ANALYTICAL METHOD FOR THE STIFFNESS OF PILES WITH FREE LENGTH IN TERMS OF THE STIFFNESS OF FULLY EMBEDDED PILES

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ABSTRACT

A method is introduced to find the stiffness of a single pile with free length above the ground line, in terms of the stiffness of a pile having the same properties and embedded in a similar soil. The head of the second pile coincides with the ground line. The method is valid for non-linear as well as linear analyses. It is also valid for the case of dynamic analysis assuming that the impedance function of the pile, with its head at the ground level is given.

INTRODUCTION

Poulos [2,3] introduced a method to find the horizontal displacement at the head of a free head pile subjected to a horizontal load acting at a distance e above the ground line. Poulos method applies flexibility approach. Hereafter a method is introduced to find the stiffness of a partially embedded pile in terms of the stiffness of a fully embedded pile with the same properties.

HORIZONTAL STIFFNESS

To find the stiffness of a single pile with length e above the ground surface in terms of the stiffness of another pile with the same properties and having the same soil conditions, we proceed as follows.

Assume the translation, rocking and coupling stiffness K_{uu}, K_{rr}, and K_{ur} of the second pile to be known, for the case in which the pile head coincides with the ground line.

The force **F** and moment **M** needed to produce unit translation and unit rotation, respectively, at the head of a free head pile, with its head at the ground line, are related to K_{uu}, K_{rr}, and K_{ur} as follows.

If the head of a fixed head pile is forced to move in the horizontal direction by unity, with zero rotation, then the force and moment developed at the head are

$$F = K_{uu}, \quad M = K_{ur}$$

for the moment M to be zero, a moment $M = -K_{ur}$ must be applied to the head. This moment produces a horizontal force equals to $K_{ur}\ K_{ru}/K_{rr}$ and rotation equals to K_{ur}/K_{rr}. Then the force required to produce unit translation in the pile head, with zero moment associated with it, is given by

$$\underline{F} = K_{uu} - \frac{K_{ur}\,K_{ru}}{K_{rr}} \qquad (1)$$

This is associated with the following boundary conditions

$$M = 0 \quad , \quad u = 1, \quad \theta = \frac{K_{ur}}{K_{rr}} \qquad (2)$$

Working out the same way, we find that the moment needed to produce a unit rotation in a free head pile is given by

$$\underline{M} = K_{rr} - \frac{K_{ur}\,K_{ru}}{K_{uu}} \qquad (3)$$

with the boundary conditions

$$F = 0 \ , \theta = 1 \ , u = \frac{\underline{K}_{ur}}{\underline{K}_{uu}} \tag{4}$$

Going back to our original problem, we find that in the case of a pile with length e above the ground line, if the head of the pile is displaced in the horizontal direction by unity, with no rotation, then a force F and a moment M developed ,in the pile, at the ground line, figure (1).

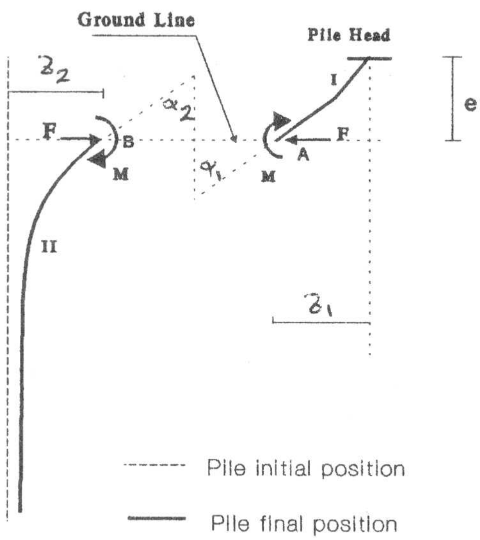

Figure 1. Pile force and continuity conditions.

This force F and the moment M are related to the stiffness $\underline{K}_{uu}$ and $\underline{K}_{ur}$ of the pile by

$$\underline{K}_{uu} = F$$
$$\underline{K}_{ur} = -(M + eF) \tag{5}$$

The translation δ_1 and rotation α_1 the upper part of the pile, part I, at point A, are given in terms of the force F, the moment M and the properties of the pile by

$$\delta_1 = \frac{Me^2}{2EI} + \frac{Fe^3}{3EI} \tag{6}$$

$$\alpha_1 = \frac{Me}{EI} + \frac{Fe^2}{2EI} \tag{7}$$

The moment M and the force F are considered positive in the directions shown in the figure.

The translation δ_2 and rotation α_2 caused by the force F and the moment M in the lower part of the pile, part II, at point B, are given by , using equations (1) through (4).

$$\delta_2 = \frac{F}{\underline{F}} - \frac{M}{\underline{M}} \frac{\underline{K}_{ru}}{\underline{K}_{uu}} \tag{8}$$

$$\alpha_2 = -\frac{F}{\underline{F}} \frac{\underline{K}_{ur}}{\underline{K}_{rr}} + \frac{M}{\underline{M}} \tag{9}$$

From the continuity of the pile, we must have

$$\delta_1 + \delta_2 = 1$$
$$\alpha_1 + \alpha_2 = 0 \tag{10}$$

Using (6) with (8) and (7) with (9), we get

$$\frac{F}{\underline{F}} + \frac{M}{\underline{M}} \frac{\underline{K}_{ru}}{\underline{K}_{uu}} = 1 - \frac{Me^2}{2EI} - \frac{Fe^3}{3EI} \tag{11}$$

$$\frac{F}{\underline{F}} \frac{\underline{K}_{ur}}{\underline{K}_{rr}} + \frac{M}{\underline{M}} = -\frac{Me}{EI} - \frac{Fe^2}{2EI} \tag{12}$$

Solving (11) and (12) together, we get F and M, using them with (5), we can find $\underline{K}_{uu}$ and $\underline{K}_{ur}$

ROCKING STIFFNESS

To find the rocking stiffness, $\underline{K}_{rr}$ and the coupling stiffness $\underline{K}_{ru}$, it is enough to replace the first term on the right hand side of (11) by e, the free length of the pile above the ground line, and add one to the R.H.S. of (12).

$\underline{K}_{rr}$ and $\underline{K}_{ru}$ are given by

$$\underline{K}_{rr} = M + e.F$$
$$\underline{K}_{ru} = -F \qquad (13)$$

Equations (11) and (12) can be written in the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} F \\ M \end{Bmatrix} = \begin{Bmatrix} \Delta \end{Bmatrix} \qquad (14)$$

where,

$$A_{11} = \frac{1}{F} + \frac{e^3}{3EI}$$

$$A_{12} = \frac{1}{M}\frac{K_{ru}}{K_{uu}} + \frac{e^2}{2EI}$$

$$A_{21} = \frac{1}{F}\frac{K_{ur}}{K_{rr}} + \frac{e^2}{2EI}$$

$$A_{22} = \frac{1}{M} + \frac{e}{EI}$$

and,

$\Delta_1 = 1$, $\Delta_2 = 0$, in case of calculating $\underline{K}_{uu}$ and $\underline{K}_{ur}$,

and

$\Delta_1 = e$, $\Delta_2 = 1$, in case of calculating $\underline{K}_{rr}$ and $\underline{K}_{ru}$

Solving (14), we get

$$\begin{Bmatrix} F \\ M \end{Bmatrix} = C \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} \begin{Bmatrix} \Delta \end{Bmatrix} \qquad (15)$$

where,

$$C = \frac{1}{A_{11}\,A_{22} - A_{12}\,A_{21}}$$

VERTICAL STIFFNESS

The vertical stiffness $\underline{K}_{vv}$ is given by equating the sum of the vertical deformation in the part of the pile over the ground surface , and the settlement in the part of the pile under the ground to unity. This leads to

$$\underline{K}_{vv} = \frac{K_{vv}}{\left(1 + K_{vv}\dfrac{e}{EA}\right)} \qquad (16)$$

where A is the cross section area of the pile.

METHOD VERIFICATION

Assuming the method introduced is correct, then it must lead to the same results given by Poulos [2]. This, of course requires that the flexibility coefficients used by Poulos , to be introduced in terms of the stiffness coefficients used in this work and vice versa. The relation among the flexibility coefficients and the stiffness coefficients are given by

$$\frac{1}{E_s}\begin{bmatrix} \dfrac{I_{uh}}{L} & \dfrac{I_{um}}{L^2} \\ \dfrac{I_{rh}}{L^2} & \dfrac{I_{rm}}{L^3} \end{bmatrix} \begin{Bmatrix} K_{uu} \\ K_{ur} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \qquad (17\text{-A})$$

and

$$\frac{1}{E_s}\begin{bmatrix} \dfrac{I_{uh}}{L} & \dfrac{I_{um}}{L^2} \\ \dfrac{I_{rh}}{L^2} & \dfrac{I_{rm}}{L^3} \end{bmatrix} \begin{Bmatrix} K_{ru} \\ K_{rr} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \qquad (17\text{-B})$$

where,

- L the embedded pile length
- E_s modulus of elasticity of the soil
- I_{ij} flexibility coefficient; a measure of the displacement (rotation) of the pile's head due to unit force (moment) applied at the head

Using (1) with (5), (13), and (14) we find that the force applied at a distance e above the ground line and required to produce unit displacement in a free head pile is given by

$$F_e = F_1 - \frac{(M_1 + e F_1) F_2}{(M_2 + e F_2)} \qquad (18)$$

where F_1 and M_1 result from solving (14) with $\Delta = \{1\ 0\}^T$ while F_2 and M_2 result from solving (14) with $\Delta = \{e\ 1\}^T$

Figure (2) shows the force required to produce a unit displacement in a free head pile with free length e, as calculated using the present method and Poulos method. Since the two methods are analytical then they give exactly the same results as expected.

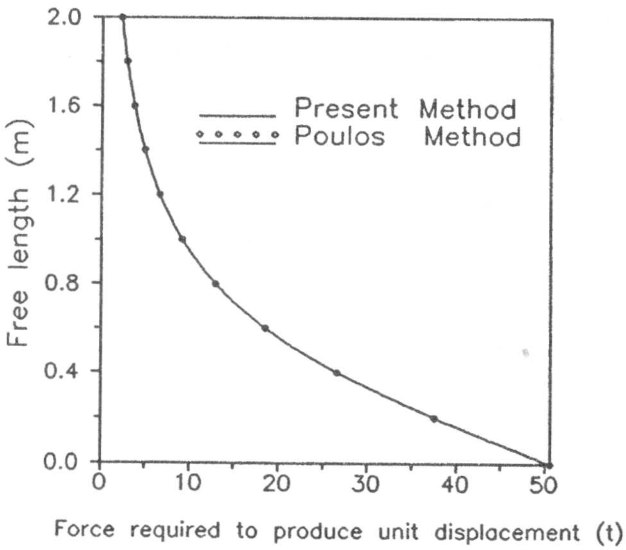

Figure 2. Force_displacement relationship, free head pile.

EXAMPLE

As an illustrative example, the following cases are introduced. Table (1) shows the properties of four different piles. Their stiffness computed at the ground level is given, together with their modulus of rigidity. For all the cases considered Poisson's ratio is taken to be 0.25. The effect of the free length on the stiffness is introduced in the figures that follows.

Table 1. Properties of piles.

Diameter(m)	E_{pile}/G_{soil}	$EI(t.m^2)$	$K_{uu}(t/m)$	$K_{ur}(t/m)$	$K_{rr}(t.m^2)$
1.0	10000	68690	2310	5960	29330
0.5	10000	4290	1150	1490	3670
1.0	250	68690	36200	35200	71240
0.5	250	4290	18100	8800	8900

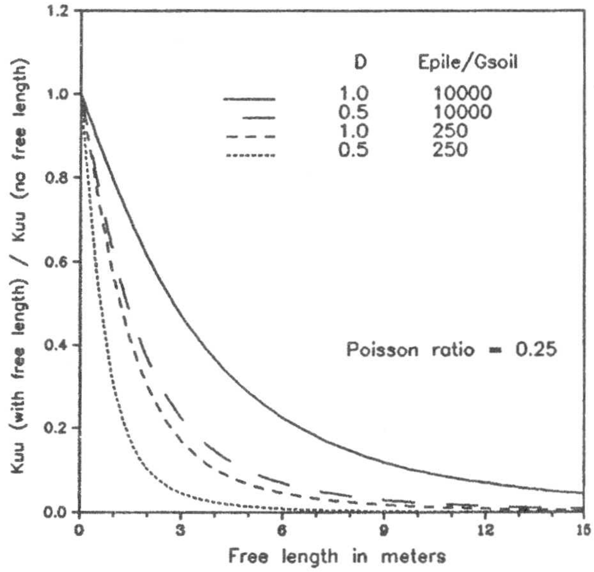

Figure 3. Variation of $\underline{K}_{uu}$ with the free length.

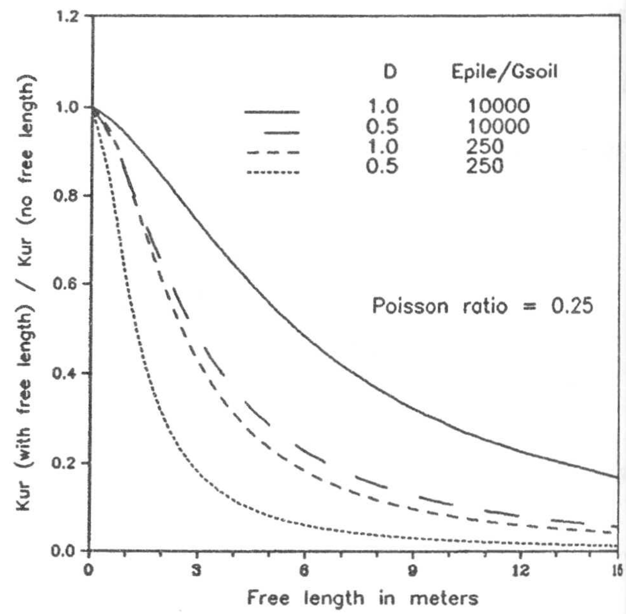

Figure 4. Variation of $\underline{K}_{ur}$ with the free length.

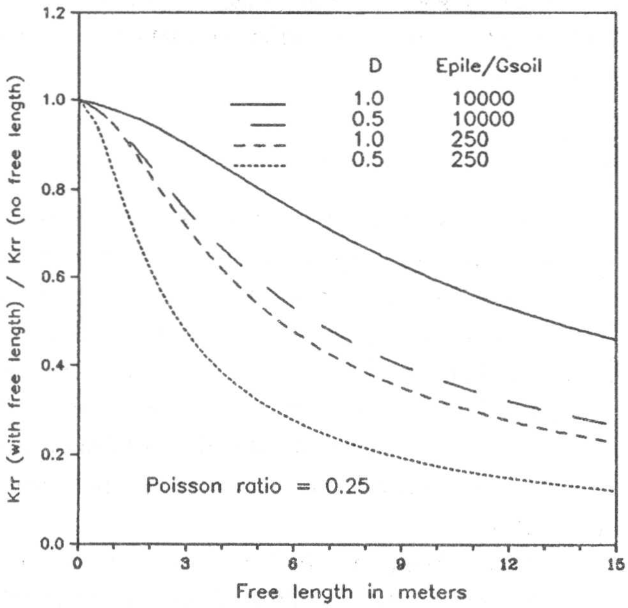

Figure 5. Variation of K_{rr} with free length.

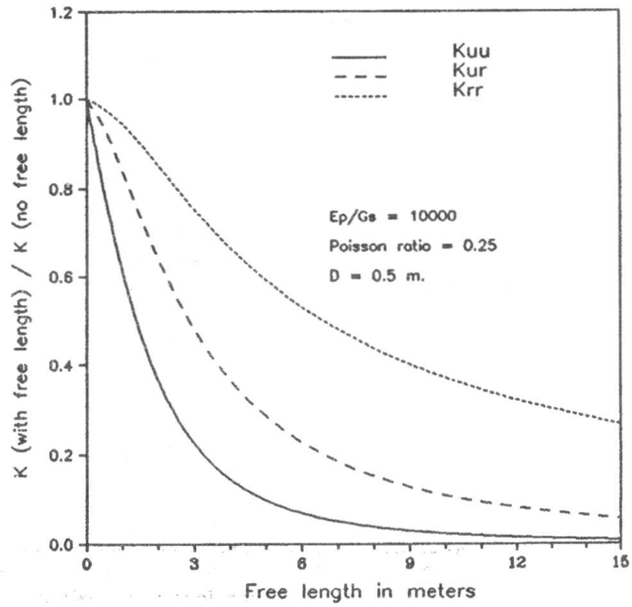

Figure 7. Variation of stiffness coefficients with free length.

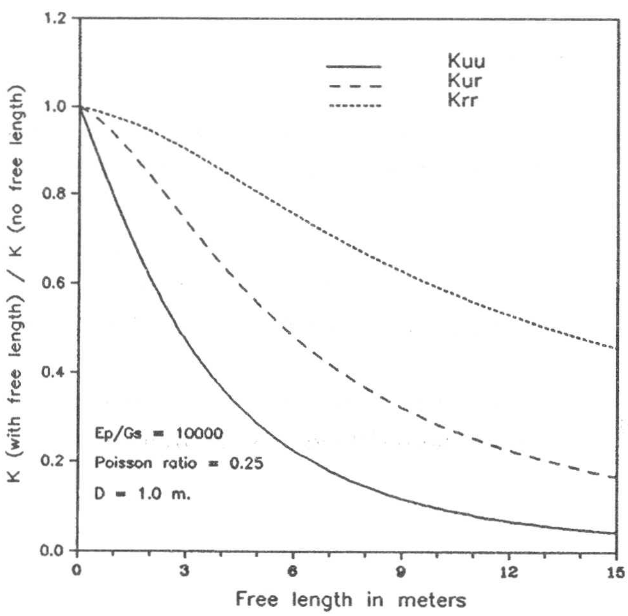

Figure 6. Variation of stiffness coefficients with free length.

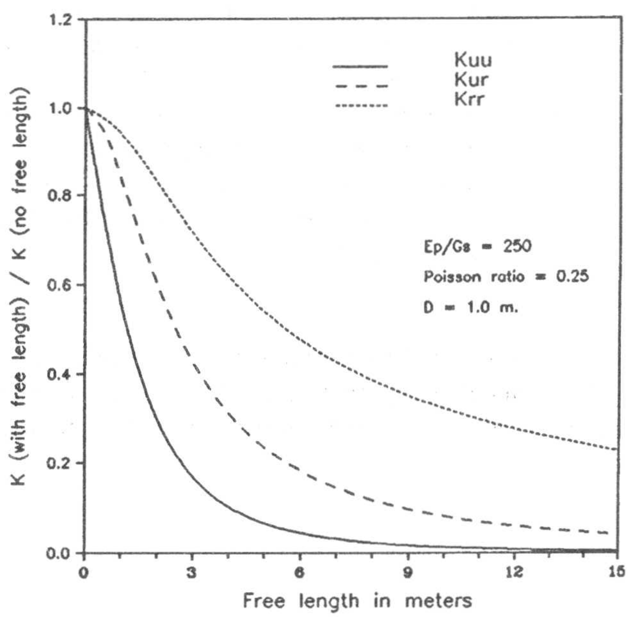

Figure 8. Variation of stiffness coefficients with free length.

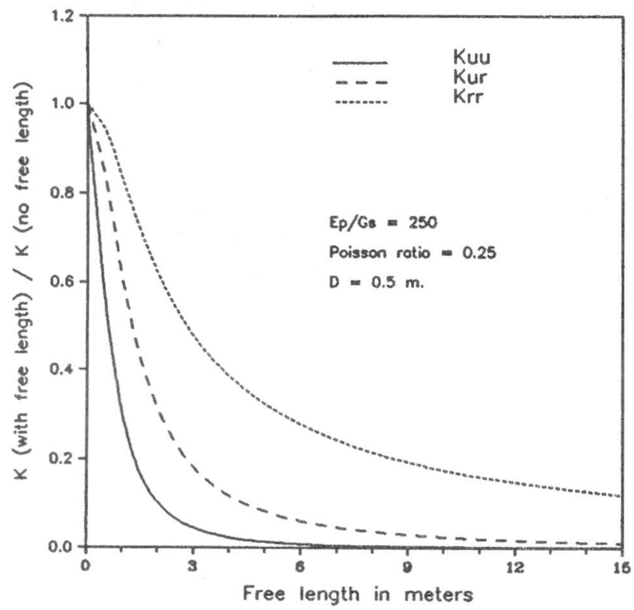

Figure 9. Variation of stiffness coefficients with free length.

From Figures (3) through (9) it may be concluded that:

1. Increasing the ratio K_{ij}/EI (i,j = u,r), increases the rate of reduction in the stiffness coefficients with increasing the free length. This may be clear, since reducing the flexural rigidity of the pile allows a greater part of the deflection to take place in the part of the pile above the ground level.
2. The variation in K_{rr} with increasing the free length is less sensitive than the variation in K_{uu}, which is greater than that of K_{ur}.

A better look at the analytical method introduced or a detailed parametric study may be needed to have a clear understanding of the relative importance of the different quantities mentioned, in the equations introduced, on the variation of the stiffness coefficients.

CONCLUSION

A method is introduced to calculate the stiffness of a pile with a free length, in terms of the stiffness coefficients of the same pile, when being fully embedded in a soil with the same properties. The method is independent of the properties of the soil and the pile. It is valid for both linear and non-linear analysis. Also it is valid for dynamic analysis, since it is based on fulfilling the continuity conditions of the pile.
The method eliminates the need of developing sophisticated programs to calculate the required stiffness coefficients, since there is a huge amount of published data ,in the form of charts and tables, for the stiffness coefficients of fully embedded pile. They may be converted using the method introduced to obtain any needed quantities.

REFERENCES

[1] James M. Gere and William Weaver, Jr., *Analysis of Framed Structures*, D. Van Nostrand Company, Inc., 1965

[2] Poulos, H.G., Behavior of Laterally Loaded Piles: I-Single Piles, *Journal of Soil Mechanics and Foundation Engineering*, Vol. 97, No. SM5, May 1971.

[3] Poulos, H.G. and Davis E.H., *Pile Foundation Analysis and Design*, John Wiley & Sons, 1980.