BENDING OF PLATES UNDER CERTAIN BOUNDARY CONDITIONS

Bahia N. El-Hamaky and Ali Okaz
Mathematical and Physical Sciences Dept., Faculty of Engineering,
Alexandria University, Alexandria, Egypt.

ABSTRACT

This work gives the solution to the problem of bending plates with mixed (discontinuous) boundary conditions in the case of elastic clamping and elastic support along the boundary. Muschelisvili's method has been used to determine the deformation of the plate for the case of non-vanishing Winker's constant and the constant of elastic clamping.

INTRODUCTION

During the last few years, there has been an apparent growing interest in the problems of structural analysis of plates with mixed (discontinuous) boundary conditions. These problems can be solved in many ways: by reducing the problem to a system of Fredholm integral equations of the first kind [1], by the method of analytic functions [2], or directly by integrating the integral equations of structural analysis of plates considering both elastic clamping and elastic support along the boundary. The present article contains the solution to the problem of bending plates with mixed boundary conditions, the case of plate in which part of it is elastically clamped and the rest is elastically supported. The solution is reduced to a system of Fredholm equations of the 2nd kind which in certain particular cases, becomes Fredholm equations of the 1st kind. The formulation of the differential equations is given in section (1) while the transformations to integral equations are presented in section (2).

In considering the case of bending a plate by a distributed load acting perpendicular to its middle plane, let us assume that this middle plane is horizontal and contain the x-and y-axes, while the z-axis is directed vertically downward. We denote by $q=q(x,y)$ the intensity of the load which is a function of x and y. Let us assume some approximations as follows:

1. The plate's thickness (h) is constant and its medium remains also constant so that the points make only the displacements $u = u(x,y)$.
2. The deformation τ_z is very small independent on z and can be neglected.
3. The perpendicular to the plane remains so, even after deformation.
4. There will be no change in the plate's energy because the stresses τ_z, τ_{yz}, and τ_{xz} are equal to the remaining stresses.

The above assumptions are considered in order to apply the thin plate theory of small deflections.

1. *The differential equations of the boundary value problem:*

Suppose that the coordinate of points on the boundary have the 4th derivatives w.r.t. the arcs length s. The boundary can be divided into 2n arcs L_k which take the limits from a_1 to a_{n+1} where $k = 1,2,3,\dots 2n$, with $a_{2n+1} = a_1$

Then:

$$L^{(1)} = \sum_{k=1}^{n} L_{2k-1} \quad , \quad L^{(2)} = \sum_{k=1}^{n} L_{2k}$$

This means that the summation of the parts of the curves L_{2k-1} and L_{2k} are denoted by $L^{(1)}$ and $L^{(2)}$ respectively. According to Kirchoff's theory of the bending plates [3], the function u(x,y) must satisfy the equation:

$$\Delta\Delta u = \frac{q}{D} \tag{1.1}$$

The boundary conditions are:

$$\left.\begin{matrix} M(u)=C_a\alpha \\ \\ u=0 \end{matrix}\right\} \text{on } L^{(1)} \qquad \left.\begin{matrix} K(u)=0 \\ \\ M(u)=0 \end{matrix}\right\} \text{on } L^{(2)} \qquad (1.2)$$

where C_1 is the Winkler's constant, α is the angle of rotation of the plate (unclamped), M and K are the momment and force functions of u given by:

$$M(u)=\sigma\,\Delta u+(1-\sigma)[u_{xx}\cos^2\theta+2u_{xy}\cos\theta\sin\theta+u_{yy}\sin^2\theta] \qquad (1.3)$$

and

$$K(u)=\frac{d\Delta u}{dn}+(1-\sigma)\frac{d}{ds}[u_{xy}\cos 2\theta+\frac{1}{2}(u_{yy}-u_{xx})\sin 2\theta] \qquad (1.4)$$

and the expressions for K and M multiplied by -D, where D is the flexural rigidity of the plate, n is the outward normal to L, θ is the angle between n and x-axes, σ is Poisson's ratio. We can write the above conditions as follows:

$$\left.\begin{matrix} G_1(u)=M(u)+C_1\alpha=0 \\ \frac{\partial u}{\partial s}=0 \end{matrix}\right\} \text{on } L^{(1)} \qquad (1.5)$$

We can obtain two special cases from this boundary conditions; if $C_1=\infty$, the edge of the plate is clamped, this means the boundary condition are

$$u=0 \quad , \quad \frac{\partial u}{\partial n}=0$$

When $C_1 = 0$, the edge of the plate is simply supported and the boundary conditions will be written in the form $u = 0$, $M = 0$

Because $u = 0$ on $L^{(1)}$ we can substitute $\frac{\partial u}{\partial s}=0$ also;

The angle α can be approximated as follows:

$$\tan\alpha = \frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\cos(x,n) + \frac{\partial u}{\partial y}\cos(n\ ,\ y)$$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

Because α is considered to be small i.e $\tan\alpha \simeq \alpha$ then

$$\tan\alpha = \alpha = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \qquad (1.6)$$

so the boundary conditions (1.2) and (1.5) can be written as:

$$G_1(u) = -D[\sigma\Delta u+(1-\sigma)[u_{xx}\cos^2\theta+2u_{xy}\cos\theta\sin\theta+u_{yy}\sin^2\theta]]$$

$$+ C_1(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta) = 0$$

$$\frac{\partial u}{\partial s} = 0 \qquad (1.7)$$

After multiply the 1st part of equation (1.2) by i and integrating over s, and adding the result to the 2nd part of (1.2), one obtains:

$$G_2(U) = M(U) + \int_s K(U)ds + iC_{2k} \qquad (1.8)$$

The general solution of equation (1.1) takes the form [5]:

$$u = U + W \qquad (1.9)$$

where W is the special solution and U is the harmonic function satisfying the harmonic equation:

$$\Delta\Delta U = 0 \qquad (1.10)$$

The function W is supported to be a convergent integral function and its partial derivatives up to the 3rd order are continuous functions in the region (S+L), implying that the load function $q = q(x,y)$ is also a convergent integral function. This is enough for the load function q to satisfy the Hölder-condition [4] in the region mentioned above. Instead of using u in equations (1.7) & (1.8), we use its sibstitute equation (1.9), where U takes the form [5]:

$$U = 2\text{Re}[\bar{z}\phi(z) + x(z)] \qquad (1.11)$$

where $\varphi(z)$ and x(z) are homogeneous functions,

$$\psi(z) = [(dx(z)/(dz)] = x'(z)$$

This form of U can take an easier and preferable form as:

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = z\bar{\varphi}(\bar{z}) + \varphi(z) + \bar{\psi}(\bar{z}) \quad (1.12)$$

The function $\varphi(z)$ is determined as well by an arbitrary complex constant especially if $U(x,y)=0$ in S, and this together with equation (1.12) implies that [6,7]:

$$\varphi(z) = iCZ + \gamma$$
$$\chi(z) = \bar{\gamma}Z + b \quad (1.13)$$
$$\chi(z) = -\bar{\gamma}$$

where b is another arbitrary complex constant, then one can apply without approximation, the general condition on the function $\varphi(z)$ which is $\mathrm{Im}\, \varphi'(0) = 0$. We would now like to solve this problem through the substitution (1.9) into (1.7) & (1.8), we obtain:

$$G_1(W) = -D[\sigma\Delta U + (1-\sigma)[U_{xx}\cos^2\theta + 2U_{xy}\cos\theta\sin\theta + U_{yy}\mathrm{Sin}^2\theta]] \quad (1.14)$$
$$+ C_1(U_x\cos\theta + U_y\sin\theta)$$

$$\frac{\partial U}{\partial s} = -\frac{\partial W}{\partial s} \quad \text{on} \quad L^{(1)} \quad (1.15)$$

$$M(U) + i\int_s K(U)ds = -[M(W) + i\int_s K(W)ds] + iC_{2k}, \text{on } L^{(2)} \quad (1.16)$$

Substituting equation (1.11) into (1.15) & (1.16) we obtain:

$$2(\sigma-1)\mathrm{Re}[\frac{d}{dt}(-m\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}] + 2C_1\mathrm{Im}[\bar{t}'(t\overline{\varphi'(t)} + \varphi(t) + \overline{\psi(t)}]$$
$$= G_1(W) \quad (1.17)$$

$$2\mathrm{Re}[\bar{t}'(t\bar{\varphi}(\bar{t}) + \varphi(t) + \overline{\psi t)}] = -\frac{\partial W}{\partial s} \quad (1.18)$$

Dividing eq. (1.17) over C_1 and multiplying (1.18) by i, then adding the result, to equation (1.18) we get:

$$\frac{2(1-\sigma)}{C_1}\mathrm{Re}[\frac{d}{dt} - m\phi(t) + \dot{\phi}'(t) + \overline{\psi(t)}]$$

$$-2\mathrm{Im}[t\overline{\phi'(t)} + \phi(t) + \overline{\psi(t)}] = \frac{1}{C_1}G_1(W) - i\frac{\partial w}{\partial s} \quad (1.19)$$

$$2\mathrm{Re}[\bar{t}(t\overline{\phi'(t)} + \phi(t) + \overline{\psi(t)})] = -\frac{\partial W}{\partial s} \quad (1.20)$$

After substituting eq. (1.11) into (1.8), the boundary condition on $L^{(2)}$ takes the form:

$$\frac{d}{dt}[-m\phi(t) + t\overline{\phi'(t)} + \overline{\psi(t)}] = \frac{1}{2(1-\sigma)}[M(W) + i\int_s K(W)ds] + iC_{2k} \quad (1.21)$$

Integrating over t, one gets:

$$-m\phi(t) + t\overline{\phi}(t) + \overline{\psi(t)}$$
$$= \frac{1}{2(1-\sigma)}[\int_s [M(W) + i\int_s K(W)ds]dt + iC_{2k}t + \beta_{2k} \quad (1.22)$$

where t is a variable point on the curve $L_{2k} = a_{2k}a_{2k+1}$, and s the length of the arc connecting the point a_{2k} to the point a_{2k+1}, C_{2k} are unknown real constants on the curve L_{2k} and C_{2k} are generally complex undermined constants on L_{2k}.

The purpose of this work is to find the functions $\varphi(z)$ & $\chi(z)$ which both are homogeneous in S and satisfy the conditions (1.20), (1.21) and (1.23). Integrating equation (1.2) over s, we obtain

$$\frac{2(1-\sigma)}{C_1}\mathrm{Re}[\bar{t}(-m\phi(t) + t\overline{\phi'(t)} + \overline{\psi(t)}$$
$$-\int_{s_{2k-1}}^{s} \bar{t}(-m\phi(t) + t\overline{\phi(t)} + \overline{\psi(t)})ds$$
$$+ 2i\int_{s_{2k-1}}^{s} \bar{t}[t\dot{\phi}(t) + \phi(t) + \overline{\psi(t)}]ds \quad (1.23)$$
$$= \int_{s_{2k-1}}^{s}(\frac{1}{C_1}G_1(W) - i\frac{\partial W}{\partial s})ds + C_{2k}$$

2. *The transformation to an integral equation*

We will consider the two functions $\phi(z)$ and $\psi(z)$ as follows [8].

$$\phi(z) = \frac{1}{2\pi i}\int_L \frac{\omega(t)dt}{t-z} \quad (2.1)$$

$$\psi(z)=\frac{-m}{2\pi i}\int_L\frac{\overline{\omega(t)}}{(t-z)}dt+\frac{1}{2\pi i}\int_L\frac{\omega(t)\overline{dt}}{(t-z)}-\frac{1}{2\pi i}\int_L\frac{\bar{t}\omega(t)dt}{(t-z)^2} \qquad (2.2)$$

where $\omega(t) = \omega_1(t) + i\omega_2(t)$, is function of t, satisfies Hölder-condition on the curve L. Suppose that $\omega(t)$ is continuous function and $\omega'(t)$ is its first derivative, so it is possible to transform the boundary conditions (1.20), (1.21) and (1.23) one finds:

$$\frac{1}{2}(1-m)(\bar{t}_o\omega(t_o)+\dot{t}_o\overline{\omega(t_o)})+2\mathrm{Re}[\bar{t}_oK(t_o)-\frac{(1+m)\dot{t}_o}{2}\int_L\overline{\omega(t)}d\log\frac{\overline{t-t_o}}{t-t_o}$$

$$+\frac{(1+m)}{2\pi i}\int_L\frac{\bar{t}_o\omega(t)-\dot{t}_o\overline{\omega(t)}}{(t-t_o)}dt]=g_1(W) \qquad (2.3)$$

$$\frac{1}{2}[\bar{t}_o\omega(t_o)+\dot{t}\overline{\omega(t_o)}]-\frac{1}{m}\mathrm{Re}\left[\bar{t}_oK(t_o)+\int_{s_{2k-1}}^{s}\bar{t}(m\omega(t_o)-K(t_o))ds\right]$$

$$-\frac{iC_1}{2(1-\sigma)m}[\int_{s_{2k}}^{s}\dot{t}\left\{\frac{1}{2}(1-m)\omega(t_o)+\frac{(1+m)}{2\pi i}\int_L\frac{\omega(t)dt}{t-t_o}+K(t_o)\right\}ds]$$

$$=\frac{-C_1}{2m(1-\sigma)}[g_3+\alpha_{2k-1}] \qquad L^{(1)} \qquad (2.4)$$

and

$$-m\omega(t)+K(t_o)=f(t)+iC_{2k}t+\beta_{2k} \quad \text{on} \quad L^{(2)} \qquad (2.5)$$

where

$$K(t_o)=\frac{m}{2\pi i}\int_L\omega(t)d\log\frac{\overline{t-t_o}}{t-t_o}-\frac{1}{2\pi i}\int_L\overline{\omega(t)}d\frac{t-t_o}{\overline{t-t_o}} \qquad (2.6)$$

$$f(t)=\frac{1}{2(1-\sigma)}[\int_{s_{2k}}M(W)+i\int_{s_{2k}}K(W)ds]dt \qquad (2.7)$$

It is easy to prove that the equations (2.3), (2.4) and (2.5), which are used to determine ω_1 (t) and ω_2 (t) represent a unique group of integral equations whose parameters are discontinuous and whose right hand sides contain unknown functions as well as fixed constants α_{ak-1}, C_{2k} and β_{2k}. From (2.3) and (2.4), we obtain:

$$\frac{1}{2\pi i}\int_L\frac{\bar{t}_o\omega(t)-\dot{t}_o\overline{\omega(t)}}{t-t_o}dt+\frac{1}{m}\mathrm{Re}[\bar{t}_oK(t_o)$$

$$+\frac{(1-m)}{2\pi i(1+m)}\int_{s_{2k}}^{s}\overline{t''}(m\omega(t)-K(t)ds]$$

$$-\frac{\bar{t}_o}{2\pi i}\int_L\bar{\omega}d\log\frac{\overline{t-t_o}}{t-t_o}+\frac{iC_1(1-m)}{2(1-\sigma)(1+m)m}$$

$$\times\left[\int_{s_{2k}}^{s}\bar{t}'[\frac{1}{2}(1-m)\omega(t_o)+\frac{(1+m)}{2\pi i}\int_L\frac{\omega dt}{t-t_o}+K(t_o)]ds\right] \qquad (2.8)$$

$$=g^{+}+\overset{+}{\alpha}_{2k-1} \qquad L^{(1)}$$

where

$$g^{+}=g_1+\frac{(1-m)}{2m(1-\sigma)}g_3 \qquad (2.9)$$

$$\overset{+}{\alpha}_{2k-1}=\frac{(1-m)}{2m(1-\sigma)}\alpha_{2k-1} \qquad (2.10)$$

The equation (2.3), (2.4) and (2.5) are used to determine ω_1 (t) and $\omega_2(t)$, $[\omega(t) = \omega_1(t) + i\omega_2(t)]$ represent a unique group of integral equations whose parameters are discontinuous and whose right hand sides contain besides the known functions, the undetermined contants α_{2k+1}, C_{2k} and β_{2k}.

By using the matrix notation, we can write down this system of integral equations in the form of a single equality as follows:

$$A(t_o)\Omega(t_o)+\frac{B(t_o)}{\pi i}\int_L\frac{\Omega(t)dt}{t-t_o}+M(t_o)=F(t_o)+\beta(t_o) \qquad (2.11)$$

where $\Omega(t) = (\omega_1, \omega_2)$ is the required vector, A and B are known matrices, $\beta = (\beta_1, \beta_2)$ is a vector whose components contain the undetermined constants

$$A=\begin{bmatrix}0 & 0\\ -\sin\theta & \cos\theta\end{bmatrix}, B=\begin{bmatrix}-i\cos\theta & -i\sin\theta\\ 0 & 0\end{bmatrix} \quad L^{(1)} \qquad (2.12a)$$

and

$$A = \begin{bmatrix} -m & 0 \\ 0 & -m \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad L^{(2)} \quad (2.12b)$$

We also obtain the following relations:

$$M_1(t) = \frac{1}{m}\mathrm{Re}[\bar{t}_o K(t_o) + \frac{(1-m)}{m(1+m)}\int_{s_{2k-1}}^{s} \bar{\bar{t}}(m\omega(t_o) - K(t_o))ds]$$

$$+\frac{iC_1(1-m)}{m(1-\sigma)(1+m)}\int_{s_{2k-1}}^{s} \bar{\bar{t}}[\frac{1}{2}(1-m)\omega(t_o) + \frac{1+m}{2\pi i}\int_L \frac{\omega dt}{t-t_o} + K(t_o)]ds$$

$$+\frac{\dot{t}_o}{2\pi i}\int_L \bar{\omega} d\log\frac{t-t_o}{\bar{t}-\bar{t}_o} \qquad L^{(1)} \qquad (2.13a)$$

$$M_2(t) = -\frac{1}{m}\mathrm{Re}(\bar{t}_o K(t_o) + \int_{s_{2k-1}}^{s} \bar{\bar{t}}[m\omega(t_o) - K(t_o)ds]$$

$$-\frac{iC_1}{m(1-\sigma)}\int_{s_{2k-1}}^{s} \bar{\bar{t}}[\frac{1}{2}(1-m)\omega(t_o) + \frac{(1+m)}{2\pi i}\int_L \frac{\omega dt}{t-t_o} + K(t_o)]ds \quad (2.13b)$$

$$F_1 = g^+, F_2 = -\frac{C_1}{2m(1-\sigma)}g_3 \qquad (2.14)$$

$$\beta_1 = \overset{+}{\alpha}_{2k-1}, \beta_2 = -\frac{C_1}{2m(1-\sigma)}\alpha_{2k-1} \qquad (2.15)$$

where

$$M = M_1 + i\,M_2,\ F = F_1 + iF_2,\ B = B_1 + iB_2 \quad (2.16)$$

the following relations can also be obtained:

$$F_1 + iF_2 = g^+ - \frac{iC_1}{2m(1-\sigma)}g_3 \qquad L^{(1)} \ (2.17)$$

$$F_1 + iF_2 = f(t) \qquad L^{(2)} \qquad (2.18)$$

$$\beta_1 + i\beta_2 = \overset{+}{\alpha}_{2k-1} - \frac{iC_1}{2m(1-\sigma)}\alpha_{2k-1} \qquad L^{(1)} \ (2.19)$$

$$\beta_1 + i\beta_2 = iC_{2k}t + \beta_{2k} \qquad L^{(2)} \quad (2.20)$$

It can be seen that $M = (M_1, M_2)$ represents a vector whose components are integral parameters of Fredholm type and the components of the vector $F = (F_1, F_2)$ are known through the particular solution W and vanish when $W = 0$ .It is also easy to write down the explicit expressions for the components of M and F. Denote:

$$S^* = A + B, \quad D^* = A - B, \quad g = S^{*-1}D$$

from expressions (2.12a) & (2.12b), we have

$$g = \begin{bmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \qquad L^{(1)} \ (2.21a)$$

$$g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad L^{(2)} \qquad (2.12b)$$

As can easily be verified, let $\det S^* \neq 0$, $\det D^* \neq 0$, everywhere on L, and in consequence to system (2.11) may be applied the existing theory of systems of singular integral equations with discontinuous coefficients [9]. For the application of this theory it is necessary to determine the roots of the equation (I is a unit matrix)

$$\det\{g^{-1}(t+0)\,g\,(t-0) - I\lambda\} = 0$$

at the points of discontinuity of the coefficients of the integral equations. These are the points of change of the boundary conditions a_k ($k = 1, \ldots, m$). From (2.12a), (2.12b) we obtain:

$$\det\{g^{-1}(b_k + 0)\,g\,(b_k - 0) - I\lambda\} = \lambda^2 - 1$$

$$\det\{g^{-1}(c_j + 0)\,g(c_j - 0) - I\lambda\} = \lambda^2 - 1$$

$$(k = 1, \ldots, m_1,\ j = 1, \ldots, m_2)$$

CONCLUSIONS

The advantage of the present study is the transformation of the differential equation into a singular integral of Fredholm type, containing the unknown complex valued function W(t). From this case, it is also easy to obtain the solution of two cases: (i) if $C_1 = 0$ the plate is simply supported [8] and (ii) if $C_1 = \infty$, the plate is fixed, these two cases were studied by [8].

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