A GENERALIZED THERMOELASTIC PROBLEM FOR A SOLID SPHERE

Farid A. Hamza
Department of Mathematics Faculty of Science,
University of Alexandria, Alexandria, Egypt.

ABSTRACT

In this work the one dimensional problem of a solid sphere whose surface is traction free and subjected to a given time dependent temperature distribution is considered within the context of the theory of generalized thermoelasticity with one relaxation time. Potential function and Laplace transform techniques are used to derive the solution in the transformed domain. The inversion of the Laplace transform is carried out using the inversion formula of the transform together with some techniques for the acceleration of convergence. Numerical results are given and represented graphically. Comparison is made with the solution of the corresponding coupled problem.

NOMENCLATURE

λ, μ	Lame's constants
ρ	density
$c_1^2 = (\lambda+2\mu)/\rho$	square of the velocity of isothermal longitudinal waves
$c_2^2 = \mu/\rho$	square of the velocity of transverse waves
c_E	specific heat at constant strain
k	thermal conductivity
t	time
T	absolute temperature
T_o	reference temperature chosen so that $\lvert (T-T_o)/T_o \rvert << 1$
u'	component of the displacement vector in radial direction
α_t	coefficient of linear thermal expansion
β	c_1 / c_2
γ	$(3\lambda+2\mu)\alpha_t$
b	$\gamma T_o/\mu$
g	$\gamma/\rho c_E$
ϵ	$\gamma^2 T_o/\rho c_E(\lambda+2\mu)$
η	$\rho c_E/k$
σ'_{ij}	components of stress tensor
τ'_o	relaxation time

INTRODUCTION

The theory of generalized thermoelasticity with one relaxation time was developed by Lord and Shulman [1] for isotropic media and extended by Dhaliwal and Sherief to anisotropic bodies. This theory is based on a modified law of heat conduction to replace Fourier's law. The equation of heat conduction of this theory is hyperbolic and therefore eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and coupled theories of thermoelasticity. For many problems involving steep heat gradients and when short time effects are sought this theory is indispensable. The fundamental solutions for the spherically symmetric and the cylindrically symmetric spaces were obtained by Sherief [3] and by Sherief and Anwar [4], respectively. A two dimensional axisymmetric problem for a thick plate was considered by Sherief and Hamza in [5] where also a full discussion of wave propagation in generalized thermoelasticity is presented.

FORMULATION OF THE PROBLEM

We consider a homogeneous isotropic thermoelastic solid sphere of radius a' whose surface is taken to be traction free and subjected to a known time dependent temperature distribution. Due to the symmetry of the problem all functions are dependent on the radial distance r only and the displacement vector will have only one non-vanishing component in the radial direction.

To simplify the formulation of the problem, we introduce the following non-dimensional variables

$$r = c_1\eta r,\ a = c_1\eta a',\ u = c_1\ \eta u',$$

$$t = c_1^2\eta t', \ \tau_o = c_1^2\eta\tau'_o,$$

$$\sigma'_{rr} = \sigma_{rr}/\mu \quad \text{and} \quad \theta = (T-T_o)/T_o .$$

In terms of these non-dimensional variables, the equation of motion takes the form [3]

$$\beta^2 \frac{\partial^2 u}{\partial t^2} = \beta^2 \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r} + \frac{2u}{r}\right] - b\frac{\partial \theta}{\partial r}. \qquad (1)$$

The equation of heat conduction can be written as

$$\nabla^2\theta = \left\{\frac{\partial}{\partial t} + \tau_o\frac{\partial^2}{\partial t^2}\right\}\theta + g\left\{\frac{\partial}{\partial t} + \tau_o\frac{\partial^2}{\partial t^2}\right\}\left[\frac{\partial u}{\partial r} + \frac{2u}{r}\right], \qquad (2)$$

where ∇^2 is Laplace's operator given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}.$$

The constitutive relation takes the form

$$\sigma_{rr} = \beta^2 \frac{\partial u}{\partial r} + 2(\beta^2 - 2)\frac{u}{r} - b\theta. \qquad (3)$$

Introducing the thermoelastic potential function ϕ defined by the relation

$$u = \frac{\partial \phi}{\partial r}, \qquad (4)$$

and integrating the first resulting equation with respect to r, we obtain

$$\beta^2 \nabla^2 \phi - b\theta = \beta^2 \frac{\partial^2 \phi}{\partial t^2}, \qquad (5)$$

$$\nabla^2 \theta = \left\{\frac{\partial}{\partial t} + \tau_o \frac{\partial^2}{\partial t^2}\right\}[\theta + g\nabla^2\phi], \qquad (6)$$

$$\sigma_{rr} = \beta^2\nabla^2\phi - \frac{4}{r}\frac{\partial \phi}{\partial r} - b\theta, \qquad (7)$$

Using equation (5), equation (7) can be written in a simpler form as

$$\sigma_{rr} = \beta^2 \frac{\partial^2 \phi}{\partial t^2} - \frac{4}{r}\frac{\partial \phi}{\partial r}, \qquad (8)$$

We shall assume that all the initial conditions of the problem are homogeneous. The boundary conditions of the problem are taken as follows:

(i) The mechanical boundary condition that the surface of the sphere is traction free that is

$$\sigma_{rr}(a,t) = 0 . \qquad (9)$$

(ii) The thermal boundary condition

$$q_r(a,t) = h\,[\theta(a,t) - F(t)] , \qquad (10)$$

where q_r is the component of the heat flux vector in the radial direction, h is the coefficient of heat transfer on the surface of the sphere and the function F(t) is the temperature of the surrounding medium.

SOLUTION IN THE TRANSFORMED DOMAIN

Taking the Laplace transform defined by the relation

$$\bar{f}(s) = \mathscr{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt,$$

of both sides of equations (5), (6) and (8) and using the homogeneous initial conditions, we get

$$(\nabla^2 - s^2)\bar{\phi} = c\bar{\theta}, \qquad (11)$$

$$(\nabla^2 - s - \tau_o s^2)\bar{\theta} = g(s + \tau_o s^2)\nabla^2\bar{\phi}, \qquad (12)$$

$$\bar{\sigma}_{rr} = \beta^2 s^2 \bar{\phi} - \frac{4}{r}\frac{\partial \bar{\phi}}{\partial r}, \qquad (13)$$

where $c = b/\beta^2$.

Eliminating $\bar{\theta}$ between equations (11) and (12), we obtain the following equation for $\bar{\phi}$

$$\{\nabla^4 - [(1+\tau_o+\epsilon\tau_o)s^2 + (1+\epsilon)s]\nabla^2 + s^3(1+\tau_o s)\}\bar{\phi} = 0.$$

This equation can be factorized as

$$(\nabla^2-k_1^2)(\nabla^2-k_2^2)\,\bar{\phi} = 0\,, \tag{14}$$

where $\pm k_1^2$ and $\pm k_2^2$ are the roots of the characteristic equation

$$k^4-[(1+\tau_o+\epsilon\tau_o)s^2+(1+\epsilon)s]\,k^2+s^3\,(1+\tau_o s)=0\,.$$

The solution of equation (14) can be written in the form

$$\bar{\phi} = \bar{\phi}_1 + \bar{\phi}_2\,,$$

where $\bar{\phi}_i$ is the solution of the equation

$$(\nabla^2-k_i^2)\bar{\phi}_i = 0,\; i=1,2.$$

These equations can be solved to give

$$\bar{\phi} = \frac{c}{\sqrt{r}}\sum_{i=1}^{2} A_i I_{1/2}(k_i r), \tag{15}$$

where I_n denotes the modified Bessel function of order n and A_i, i=1,2 are parameters depending on s to be determined from the boundary conditions.
Substituting from equation (15) into equation (11), we obtain

$$\bar{\theta} = \frac{1}{\sqrt{r}}\sum_{i=1}^{2} A_i\,(k_i^2-s^2)\,I_{1/2}(k_i r). \tag{16}$$

From equations (13) and (15), it follows that

$$\bar{\sigma}_{rr} = \frac{c}{\sqrt{r}}\sum_{i=1}^{2} A_i\left\{\beta^2 s^2 I_{1/2}(k_i r) - \frac{4k_i}{r} I_{3/2}(k_i r)\right\}, \tag{17}$$

where, we have used the recurrence relation of the modified Bessel functions, namely

$$\frac{d I_{1/2}(kr)}{dr} = \frac{1}{2r} I_{1/2}(kr) + k\,I_{3/2}(kr)$$

Finally, to obtain the displacement we substitute from (15) into (4), to obtain

$$\bar{u} = \frac{c}{\sqrt{r}}\sum_{i=1}^{2} A_i k_i I_{3/2}(k_i r). \tag{18}$$

The boundary conditions (9) together with equation (17), yield the equation

$$\sum_{i=1}^{2} A_i\left\{\beta^2 s^2 a I_{1/2}(k_i a) - 4k_i I_{3/2}(k_i a)\right\}=0. \tag{19}$$

Applying the Laplace transform to both sides of (10), this condition takes the form

$$\bar{q}_r(a,s) = h\left[\bar{\theta}(a,s) - \bar{F}(s)\right]. \tag{20}$$

We shall use the generalized Fourier's law of heat conduction in non-dimensional form, namely

$$q_r + \tau_o\frac{\partial q_r}{\partial t} = -\frac{\partial \theta}{\partial r}. \tag{21}$$

Taking Laplace transforms of (20)-(21) and eliminating $\bar{q}_r$, we obtain

$$\frac{\partial\bar{\theta}}{\partial r} = h\,(1+\tau_o s)\,(\bar{F}-\bar{\theta})\,,\ \text{for } r=a. \tag{22}$$

Using equations (16) and (22), we obtain

$$\sum_{i=1}^{2} A_i\,(k_i^2-s^2)\left[k_i I_{3/2}(k_i a) + h(1+\tau_o s) I_{1/2}(k_i a)\right] = \sqrt{a}\,h(1+\tau_o s)\bar{F}(s) \tag{23}$$

Equations (19) and (23) can be solved to give

$$A_1 = \frac{h(1+\tau_o s)\sqrt{a}\bar{F}(s)}{\Lambda}\left[\beta^2 s^2 a I_{1/2}(k_2 a) - 4k_2 I_{3/2}(k_2 a)\right],$$

$$A_2 = \frac{-h(1+\tau_o s)\sqrt{a}\bar{F}(s)}{\Lambda}\left[\beta^2 s^2 a I_{1/2}(k_1 a) - 4k_1 I_{3/2}(k_1 a)\right],$$

where

$$\Lambda = k_1^2\,[\beta^2 s^2 a\, I_{1/2}(k_2 a) - 4 k_2 I_{3/2}(k_2 a)][k_1 I_{3/2}(k_1 a) + h(1+\tau_o s) I_{1/2}(k_1 a)]$$
$$- k_2^2\,[\beta^2 s^2 a\, I_{1/2}(k_1 a) - 4 k_1 I_{3/2}(k_1 a)][k_2 I_{3/2}(k_2 a) + h(1+\tau_o s) I_{1/2}(k_2 a)]$$
$$+ s^2 [\beta^2 s^2 a + h(1+\tau_o s)][k_2 I_{1/2}(k_1 a) I_{3/2}(k_2 a) - k_1 I_{1/2}(k_2 a) I_{3/2}(k_1 a)].$$

This completes the solution of the problem in the Laplace transform domain.

SOLUTION IN THE PHYSICAL DOMAIN

We shall now outline the method used to invert the Laplace transforms in the above equations. Let $\bar{f}(s)$ be the Laplace transform of a function f(t). The inversion formula for Laplace transforms can be written as

$$f(t) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}\bar{f}(s)\,ds,$$

where d is an arbitrary real number greater than all the real parts of the singularities of $\bar{f}(s)$.
Taking s = d + i y , the above integral takes the form

$$f(t) = \frac{e^{dt}}{2\pi}\int_{-\infty}^{\infty} e^{ity}\,\bar{f}(d+iy)\,dy.$$

This integral can be approximated by [6]

$$f(t) = \frac{e^{dt}}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikt\Delta y}\bar{f}(d+ik\Delta y)\,\Delta y + E_D,$$

where E_D , the discretization error, can be made arbitrarily small by choosing d large enough [6].
Taking $\Delta y = \pi/L$, we obtain

$$f(t) = \frac{e^{dt}}{2L}\sum_{k=-\infty}^{\infty} e^{ik\pi t/L}\,\bar{f}(d+ik\pi/L) + E_D$$

This can be written in the form

$$f(t) = \frac{e^{dt}}{L}\left[\tfrac{1}{2}\bar{f}(d) + \mathrm{Re}\sum_{k=1}^{\infty} e^{ik\pi t/L}\,\bar{f}(d+ik\pi/L)\right] + E_D,$$

The above series can be shown to be convergent for $0 \le t \le 2L$ [7].

This series can be written more concisely in the form

$$f(t) = f_\infty(t) + E_D,$$

where

$$f_\infty(t) = \tfrac{1}{2} c_0 + \sum_{k=1}^{\infty} c_k, \quad \text{for } 0 \le t \le 2L, \tag{24}$$

and

$$c_k = \frac{e^{dt}}{L}\mathrm{Re}\left[e^{ik\pi t/L}\,\bar{f}(d+ik\pi/L)\right], \tag{25}$$

E_D, the discretization error, can be made arbitrarily small by choosing d large enough [7] .

As the infinite series in (24) can only be summed up to a finite number N of terms, the approximate value of f(t) becomes

$$f_N(t) = \tfrac{1}{2} c_0 + \sum_{k=1}^{N} c_k, \quad \text{for } 0 \le t \le 2L. \tag{26}$$

Using the above formula to evaluate f(t), we introduce a truncation error E_T that must be added to the discretization error to produce the total approximation error.

We shall now describe the ϵ-algorithm which is used to accelerate the convergence of the series in (24). Let N be an odd natural number and let

$$s_m = \sum_{k=1}^{m} c_k,$$

be the sequence of partial sums of (24). We define the ϵ-sequence by

$$\epsilon_{0,m} = 0 \;, \; \epsilon_{1,m} = s_m \quad , \quad m = 1,2,3,\ldots$$

and

$$\epsilon_{n+1,m} = \epsilon_{n-1,m+1} + \frac{1}{\epsilon_{n,m+1} - \epsilon_{n,m}} , n,m = 1,2,3,\ldots$$

It can be shown that [7] the sequence

$$\epsilon_{1,1} \,,\; \epsilon_{3,1} \,,\; \ldots \,,\; \epsilon_{N,1}$$

converges to $f(t) + E_D - c_0/2$ faster than the sequence of partial sums

$$s_m, \quad m = 1,2,3, \ldots$$

The actual procedure used to invert the Laplace Transforms consists of using equation (26) together with the ϵ-algorithm. The values of d and L are chosen according the criteria outlined in [7].

NUMERICAL RESULTS

In order to illustrate the above results graphically the function F(t) which is the value of the temperature of the surrounding medium was taken as:

$$F(t) = F_o \exp(-t)\, H(t),$$

where H(t) is the Heaviside unit step function and F_o is a non -dimensionalization constant. At room temperature the value of F_o is approximately equal to 1/300 [6].

The copper material was chosen for purposes of numerical evaluations. The constants of the problem were taken as [3]

$$\epsilon=0.0168, \beta^2=3.5, \tau_o=0.02, a=1 \text{ and } h=1.$$

The computations were carried out for two values of time, namely t=0.1 and t= 0.5, respectively. The numerical technique outlined above was used to invert the Laplace transforms in the above equations. The computations were performed for both the coupled theory ($\tau_o = 0$) and for the generalized theory ($\tau_o \neq 0$). The temperature Θ, the stress component σ_{rr} and the displacement component u are evaluated. The graphs of these functions are shown in Figures (1-3), respectively.

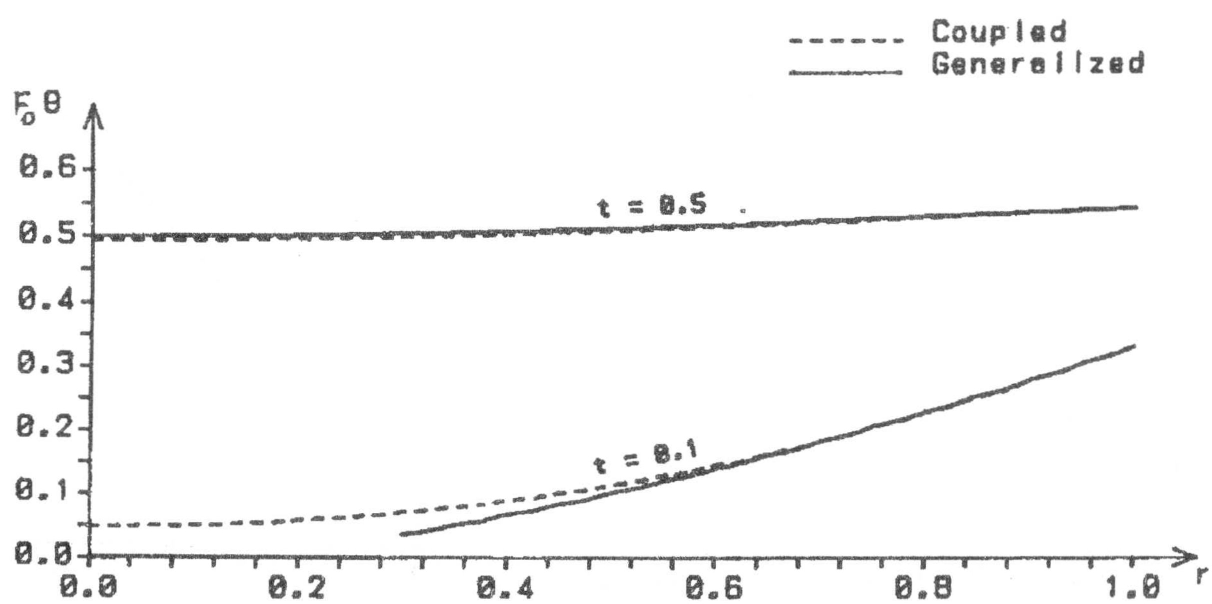

Figure 1. Temperature distribution.

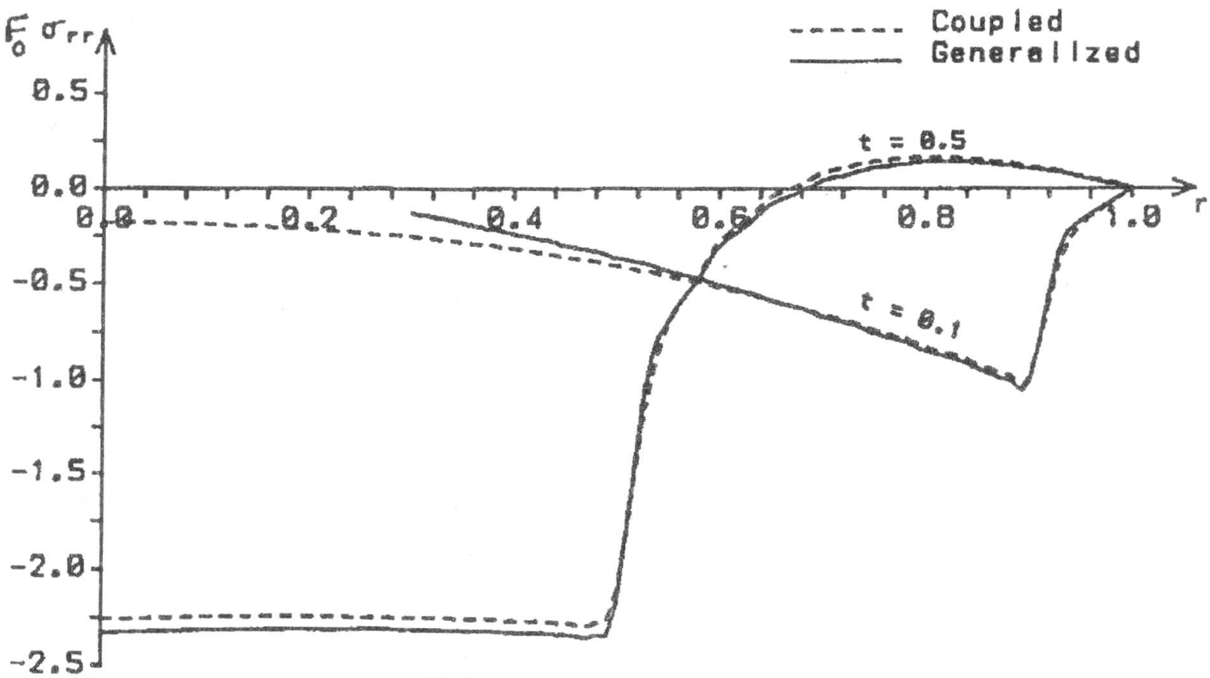

Figure 2. Radial stress distribution.

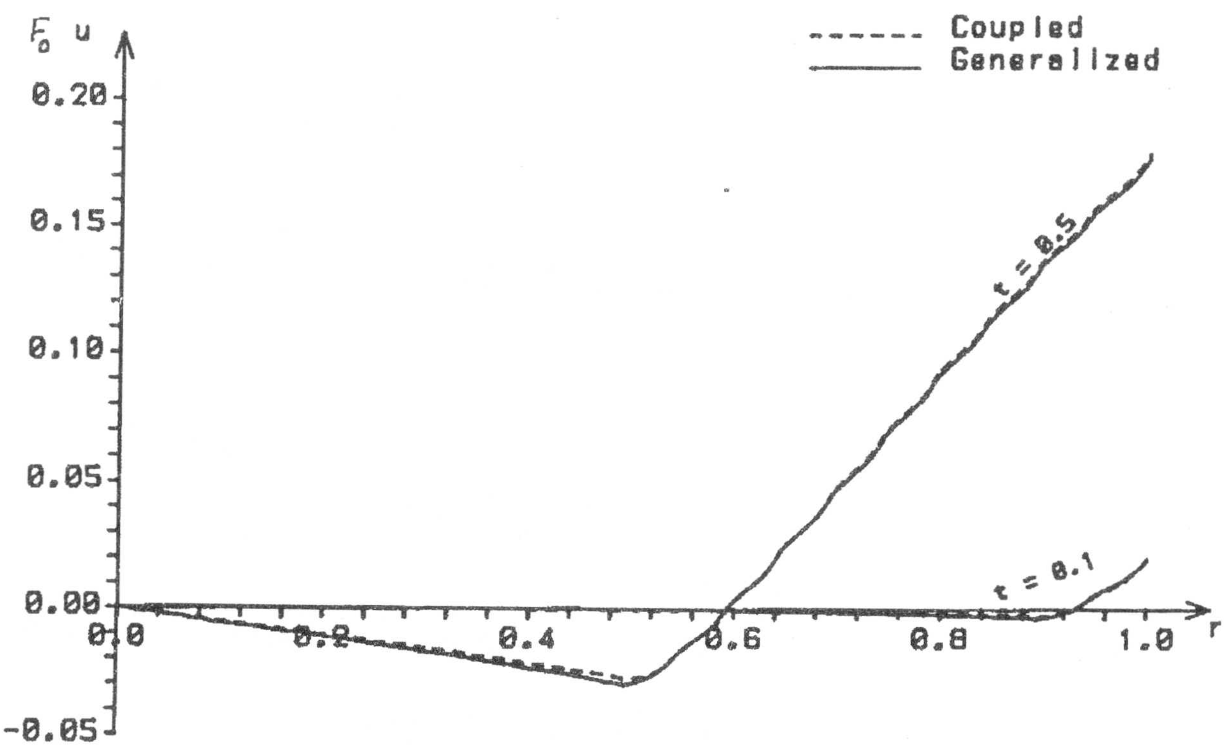

Figure 3. Displacement distribution.

It was found that for large values of time the results obtained by using either the coupled or the generalized theories are quite similar. The case is quite different when we consider small values of time. Since the coupled theory predicts infinite speeds of wave propagation the effect of heating at the boundary is transmitted instantaneously to all parts of the medium so the solution is not identically zero for any value of $t > 0$ (though it may be very small). For the generalized theory, however, the waves take a finite time to be transmitted.

It should be noted here that Figures. (1-3) are multiplied by a factor of F_o. The maximum value of Θ reported in Figure (1) is about 0.5 for $t = 0.5$. This should be understood to mean a value of 0.5 times F_o or 0.001667 in accordance with the fact that we are dealing with a linear theory.

It was shown in [5] that in generalized thermoelasticity, the thermal and mechanical effects are transmitted by the action of two waves. The first of these waves is mainly thermal in nature and has a velocity of 7.07 units, while the second wave is mainly mechanical with a velocity equal to unity approximately. Thus, for $t=0.1$ the thermal waves travels a distance of 0.707 starting from the surface of the sphere ($r = 1$) and the wave front is at $r=0.293$. This explains the jump in Figure. (1) where it is clear that for $r \leq 0.293$ the temperature is identically zero.

The other two curves are more affected by the arrival of the mechanical wave. We note a change in each of these functions at the location of the front of the mechanical wave. $r = 0.9$, $r = 0.5$ corresponding to $t=0.1$ and $t=0.5$, respectively.

REFERENCES

[1] H. Lord and Y. Shulman, A Generalized Dynamical Theory of Thermoelasticity, *J. Mech. Phys.* Solid, vol. 15 , pp. 299-309, 1967.

[2] R. Dhaliwal and H. Sherief, Generalized Thermoelasticity for Anisotropic Media, *Quart. Appl. Math.*, vol. 33 , pp. 1-8, 1980.

[3] H. Sherief, Fundamental Solution of the Generalized Thermoelastic Problem for Short Times, *J. Therm. Stress.*, vol.9, pp. 151-164, 1986.

[4] H. Sherief and M. Anwar, Problem in Generalized Thermoelasticity, *J. Therm. Stress.*, vol. 9, pp. 165-181, 1986.

[5] H. Sherief and F. hamza, A Generalized Thermoelastic Problem of a Thick Plate Under Axisymmetric Temperature Distribution, (Accepted for Publication in the *J. Therm. Stress.*)

[6] H. Sherief and M. Anwar, A Problem in Generalized Thermoelasticity for an Infinitely Long Annular Cylinder Composed of Two Different Materials, *Acta Mechanica*, vol. 80, pp. 137-149, 1989.

[7] G. Honig and U. Hirdes , A Method for the Numerical Inversion of the Laplace Transform, *J. Comp. Appl. Math*, vol 10, pp. 113-132, 1984.