/H = 9$, which may be considered as a shell, and showed that the approximation is satisfactory. The results are extended to different values of R_i/H, and also for different values of normalized stiffness of the elastic foundation $\eta E/H$.

It is assumed that, the present transient thermal stress problem is quasi-static, i.e, the inertia effects are negligible. Also the thermoelastic coupling effects and the dependence of thermoelastic coefficients on temperature are negligible. By taking the advantage of the linearity of the material, the principle of superposition is used resulting in a singular integral equation which is solved numerically. The main results of the fracture problem are the stress intensity factors which are given as a function of Fourier number, crack length, Biot number, and the normalized stiffness of the elastic foundation.

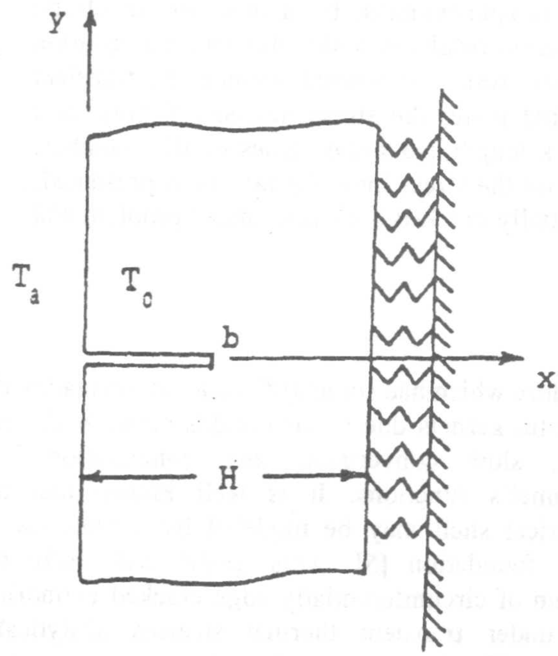

Figure 1-a. Edge cracked plate on an elastic foundation intial at T_o, cooled by ambiant temperature T_o at x=0 and insulated at x=H.

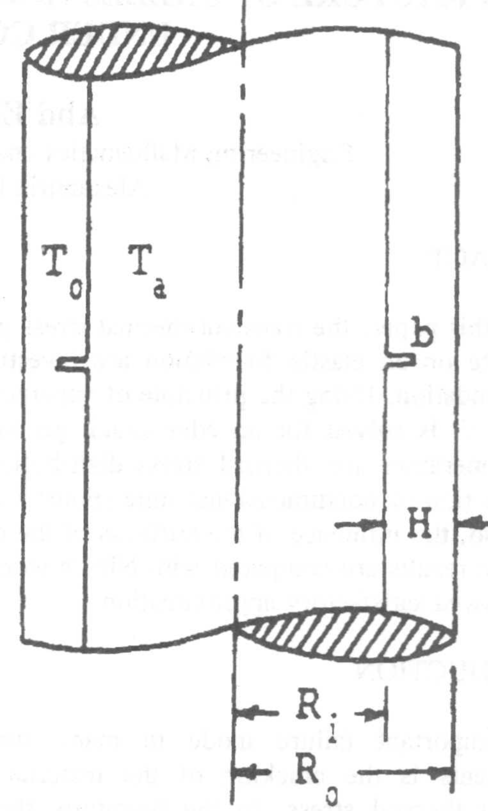

Figure 1-b. Geometry of a cylindrical shell of thickness $H=R_o-R_i$ containing axisymmetric circumferential crack.

MATHEMATICAL FORMULATION

For analyzing the cylindrical shell cracked problem depicted in Figure (1-b), it is approximated by a plate on an elastic foundation with stiffness related to [5]

$$\eta = EH/R_n^2 \tag{1}$$

where E Young's modulus, H the thickness of the cylinder (R_o - R_i), R_n the mean radius of the cylinder (R_i + H/2).

The principle of superposition technique is used to formulate the problem. First the transient temperature distribution is obtained to use it in the uncracked problem to determine the transient thermal stress distribution. Then, applying the equal and opposite of these thermal stresses to the crack surface to solve the crack problem.

TEMPERATURE DISTRIBUTION

Referring to Figure (1), the temperature distribution is obtained by solving the diffusion equation

$$\frac{\partial^2 \theta(x,t)}{\partial x^2} = \frac{1}{D}\frac{\partial \theta(x,t)}{\partial t} \tag{2}$$

where

$$\theta(x,t) = T(x,t) - T_o \qquad (3)$$

and T(x,t), T_o are the temperature in the plate at time t, and initial temperature, respectively. D is the thermal diffusivity.

The plate is assumed to be suddenly cooled by convection at the surface x = 0, with heat transfer coefficient h, and ambient temperature T_a, while it is insulated at the surface x = H. The initial and boundary conditions are

$$\theta(x,0) = 0 \qquad (4)$$

$$\acute{k}\frac{\partial\theta(0,t)}{\partial x} = h[\theta(0,t) - \theta_o] \qquad (5)$$

$$\frac{\partial\theta(H,t)}{\partial x} = 0 \qquad (6)$$

where

$$\theta_o = T_a - T_o \qquad (7)$$

The solution of equation (2) with the conditions (4-6) is given by [3]

$$\frac{T(x^*,\tau) - T_a}{T_o - T_a} = 2\sum_{n=1}^{\infty}\frac{\sin\lambda_n\cos\lambda_n(x^*-1)}{\lambda_n + \frac{1}{2}\sin 2\lambda_n} e^{-\tau\lambda_n^2} \qquad (8)$$

where τ^* is the Fourier number defined by tD/H, x^*=x/H, and λ_n are the eigenvalues determined from the transcendental equation

$$\lambda_n \tan\lambda_n = Bi \qquad (9)$$

where Bi is the Biot number defined by $hH/\acute{k}$.

THERMAL STRESSES IN THE UNCRACKED PROBLEM

A cylinder shell which is subjected to a radial temperature variation can be assumed to undergo uniform strain over the shell thickness. So, the beam on an elastic foundation is also subjected to uniform strain $\varepsilon_o(t)$ over the thickness H, i.e, it would remain flat under the self-equilibrating transient thermal stresses. Thus the thermal stresses and strains would satisfy the following relations :

$$\sigma_{xx}^T = 0 \quad , \quad \sigma_{yy}^T = \sigma_{zz}^T \qquad (10)$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_o(t) \qquad (11)$$

$$\int_0^H \sigma_{yy}^T\, dx = 0 \quad , \quad \int_0^H \sigma_{zz}^T\, dx = 0 \qquad (12)$$

By following [3], the thermal stresses in the plate may be expressed as

$$\frac{\sigma_{yy}^T(x^*,\tau)(1-\nu)}{\alpha E(T_o - T_a)} = 2\sum_{n=1}^{\infty}\frac{\sin^2\lambda_n}{\lambda_n^2 + \frac{\lambda_n}{2}\sin 2\lambda_n} e^{-\tau\lambda_n^2}$$

$$-2\sum_{n=1}^{\infty}\frac{\sin\lambda_n\cos\lambda_n(x^*-1)}{\lambda_n + \frac{1}{2}\sin 2\lambda_n} e^{-\tau\lambda_n^2} \qquad (13)$$

THE CRACK PROBLEM

The crack problem shown in Figure (1-a) may be formulated by using the stress given by equation (13) with an opposite sign acting on the crack surface. The governing differential equations for the displacements in the plane elasticity are

$$(\kappa - 1)\nabla^2 u + 2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\,\partial y}\right) = 0 \qquad (14)$$

$$(\kappa - 1)\nabla^2 v + 2\left(\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 v}{\partial y^2}\right) = 0 \qquad (15)$$

where u, v are the displacement components in x, y directions, $\kappa = (3 - 4\nu)$ for plane strain. Because of symmetry, the problem is considered for $0 < y < \infty$ and subjected to the following boundary and mixed boundary conditions

$$\sigma_{xx}(0,y) = 0 \qquad (16)$$

$$\sigma_{xy}(0,y) = 0 \tag{17}$$

$$\sigma_{xx}(H,y) = -\eta u(H,y) \tag{18}$$

$$\sigma_{xy}(H,y) = 0 \tag{19}$$

$$v(x,0) = 0, \quad 0<x<a, \quad b<x<H \tag{20}$$

$$\sigma_{yy}(x,0) = p(x) = -\sigma_{yy}^{T}, \quad a<x<b \tag{21}$$

After lengthy but straight forward analysis [2], we would obtain the following singular integral equation of Cauchy-type

$$\int_a^b \frac{\phi(s)}{s-x}ds + \int_a^b k(x,s)\phi(s)ds = \frac{\pi(\kappa+1)}{4\mu}p(x), a<x<b \tag{22}$$

where μ is the shear modulus, $\phi(s)$ is the unknown function called density function defined by

$$\phi(s) = \frac{\partial v(x,0)}{\partial x}, \quad 0<x<H \tag{23}$$

The expression of the kernel k(x,s) includes the generalized Cauchy kernels, and can be found in Appendix A.

Following [6] the unknown function $\phi(s)$ for an edge crack is given by

$$\phi(s) = \frac{g(s)}{(b-s)^{1/2}}, \quad 0<s<b \tag{24}$$

where g(s) is the unknown bounded function with $g(0) \neq 0$, $g(b) \neq 0$. The main quantity of interest is the stress intensity factor defined by

$$K(b) = \lim_{x\to b} \sqrt{2(x-b)}\ \sigma_{yy}(x,0) \tag{25}$$

By using asymptotic analysis [6], equation (25) may be reduced to

$$K(b) = -\frac{4\mu}{\kappa+1}\sqrt{2}\ g(b) \tag{26}$$

The stress intensity factor can be calculated from equation (26) after solving the singular integral equation (22) for the unknown function g(s) numerically. The numerical solution is obtained by using expansion method procedure described in [7].

RESULTS AND CONCLUSION

Figures (2-3) show the normalized transient temperature and thermal stress distributions for the most dangerous case (Bi = ∞) which corresponds to unit step cooling temperature at the surface boundary x = 0. The results are plotted against the dimensionless distance $x^* = x/H$, for different values of dimensionless time (Fourier No. $\tau = tD/H^2$). Also, the results for the case of transient thermal hollow cylinder problem [4] for $R_i/H = 9$ are presented in the same figures for comparison. It can be seen that the plate on an elastic foundation provides a good approximation for a cylindrical shell.

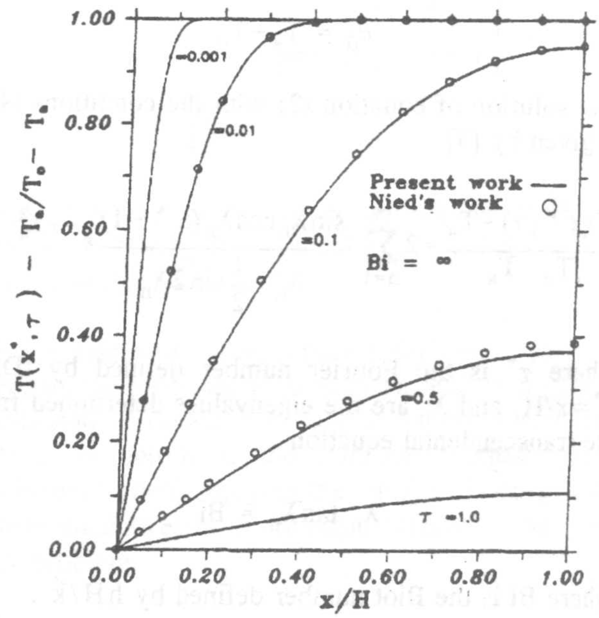

Figure 2. Transient temperature distributions for Bi=∞.

The stress intensity factor obtained from equation (26) is normalized by

$$K^*(b) = \frac{K(b)(1-\nu)}{E\alpha(T_o-T_a)\sqrt{b}} \tag{27}$$

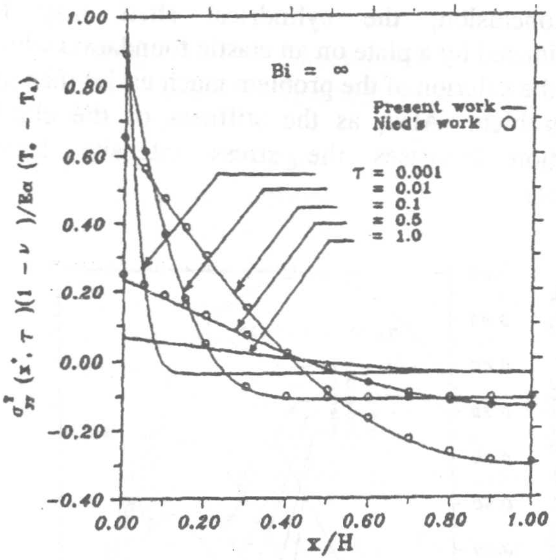

Figure 3. Transient thermal stress distribution for Bi=∞.

and are shown in Figures (4)-(8). In Figure (4), the normalized stress intensity factors are plotted against the normalized time τ, for Bi = ∞, normalized stiffness of the elastic foundation $\eta H/E = 0.01108$, and different values of the normalized crack length b/H. The normalized stiffness $\eta H/E = 0.01108$ corresponds to cylindrical shell of $R_i/H = 9$ through equation (1). The results are compared with Nied's results [4] and showed good approximation too.

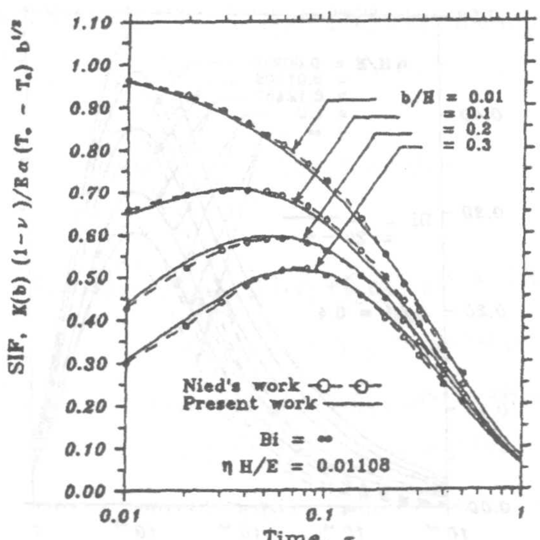

Figure 4. Normalized stress intensity factors for different values of normalized crack length, Bi=∞, $\eta H/E = 0.01108$, $R_i/H = 9$.

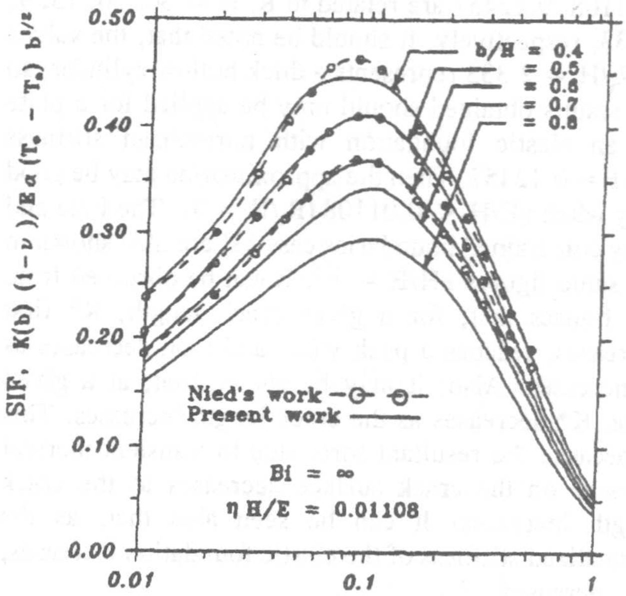

Figure 4 cont.

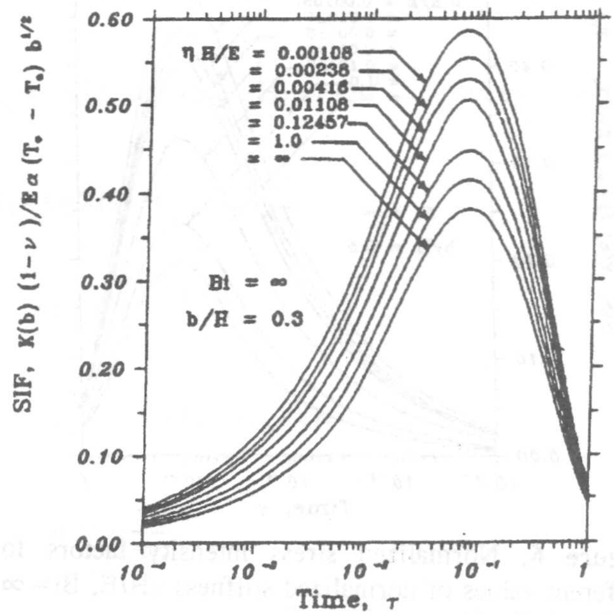

Figure 5. Normalized stress intensity factors for different values of normalized stiffness $\eta H/E$, Bi=∞, b/H = 0.3

Figures (5)-(6) demonstrate K^* versus τ, for two different values of b/H = 0.3, 0.5 and different values of normalized stiffness of the elastic foundation $\eta H/E$. The values of $\eta H/E$ = 0.00108, 0.00238, 0.00418, 0.01108, 0.12457 are related to R_i/H = 30, 20, 15, 9, 2.333, respectively. It should be noted that, the values of $R_i/H = 2.333$ represents a thick hollow cylinder, so the results obtained should only be applied for a plate on an elastic foundation with normalized stiffness $\eta H/E = 0.12457$. Then the approximation may be good only when $\eta E/H < 0.01108$ ($R_i/H = 9$). The Free and fully constrained boundaries case [3] are also shown in the same figures $\eta H/E = \infty$. It can be observed from the figures that, for a given crack length, K* first increases, reaches a peak value and then decreases as τ increases. Also, it may be shown that, at a given time, K* decreases as the crack length increases. This is because the resultant force due to transient thermal stresses on the crack surface decreases as the crack length increases. It can be seen also that, as the normalized stiffness of the elastic foundation increases, K* decreases.

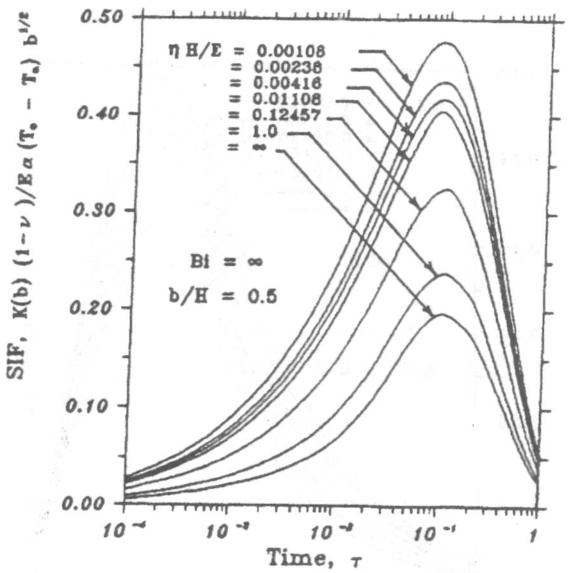

Figure 6. Normalized stress intensity factors for different values of normalized stiffness $\eta H/E$, Bi$=\infty$, $b/H = 0.5$

The influence of the Biot number on the normalized stress intensity factors can be shown in Figures (7-8). Two different values of Biot number are used Bi $= \infty$, 20. Figure (7) shows K* versus τ for normalized stiffness $\eta H/E = 0.00416$ ($R_i/H = 15$), and different values of normalized crack length b/H, while Figure (8) shows K* versus τ for $b/H = 0.4$, for different values of normalized stiffness $\eta H/E$. As we expected, the smaller Biot number, the smaller K*.

In conclusion, the cylindrical shell may be approximated by a plate on an elastic foundation which makes the solution of the problem much easier than the shell problem. Also, as the stiffness of the elastic foundation increases the stress intensity factor decreases.

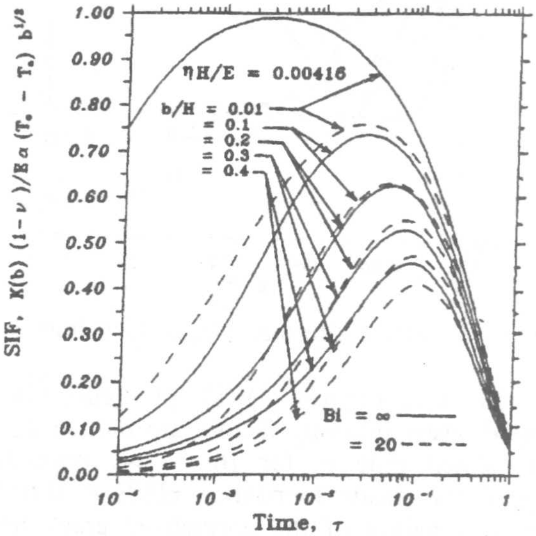

Figure 7. Normalized stress intensity factors for different values of b/H, Bi$=\infty$, 20, $\eta H/E = 0.00416$ ($R_i/H=15$).

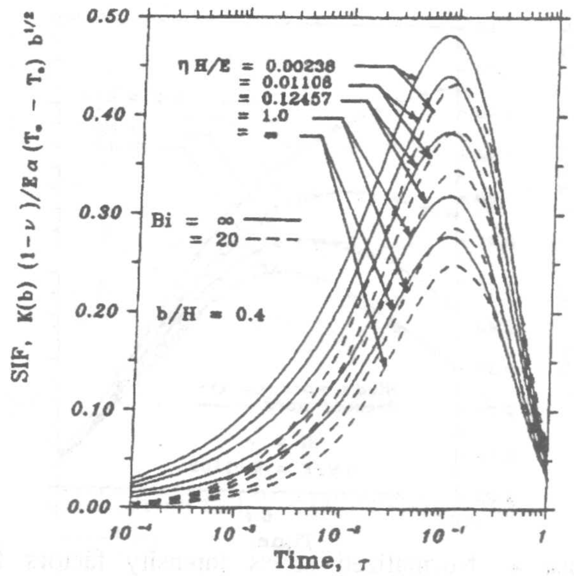

Figure 8. Normalized stress intensity factors for different values of $\eta H/E$, Bi$=\infty$, 20, $b/H = 0.4$.

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Appendix A

The kernel k(x,s) in equation (22) takes the following form

$$k(x,s) = \int_0^\infty G(x,s,\omega)\, d\omega$$

where

$$\begin{aligned} G(x,s,\omega) = -\frac{1}{2M}\{ & (I_1 r_1 + I_2 q_2)\, e^{(s+x-2H)\omega} \\ & + (I_2 q_1 + I_1 S_1)\, e^{-(s-x+2H)\omega} \\ & + (I_1 r_2 + I_2 q_3 + I_1 S_2)\, e^{(s+x-4H)\omega} \\ & + (I_1 I_2 + I_1 S_3)\, e^{-(s-x+4H)\omega} \\ & + (r_1 q_1 + I_3 S_1)\, e^{-(s+x)\omega} \\ & + (r_1 q_3 + I_3 S_2 - q_2 r_2)\, e^{(s-x-2H)\omega} \\ & + (I_1 r_1 + I_3 S_3)\, e^{-(s+x+2H)\omega} \\ & + (-I_1 I_3 - I_1 q_2)\, e^{(s-x-4H)\omega} \} \end{aligned}$$

and

$$I_1 = 1 - \rho\frac{\kappa+1}{2}$$

$$I_2 = 3 - 2(H-x)\omega + \rho\frac{\kappa+1}{2}$$

$$I_3 = 3 + 2(H-x)\omega - \rho\frac{\kappa+1}{2}$$

$$q_1 = (1-2s\omega)\left(2H\omega - \rho\frac{\kappa+1}{2}\right) - 1$$

$$q_2 = 1 + 2(s-H)\omega + \rho\frac{\kappa+1}{2}$$

$$q_3 = (-1-2s\omega)\left(1 - \rho\frac{\kappa+1}{2}\right)$$

$$S_1 = (-1+2s\omega)\left(1 + \rho\frac{\kappa+1}{2}\right)$$

$$S_2 = 1 - 2H\omega(1 + 2(s-H)\omega) + \rho\frac{\kappa+1}{2}(1 + 2s\omega - 4H\omega)$$

$$S_3 = 1 - 2(s-H)\omega - \rho\frac{\kappa+1}{2}$$

$$r_1 = 1 + \rho\frac{\kappa+1}{2}$$

$$r_2 = -2 - 4H^2\omega^2 + 4H\omega\rho\frac{\kappa+1}{2}$$

$$\rho = \frac{\eta}{2\mu\omega}$$

$$M = 1 - (2 + 4H^2\omega^2)e^{-2H\omega} + e^{-4H\omega} + \rho\frac{\kappa+1}{2}(1 + 4H\omega e^{-2H\omega} - e^{-4H\omega})$$