EFFECTIVE BUCKLING LENGTH FACTORS FOR ELASTICALLY CONNECTED COLUMNS

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ABSTRACT

The design specifications for columns include alignment charts for calculating effective buckling length factors "*K*". For braced columns these factors range between 0.5 for rigidly ended column and 1.0 for pin ended column. The value of the *K*-factor depends on the values of the rotational end restraint at the column ends. Exact theoretical expressions for elastically connected columns are developed. The value of the exact *K*-factor can be obtained using the developed expressions. Aided design curves are also produced. The values of the *K*-factors obtained from the present study are compared with those obtained from the alignment chart. Conclusions and recommendations toward a better evaluation of the *K*-factors and the column buckling loads, using the developed design curves, are included.

INTRODUCTION

The behavior of steel structures is dominated by the behavior of the compression members in these structures. These compression members suffer from loss of stability and buckle at a certain load level. The buckling load of a compression member may lead to local or global buckling and failure of the whole structure.

In a real structure the end conditions of a member that is primarily under axial compression is far from being ideally fixed or pinned. For the sake of convenience in design, the limit load of such compression members is generally expressed in terms of the limit load of a pin-ended column with length equals "l". The length of the imaginary pin-ended column is related to the actual column length "L" by a factor "K" where $l=KL$. This factor is denoted as the effective length factor.

The selection of an appropriate value for K is an important prerequisite of column design. This factor is based upon the flexural restraint at each end of the column as well as the degree of in-plane sway restraint. For a column with well defined supporting conditions, it is easy to find a theoretical value of the *K*-factor. Figure (1) shows the theoretical values for columns with different supporting conditions.

The critical buckling load of a column having an effective length KL is

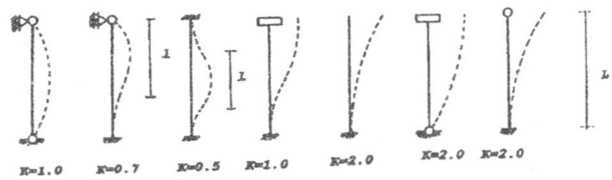

Figure 1. Theoretical effective length factors for columns with defined end conditions.

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{P_E}{K^2} \tag{1}$$

where P_E is the Euler buckling load of a pin-ended column having length L,

and

$$K = \frac{l}{L} = \sqrt{\frac{P_E}{P_{cr}}} = \frac{\pi}{L}\sqrt{\frac{EI}{P_{cr}}} = \sqrt{\frac{1}{q}} \tag{2}$$

where q is a non-dimensional axial force parameter that relates the axial critical load P_{cr} to the Euler buckling load $P_E = \dfrac{\pi^2 EI}{L^2}$.

In order to evaluate the buckling load of a column in rigidly jointed frames, braced or unbraced, an approach called the *G*-factor method [1,2,3,4] is used. This approach is based on two separate characteristic equations for braced and unbraced frames. These equations are derived for frame subassemblages with preassigned buckling mode assuming that all the columns in a story buckle simultaneously. These equations are in terms of stability functions of the column and of the stiffness distribution factors G_1 and G_2 corresponding to ends 1 and 2 of the column; respectively.

The flexural restraints at the ends 1 and 2 of a column depend on the rotational stiffnesses of all members rigidly connected to these ends. These members include any columns above and/or below and any beams. If the rigidly connected beams have zero or small axial force, which is a common case in most of regular rectangular frames, they restraint the column against buckling. The stiffnesses of these beams are classed as positive and generate positive *G*-factors.

If the other columns, above and below, have significant axial forces they will disturb the column being investigated, hence their stiffnesses are classed as negative and they generate negative *G*-factors [2]. The *G*-factors are defined by:

$$G_i = \sum(\frac{I_c}{L_c})/\sum(\frac{I_b}{L_b}) \qquad (3)$$

where i = 1 and 2 for ends 1 and 2 of the column; respectively, I_c, L_c are the moment of Inertia, and the actual length of the column, and I_b, L_b are the moment of inertia and length of all members rigidly connected to the column ends 1 and 2 and lying in the plane in which buckling of the column is being considered.

The Egyptian Code of Practice for Steel Structures and Bridges ***(ECPSSB)***, [5] has adopted the chart shown in Figure (2) to determine the *K*-factors. In connection with the use of this chart the ***ECPSSB*** has adopted the following recommendations:

(i) For column base connected to a footing or foundation by a frictionless hinge *G* is theoretically infinite, but should be taken as 10 in design practice.

(ii) If column based is rigidly attached to properly designed footing *G* approaches a theoretical value of zero but should be practically taken as 1.0.

(iii) The inertia of girder members connected to the column should be modified according to the supporting condition of their far ends.

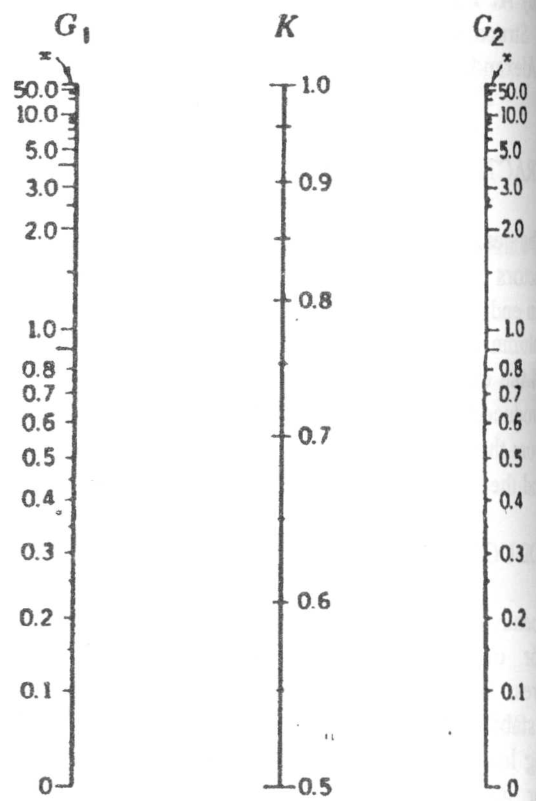

Figure 2. Alignment chart for effective length factors of braced columns.

In the case of columns with well defined supporting conditions and also for columns in rigidly jointed frames, the columns have been considered as pinned or rigid jointed to other members or to the foundations. In practice most of steel column are elastically connected to the other members or to the base. Figure (3) shows sketches for some typical types of column-base connections.

In this research the effects of joint rotational stiffness on the value of column buckling load are considered. An effective buckling length factor that takes the effect of column end joints properties is studied. The present study is limited to the case of braced column (sway prevented). The other cases of unbraced (sway permitted) columns and sway elastically restrained columns are under investigation.

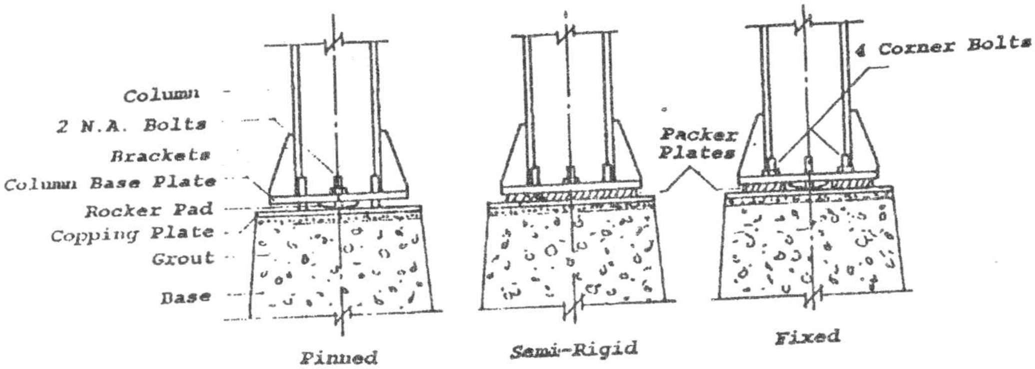

Figure 3. Typical column-base connections.

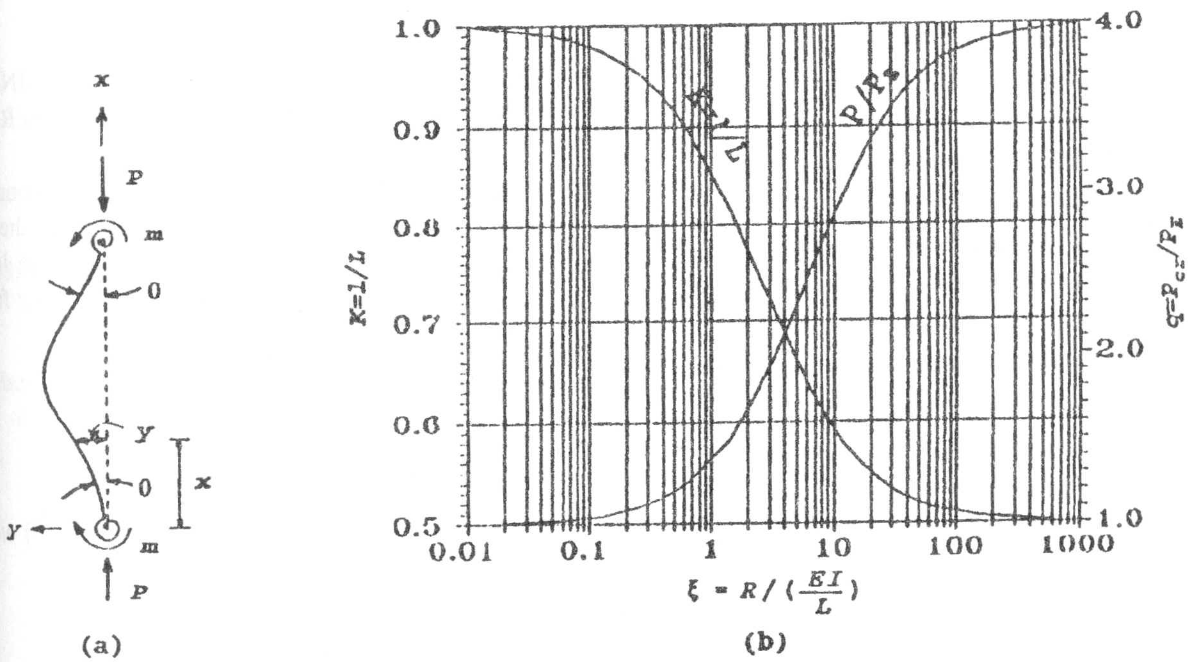

Figure 4. Column with equally rotational end restraints: (a) Deformed column, (b) Effective length factors, k and critical load parameters, $q=p_{cr}/p_E$, vs. ratio of rotational end restraint.

The following sections deal with different cases of rotational end restraint. A theoretical expression for each case is derived.

COLUMNS WITH EQUAL ROTATIONAL END RESTRAINTS AT BOTH ENDS

The column shown in Figure (4) has equal rotational springs at its ends. The stiffness of the spring at any end equal to the stiffness of joint in addition to the stiffnesses of all members connected to that end.

From practical point of view, the column critical load is the minimum load that makes the equilibrium of the deformed column neutral. This critical load is corresponding to the first mode of failure. For the deformed column shown in Figure (4-a), the moment M acting on a section at distance x from end 1 due to the applied load is

$$M = Py - m \tag{4}$$

where m is the restraining moment at column ends. The curvature-moment relationship is

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} \quad (5)$$

Substituting from Eqn. (4) into Eqn. (5), gives

$$\frac{d^2y}{dx^2} + \mu^2 y = \frac{m}{EI} \quad (7)$$

where

$$\mu^2 = \frac{P}{EI} \quad (8)$$

Solution of Eqn. (7) leads to

$$y = A\sin\mu x + B\cos\mu x + \frac{m}{p} \quad (9)$$

and

$$\frac{dy}{dx} = A\mu\cos\mu x - B\mu\sin\mu x \quad (10)$$

The constants **A** and **B** are determined from the boundary conditions as follows:

$$at \;\; x = 0 \;\; y = 0 \;\; hence \;\; B = -\frac{m}{P} \quad (11)$$

from symmetry:

$$at\, x = \frac{L}{2} \;\; \frac{dy}{dx} = 0 \; hence \; A = -\frac{m}{p}\tan\frac{\mu L}{2} \quad (12)$$

$$at\, x = 0 \;\; \frac{dy}{dx} = \theta \; hence \; \theta = -\frac{m\mu}{p}\tan\frac{\mu L}{2} \quad (13)$$

The restraining moment m is related to the slip rotation θ by

$$m = R\theta \quad (14)$$

where R represents the rotational restraining coefficient or the stiffness of the joint at the ends of the column.

The joint stiffness R can be expressed in terms of column flexural stiffness such as:

$$R = \xi\frac{EI}{L} \quad (15)$$

Substituting from Eqns. (14) and (15) into Eqn. (leads to:

$$\tan\frac{\mu L}{2} + \frac{\mu L}{\xi} = 0 \quad (1$$

Eqn. (16) represents the equilibrium equation of column shown in Figure (4-a). For any given value ξ, this equation can by solved numerically graphically to find the value of "μL" that satisfies equation. Figure (4-b) shows the relation between effective bucking length factor K, P_{cr}/P_E and values of joint stiffness parameter "ξ". From Eqns. and (8) it can be concluded that:

$$\mu L = \pi\sqrt{q} \quad (1$$

COLUMN ROTATIONALLY RESTRAINED A ONE END AND PINNED AT THE OTHER END

Figure (5-a) shows a column elastically connected end 1 and hinged at end "2". Considering the column in its deformed position, under the applied load, the moment M acting on a section at distance x from end 1 is:

$$M = Py - m_1 + \frac{m_1}{L}x \quad (1$$

substituting M from Eqn. (18) into Eqn. (5) we have

$$\frac{d^2y}{dx^2} + \mu^2 y = \frac{m_1}{EI}\left(1 - \frac{x}{L}\right) \quad (19)$$

Solution of Eqn. (19) is given as:

$$y = A\sin\mu x + B\cos\mu x + \frac{m_1}{P}\left(1 - \frac{x}{L}\right) \quad (20)$$

and

$$\frac{dy}{dx} = A\mu\cos\mu x - B\mu\sin\mu x - \frac{m_1}{P}\left(\frac{1}{L}\right) \quad (21)$$

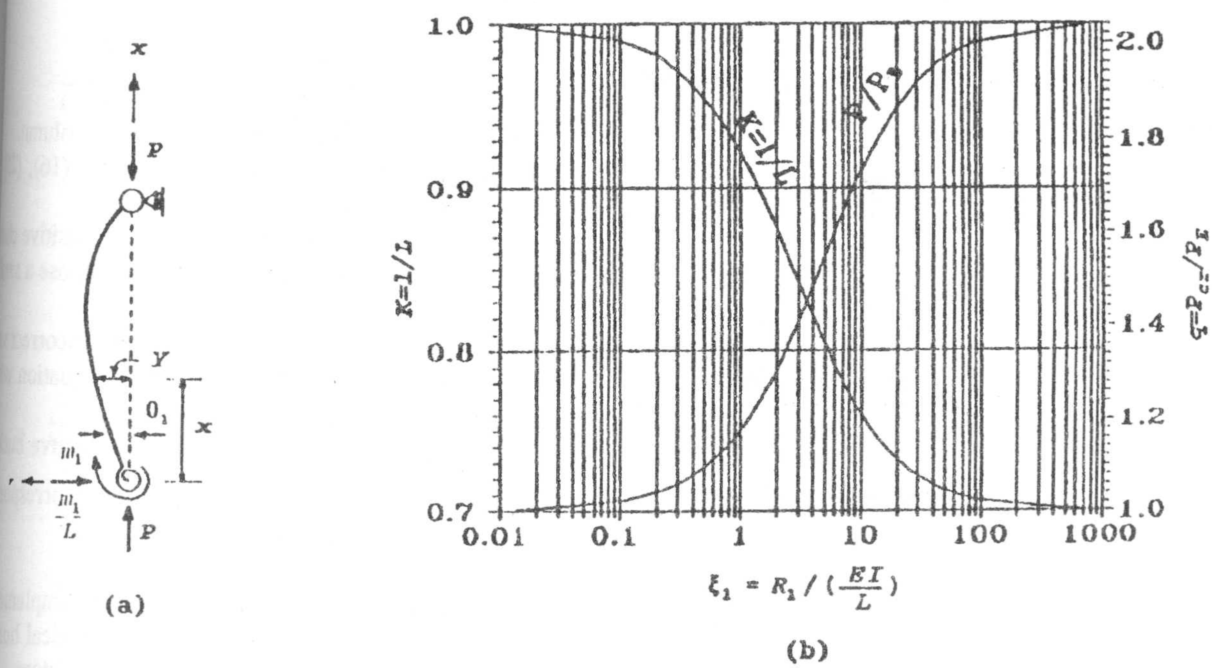

Figure 5. Colum hinged at one end rotationally restrained at the other end: (a) Deformed column, (b) Effective length factors, k and critical load parameters, $q=p_{cr'}/p_E$, vs. ratio of rotational end restraint.

The constants A, B and the slip rotation θ, at end 1 can be obtained from the boundary conditions as follows:

$$at \;\; x = 0 \;\; y=0 \; hence \; B = -\frac{m_1}{P} \tag{22}$$

$$at \;\; x = L \;\; y=0 \; hence \; A = \frac{m_1}{P \tan uL} \tag{23}$$

$$at \; x=0 \;\; \frac{dy}{dx}=\theta_1 \;\; hence \; \theta_1 = +\frac{m_1 \mu}{P}\left(\frac{1}{\tan \mu L} - \frac{1}{\mu L}\right) \tag{24}$$

Substituting from Eqns. (14) and (15) after replacing m, R, θ and ξ by m_1, R_1, θ_1, and ξ_1 into Eqn. (24) and rearranging gives:

$$\tan \mu L - \frac{\xi_1 (\mu L)}{(\mu L)^2 + \xi_1} = 0 \tag{25}$$

Eqn. (25) represents the equilibrium equation of the column shown in Figure (5-a). Solution of this equation can be obtained numerically or graphically. Figure (5-b) shows the solution of this equation for different values of ξ_1.

COLUMN WITH UNEQUAL ROTATIONAL RESTRAINTS AT ITS ENDS

In this general case the ends of the column shown in Figure (6) have two different end rotational restraints with stiffnesses R_1 and R_2 at ends 1 and 2 respectively. The moment M at distance x from end 1 is:

$$M = Py - m_1 + Sx \tag{26}$$

Where P, m_1, and S are shown in Figure (6). The value of the shearing force S is:

$$S = \frac{m_1 + m_2}{L} \quad (27)$$

Substitute the value of M from Eqn. (26) into Eqn. (5), gives:

$$\frac{d^2y}{dx^2} + \mu^2 y = \frac{m_1}{EI} - \frac{m_1 + m_2}{E\,I}\,x \quad (28)$$

The solution of Eqn. (28) leads to:

$$y = A \sin \mu x + B \cos \mu x + \frac{m_1}{P}\left(1 - \frac{x}{L}\right) - \frac{m_2}{P}\left(\frac{x}{L}\right) \quad (29)$$

Applying the following boundary conditions at the ends of the column

$$at \quad x = 0 \qquad y = 0 \qquad and \qquad \frac{dy}{dx} = \theta_1$$

$$at \quad x = L \qquad y = 0 \qquad and \qquad \frac{dy}{dx} = \theta_2$$

and noting that $m_1 = \xi_1 \frac{EI}{L}\theta_1$ and $m_2 = \xi_2 \frac{EI}{L}\theta_2$, Eqn. (29) leads to:

$$(\mu^2 L^2 + \xi_1)(\mu^2 L^2 + \xi_2)\sin^2 \mu L$$

$$- \mu L\left[2\xi_1\xi_2 + \mu^2 L^2(\xi_1 + \xi_2)\right]\sin \mu L \cos \mu L$$

$$+ \mu^2 L^2 \xi_1 \xi_2 \cos^2 \mu L - \xi_1\xi_2(\mu L - \sin \mu L)^2 = 0 \quad (30)$$

Eqn. (30) expresses the condition for neural equilibrium of the elastically restrained column shown in Figure (6).

COMPUTATION OF K-FACTORS

Eqns. (16), (25) and (30) are the equilibrium equations for the elastically restrained columns shown in Figures (4-a), (5-a) and (6); respectively. The procedure for computing the effective buckling length factors and the corresponding critical buckling load for a column with given values of E, I, L and rotational end stiffnesses R_1 and R_2 may be summarized follows:

1- Calculate $\xi_1 = \frac{R_1}{(EI/L)}$ and $\xi_2 = \frac{R_2}{(EI/L)}$
2- Choose value of $q = P_{cr}/P_E$ for the column.
3- Calculate the left hand side of Eqns. (16), (2 (30) as appropriate.
4- If the value of the left-hand side is positive ch a large value of q, if it is negative choose a sm value of q and go to step 3.
5- repeat steps 3 and 4 till obtaining the correct v of q that satisfies the equilibrium equation w certain tolerance of q (say 1%).
6- From the correct value of q, the effective buck length factor $K = \sqrt{1/q}$ and the correspon critical buckling load can be obtained.

A computer program has been coded impleme the above procedure, to calculate the critical buck loads and the effective buckling length factors.

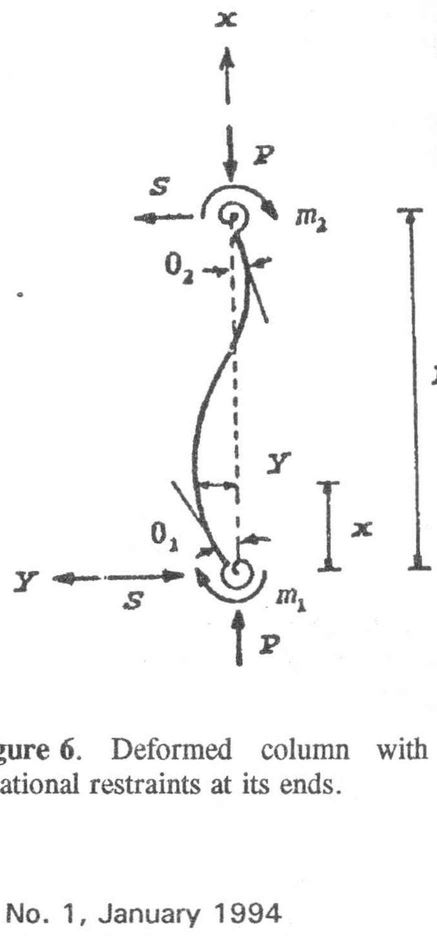

Figure 6. Deformed column with unequal rotational restraints at its ends.

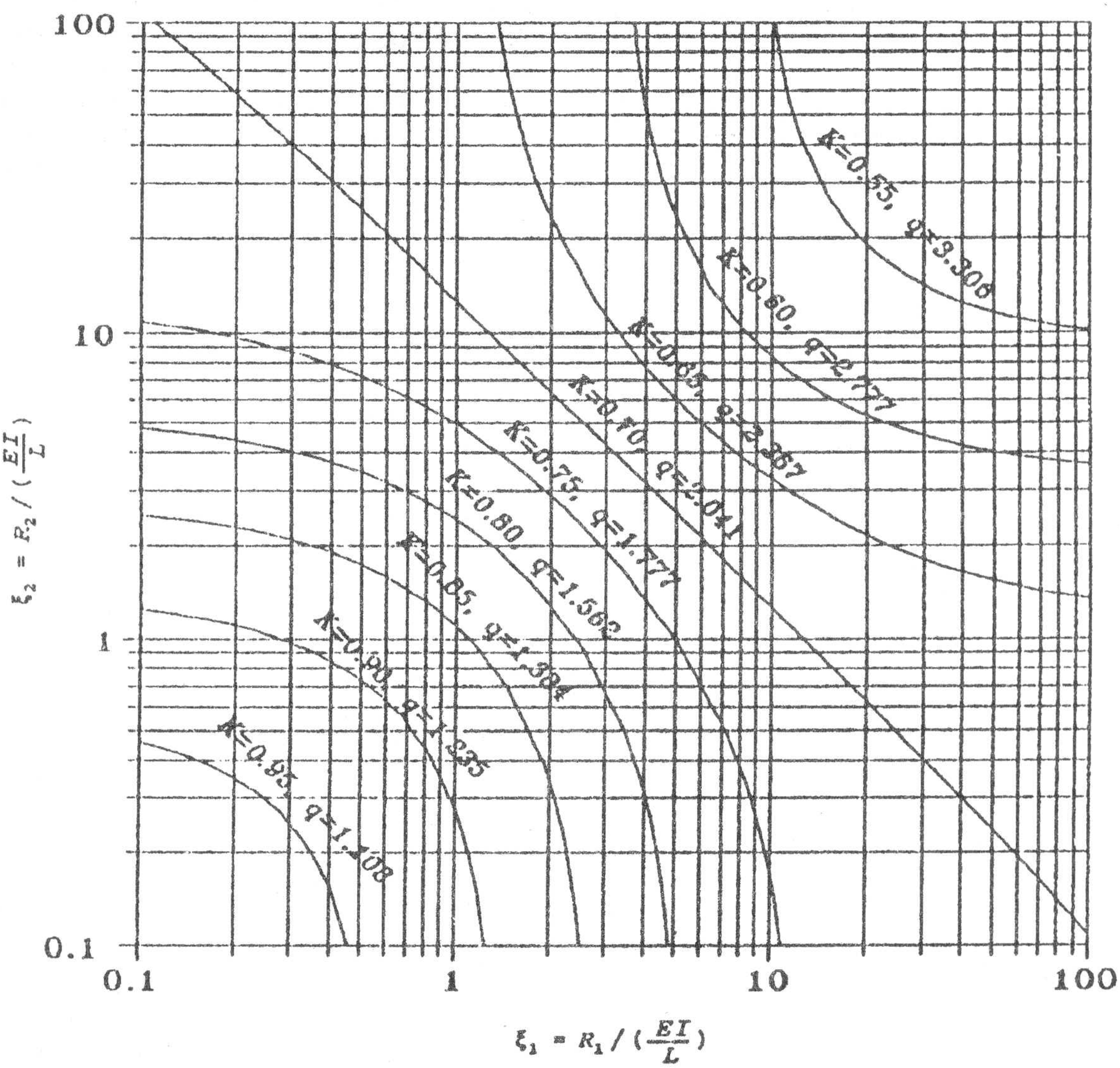

Figure 7. Aided design curves (efective length factor, k and critical load parameters, $q=p_{cr}/p_E$, vs. ratio of rotational end restraint).

DESIGN CURVES

An alternative way to the above mentioned numerical determination of K and q is by using the design curves shown in Figures (4-b), (5-b) and (7). These curves are generated by substituting in the neutral equilibrium equation (Eqn. 30) the value of q and computing the corresponding values of ξ_1 & ξ_2 that satisfy these equations. To obtain the value of q or K from these graphs for a column with given values of ξ_1 and ξ_2, read off ξ_1 and ξ_2 along the horizontal and vertical axes of Figure (7). Locate the corresponding point on the graph. The curve that can be interpolated using the two neighboring curves and runs through this point is the one representing the right values of q and K. It is worth noting that the values of K or q at the points of intersection of the design curves of Figure (7) with the diagonal line of symmetry are the same values given in Figure (4-b) for equally rotationally restrained columns. Also, it can be noted that the points of intersections of these curves with the vertical or horizontal axes give almost the same value of K or q given in Figure (5-b) for a column pinned at one end and rotationally restrained at the other end. This ensures that the graph of Figure (7) is general in predicting the critical buckling load of a column with different degrees of end restraints.

COMMENTS ON THE ALIGNMENT CHART USED BY THE EGYPTIAN CODE OF PRACTICE

The values of ξ_1 & ξ_2 used in the design aid curves shown in Figure (7) are related to G_1 & G_2 used in the alignment chart of Figure (2) that has been adopted by the Egyptian Code of Practice for Steel Structures and Bridge ***(ECPSSB)*** of 1989 as follow:

$$\xi_1 = \frac{1}{G_1} \quad \& \quad \xi_2 = \frac{1}{G_2} \qquad (31)$$

The results obtained from the design curves presented in Figure (7) are different from those obtained from the charts in Figure (2). For example if $\xi_1 = \xi_2 = 10$ (***i.e.***, $G_1 = G_2 = 0.1$) Figure (7) gives K=0.595 and q=2.825; however Figure (2) gives K=0.55 and q=3.305. This means that the charts adopted by the ***ECPSSB*** give unsafe critical load. In this case the values of K predicted by the ***ECPSSB*** is less than that given in this study by 8.2%. Also, the value of q predicted by the ***ECPSSB*** is higher than that given in this study by 15%. This observation has been noticed for all values of ξ_1 & ξ_2 but with variable ratios of the difference between the results of the ***ECPSSB*** and the present approach.

CONCLUSION

Exact expressions for the neutral equilibrium of rotationally restrained columns with variable end conditions have been derived. The minimum value that satisfies these equations gives the exact critical buckling load of the column. A procedure to obtain the solution of the neutral equilibrium equations using a computer program is presented. Aided design curves using the derived equations are produced. These curves can be used easily for accurate column design. Safe critical buckling loads for columns compared with those adopted by the Egyptian Code of Practice for Steel Structures and Bridges are obtained.

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