ON VECTOR-MATRIX DIFFERENTIAL EQUATION OF VIBRATING SYSTEMS

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ABSTRACT

The equation of motion of an n-degrees of freedom non-conservative mechanical system is represented by a Vector-Matrix differential equation. The Matrix method is usually used, where the natural frequencies and natural modes of vibrations are obtained, to solve this equation and a similarity transformation is always required to obtain the steady state solution. The purpose of this paper is to propose a solution, based on the Cayley-Hamilton theorem, to the equation of motion of n-degrees of freedom discrete non- conservative mechanical system. The similarity transformation is not required in the proposed method, and in contrary to the classical matrix method, the computations to be carried out are moderate, thus enabling use of personal computers. Simulated examples are presented to illustrate the results given in this paper.

1. INTRODUCTION

Vibrating systems are divided into two types, discrete and continuous [1]. A discrete system is one whose exact equations of motion may be expressed as a set of ordinary differential equations and, for the small oscillations with which we are concerned, the equation will be linear and have constant coefficients. Discrete systems will then be characterized by having a finite number of natural frequencies and corresponding modes of vibration. Although exactly discrete systems never occur in practice, many physical systems are discrete for most practical purposes [7,8].

Consider a discrete non-conservative mechanical system having n-degrees of freedom. A set of generalized coordinates are denoted by $x_1, x_2, \dots, x_n$ where the datums configuration, in which all these coordinates vanish, is assumed to be one of a stable or neutral equilibrium. The Lagrangian equations take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_r}\right) - \frac{\partial T}{\partial x_r} + \frac{\partial U}{\partial x_r} + \frac{\partial V}{\partial \dot{x}_r} = F_r \qquad (1.1)$$

$$r = 1, 2, \dots, n$$

where t denotes the time, T, U and V are the kinetic energy, the potential energy and the dissipation function, where

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\,\dot{x}_i\,\dot{x}_j, \quad U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}\,x_i\,x_j \qquad (1.2)$$

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}\,\dot{x}_i\,\dot{x}_j$$

In equation (1.1), the quantities $F_1, F_2, \dots, F_n$ are the generalized forces or driving forces which produce an excitation of the system. In equation (1.2), the quantities m_{ij}, k_{ij} and d_{ij} are constants, they are known as the mass, the stiffness, and the damping coefficients of the system. If we substitute the expressions for T, U and V into (1.1), then we write n equations of motion which can be written in the Vector-Matrix form as

$$M\ddot{x} + D\dot{x} + Kx = F \qquad (1.3)$$

where M is the mass matrix, K the stiffness matrix and D is the damping matrix. They have the general form

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ m_{n1} & \cdots & \cdots & m_{nn} \end{bmatrix}, \quad K = \begin{bmatrix} k_{11} & \cdots & \cdots & k_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1} & \cdots & \cdots & k_{nn} \end{bmatrix} \qquad (1.4)$$

$$D = \begin{bmatrix} d_{11} & \cdots & \cdots & d_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ d_{n1} & \cdots & \cdots & d_{nn} \end{bmatrix}$$

The displacement vector x, and the force vector F have the general form

$$x = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \qquad F = \begin{bmatrix} F_1 \\ \cdot \\ \cdot \\ \cdot \\ F_n \end{bmatrix} \tag{1.5}$$

The solution of equation (1.3) is usually obtained by the matrix method, where the natural frequencies and the corresponding natural modes of vibration are obtained, and a similarity transformation is used to the diagonal form, in the case of distinct eigenvalues, or to the Jordan canonical form, in the case of repeated eigenvalues. For a complete discussion of the matrix method the reader may referred to [1].

However, the determination of the natural modes of vibration corresponding to each natural frequency and the using of similarity transformation require a computational effort and a large memory space in the digital computer. The purpose of our paper is to propose a method, based on the Cayley-Hamilton theory which is presented in Section 2, for the solution of equation (1.3). In contrary to the classical matrix method, the computations to be carried out are moderate and the similarity transformation is avoided in the proposed method.

2. GENERAL RESULTS

In this section we state the general results which have been used for simulations.

Result 2.1

For the system described by equation (1.3) satisfying
1. The matrices M and K are symmetric
2. The matrices M and K are positive definite
3. The force vector $F = 0$.
4. The matrix $D = 0$.

Then, the frequency equation has only real positive roots.

Proof

Note that the equation (1.3) in this case has the form

$$M\ddot{x} + Kx = 0 \tag{2.1}$$

which represents the free oscillations of a conservative system. The eigenvalues λ (the square of the natural frequencies) and the eigenvectors u (the amplitude vector of the natural modes of vibration) are determined from the equation

$$(K - \lambda M)u = 0 \tag{2.2}$$

The frequency equation is given by

$$\det(K - \lambda M) = 0 \tag{2.3}$$

where $\lambda = \omega^2$, and ω denotes the frequency.
Let λ be a complex root of equation (2.3) and $u \neq 0$ is a complex vector corresponding to λ. Then λ^* is also a root of the equation (2.3) with complex vector u^*, where λ^* is the complex conjugate of λ. Define the quadratic form

$$M(u,u) = \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\, u_i\, u_j \tag{2.4}$$

Since $\lambda \neq \lambda^*$, and the orthogonality property of the eigenvectors holds in this case, it follows that

$$M(u,u^*) = 0$$

but this contradicts the fact that, if $M(u,u^*)$ is a positive definite quadratic form and $u \neq 0$ is an arbitrary complex vector, then $M(u,u^*) > 0$.
If λ is real, then also the eigenvector $u \neq 0$ that corresponds to it may be chosen real. From equation (2.2) $ku = \lambda M u$. Then the quadratic forms $k(u,u)$ and $M(u,u)$ are related by the equation

$$k(u,u) = \lambda M(u,u),$$

then $$\lambda = \frac{k(u,u)}{M(u,u)} > 0.$$

Thus, the frequency equation (2.3) has n positive real roots λ_j, to which there correspond real positive frequencies $\omega_j = \sqrt{\lambda_j}$ and real eigenvectors u_j, $j = 1, 2, \ldots, n$.

Result 2.2

For the system described by equation (1.3) satisfying
1. The matrices M, D and K are symmetric
2. The matrices M, D and K are positive-definite
3. The force vector $F = 0$.

Then the roots of the frequency equation have negative real parts.

Proof

Note that the equation (1.3) in this case has the form

$$M\ddot{x} + D\dot{x} + Kx = 0$$

which represents the free oscillations of a non-conservative system. The eigenvectors u are determined from the equation

$$\left(M\mu^2 + D\mu + K\right)u = 0 \qquad (2.5)$$

The frequency equation is given by

$$\det\left(M\mu^2 + D\mu + K\right) = 0 \qquad (2.6)$$

which has n-pair of roots. Equation (2.5) can be expanded to

$$\sum_{j=1}^{n}\left(m_{ij}\mu^2 + d_{ij}\mu + k_{ij}\right)u_j = 0, \qquad (2.7)$$
$$i = 1, 2, \ldots, n.$$

Multiplying both parts of the ith equation of (2.7) by u_i^* (u_i^* is the complex conjugate of u_i) and summing over i, we find

$$\mu^2\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}u_i^*u_j + \mu\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}u_i^*u_j + \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}u_i^*u_j = 0 \qquad (2.8)$$

Using (2.4), the equation (2.8) can be written as

$$M(u,u^*)\mu^2 + D(u,u^*)\mu + k(u,u^*) = 0. \qquad (2.9)$$

Using the fact that, if $M(u,u^*)$ is a positive definite quadratic form and $u \neq 0$ is an arbitrary complex vector, then

$$M(u,u^*) > 0.$$

Thus, any root μ of the frequency equation satisfies the quadratic equation (2.9) with positive coefficients. From this it immediately follows that Re $\mu < 0$, where Re denotes real part, Thus:

$$2\,\mathrm{Re}\,\mu = \mu + \mu^* = -\frac{D(u,u^*)}{M(u,u^*)} < 0$$

Result 2.3

For the system described by equation (1.3) satisfying
1. The matrices M, D, and K are asymmetric
2. The force vector $F \neq 0$
3. The force vector F is a function of the resonance frequency, satisfies $F = \hat{F}e^{(\mu,\mu^*)t}$, where $\hat{F}$ denotes the peak value

Then the roots of the frequency equation have a magnitude of $\sqrt{\dfrac{K(u,u^*) - \hat{F}}{M(u,u^*)}}$ and a real part of $-\dfrac{D(u,u^*)}{2M(u,u^*)}$

Proof

Note that the equation (1.3) in this case has the form

$$M\ddot{x} + D\dot{x} + Kx = F$$

which represents a non conservative system operating under a forced oscillations. The eigenvectors u are determined from the equation

$$(M\mu^2 + D\mu + K)u = F \qquad (2.10)$$

which has n-pair of roots, and can be generalized using the same argument derived in section 2.2. Thus:

$$M(u,u^*)\mu^2 + D(u,u^*)\mu + K(u,u^*) = F. \qquad (2.11)$$

Using the fact that $M(u, u^*) \neq 0$ is a definite quadratic form and $u \neq 0$ is an arbitrary complex vector; thus, the roots μ of equation (2.10) satisfy the quadratic equation (2.11). It is immediately seen:

$$M(u,u^*)\mathrm{Re}(\mu^2) + D(u,u^*)\mathrm{Re}(\mu) + K(u,u^*) = F$$
$$M(u,u^*)\mathrm{Im}(\mu^2) + D(u,u^*)\mathrm{Im}(\mu) = 0 \qquad (2.12)$$

where Im denotes imaginary part.
Since equations (2.12) are simultaneously satisfied, it is clear that:

$$Re(\mu) = -\frac{D(u,u^*)}{2M(u,u^*)}$$

$$|\mu| = \sqrt{\frac{K(u,u^*) - \hat{F}}{M(u,u^*)}}$$

Result 2.4 The solution of state equation

The system described by equation (1.3) can be easily derived in state space form (though, in case that, M-matrix is nonsingular, det (M) $\neq$ 0) as follows:

Let $y = (x, \dot{x})$, hence $\dot{y} = (\dot{x}, \ddot{x})$.

Substituting into (1.3), it can be seen:

$$\dot{y} = A y + M^{-1} F \qquad (2.13)$$

The general form of A-matrix, containing constant coefficients, including the parameters of mass M, stiffness K, damping D in matrix form, can be written as:

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}$$

where M, K and D are defined in (1.4).

The solution of (2.13) can be derived, following [4,5,6], by calculating the **transition matrix** e^{At}, as follows:

$$y = e^{At} y_0 + M^{-1} \int_0^t e^{A(t-\tau)} F(\tau)\, d\tau \qquad (2.14)$$

Result 2.5 The Cayley-Hamilton theorem

The Cayley-Hamilton method for determining the transition matrix e^{At}, can be summarized as follows:

1. Obtain the eigenvalues of A-matrix, from det $(I\lambda - A) = 0$. If A is n x n therefore we have n-roots (λ_i, i = 1, 2, ... , n). These roots may be distinct real (positive or negative) roots, pair of complex conjugate, or repeated real roots.
2. In the first two cases of eigenvalues, the transition matrix is

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i A^i \qquad (2.15)$$

where the coefficients α_i are the simultaneous solution of the following algebraic equations, represented in a general matrix form,

$$e^{\lambda_j t} = \sum_{i=0}^{n-1} \alpha_i \lambda_j^i, \qquad (2.16)$$

j = 1, 2, ... , n and α_i are exponential functions in time.

3. In case of repeated real eigenvalues the transition matrix takes the same form of (2.15), while the coefficients α_i take the derivative form of (2.16) with respect to λ_j. namely,

$$t e^{\lambda_j t} = \sum_{i=1}^{n-1} i \alpha_i \lambda_j^{i-1}. \qquad (2.17)$$

Finally, substituting for e^{At} from (2.15) into (2.14), to obtain the solution of the state space form (2.13).

3. FORMULATION OF STATE EQUATION

In a discrete non-conservative vibrating mechanical system, the equation of motion often takes the form (1.3) and this would be converted to the state space form as equation (2.13). To appreciate the solution technique, consider the mass/ spring system of n-degrees of freedom of Fig(1), it can be appreciated that the parameter matrices have the following forms:

1) M-matrix takes the diagonal form, where

$$M = \begin{bmatrix} m_{11} & & & \\ & m_{22} & & \\ & \cdots & \cdots & \\ & & & m_{nn} \end{bmatrix}$$

2) D and K matrices involve a coupling, mutual effects, but in a symmetric form, such that:

$$D = \begin{bmatrix} d_{01}+d_{11}+d_{12} & -d_{12} & & & \\ -d_{12} & d_{12}+d_{22}+d_{23} & -d_{23} & & \\ .. & .. & .. & .. & .. \\ & & & -d_{(n-1)n} & d_{(n-1)n}+d_{nn} \end{bmatrix}$$

includes mass damping coefficients in the diagonal elements, and spring damping in the co-symmetric elements.

$$K = \begin{bmatrix} k_{01}+k_{11}+k_{12} & -k_{12} & & & \\ -k_{12} & k_{12}+k_{22}+k_{23} & -k_{23} & & \\ .. & .. & .. & .. & .. \\ & & & -k_{(n-1)n} & k_{(n-1)n}+k_{nn} \end{bmatrix}$$

includes mass friction coefficients in the diagonal elements, and spring stiffness in the co-symmetric elements.

In this case A-matrix of the formula (2.13) takes the form:

$$A = \begin{bmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ -\frac{k_{01}+k_{11}+k_{12}}{m_{11}} & \frac{k_{12}}{m_{11}} & & -\frac{d_{01}+d_{11}+d_{12}}{m_{11}} & \frac{d_{12}}{m_{11}} & \\ \frac{k_{12}}{m_{22}} & -\frac{k_{12}+k_{22}+k_{23}}{m_{22}} & \frac{k_{23}}{m_{22}} & \frac{d_{12}}{m_{22}} & -\frac{d_{12}+d_{22}+d_{23}}{m_{22}} & \frac{d_{23}}{m_{22}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ & \frac{k_{(n-1)n}}{m_{nn}} & -\frac{k_{(n-1)n}+k_{nn}}{m_{nn}} & & \frac{d_{(n-1)n}}{m_{nn}} & -\frac{d_{(n-1)n}+d_{nn}}{m_{nn}} \end{bmatrix}$$

The force vector F of (1.5) (applied on each mass) takes any specified value, such that:

1. $F_i = 0$ for all $i = 1, \ldots, n$ represents free oscillation,
2. F_i = constant for some i represents the system under a step-force,
3. $F_i = \hat{F}_i \sin\omega_i t$ (or $\cos\omega_i t$) for some i represents forced oscillation,
4. $F_i = f_i(x, t)$ = any function of x, t for any i, represents the system under a feedback control,
5. $F_i = \{0;$ constant; $f(t)$; $\phi(x)$; $\Psi(x, t)$; ... etc.$\}$ for each i, a discrete type of force function; which represents a heuristic control system associated with the analysis of random functions applications.

In the analysis of this paper, the solution takes the form (2.14) and is computed by Cayley-Hamilton theory, considering that all the coefficients are positive definite and all possible values or functions of forces F_i could be involved, and i is n-degree of freedom as shown in Fig(1). The similarity transformation is not required and the computer will give the solution at each step of integration providing a time-specification such a bit small value of step length.

The state variables y_i of (2.14) give the solution response at each step of time, such that $y_i = \Phi_i(t)$ is a function of time for each $i = 1, \ldots, n$. The vector y represents the velocities $\dot{x}$ and the displacement of (1.5) for each mass m_{ii} ($m_{ij} = 0$, $i \neq j$, $i, j = 1, \ldots, n$) contained in the vibrating mechanical system. All displacements are measured from the equilibrium positions as shown in Fig(1). Therefore:

$x_i = y_i$, $\dot{x}_i = y_{n+i}$, $i = 1, \ldots, n$.

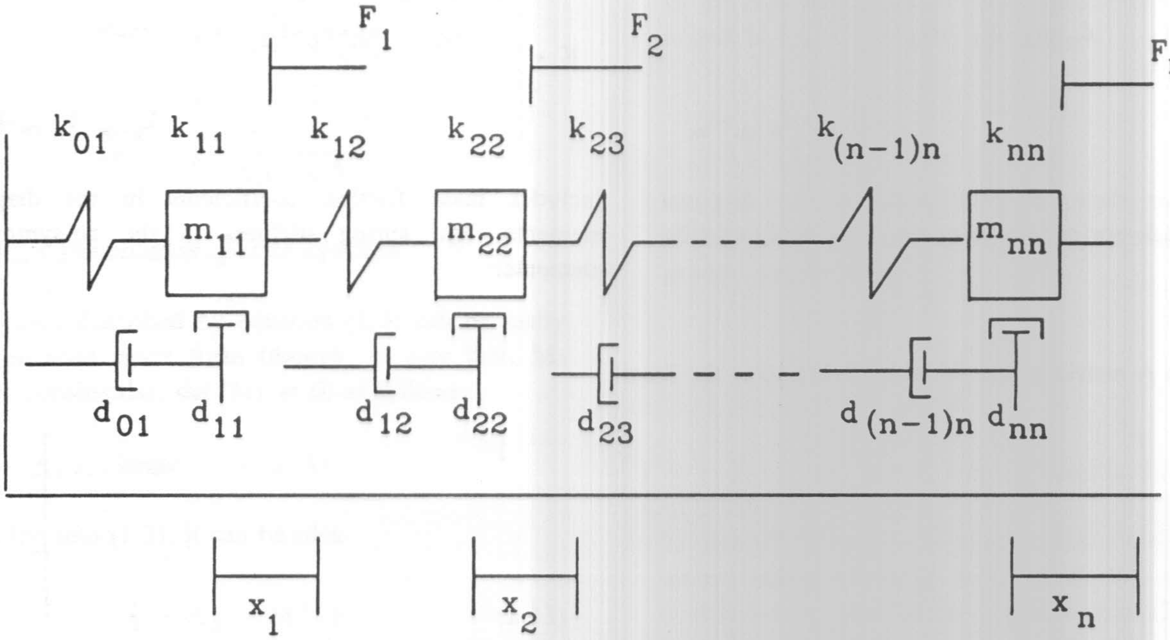

Figure 1. Mass/spring system

4. SIMULATION RESULTS

A software package of a computer program is developed by the authors. This is available for applying any type of force in a random formula. it also accepts all possible values of the parameters, and is analyzed for the mass/spring mechanical vibrating system. Another software based on the solution of the state space equation, accepting various components of parameters, state and input variables, is also developed; both are available for running on IBM-DOS PC computer.

In both packages, evaluation of the eigenvalues of the A-matrix and evaluation of the transition matrix by Cayley method are involved. Step length is to be specified and the final solution of velocities and displacements are associated with a graph program, to show their response characteristics in the time domain.

Example 4.1 Consider the system of Fig(1) to have only two masses of the following parameters (in per unit value):

$m_{11} = 2, m_{22} = 6, k_{01} = 3, k_{11} = 1, k_{12} = 12, k_{22} = 0, d_{01} = 4, d_{11} = 6, d_{12} = 0, d_{22} = 0.$

Show the step response of free oscillation assuming that the masses are respectively, in a 3 and 4 unit distance from their equilibrium position.

Solution

$$M = \begin{bmatrix} 2 & \\ & 6 \end{bmatrix}, k = \begin{bmatrix} 3+1+12 & -12 \\ -12 & 12 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$x_0 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, D = \begin{bmatrix} 4+6 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -8 & 6 & -5 & \\ 2 & -2 & & \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

The solution is

$$y = e^{At} \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} \tag{4.1}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & & -1 & \\ & \lambda & & -1 \\ 8 & -6 & \lambda+5 & \\ -2 & 2 & & \lambda \end{vmatrix} = 0,$$

the eigenvalue: $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_{3,4} = -1 \pm i$

Cayley-Hamilton method

(2.16) gives

$$e^{-t} = \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3$$
$$e^{-2t} = \alpha_0 - 2\alpha_1 + 4\alpha_2 - 8\alpha_3$$
$$e^{-(1+i)t} = \alpha_0 - (1+i)\alpha_1 + 2i\alpha_2 + 2(1-i)\alpha_3$$
$$e^{-(1-i)t} = \alpha_0 - (1-i)\alpha_1 - 2i\alpha_2 + 2(1+i)\alpha_3$$

the last two equations can be replaced by

$$e^{-t}\cos t = \alpha_0 - \alpha_1 + 2\alpha_3$$
$$e^{-t}\sin t = \alpha_1 - 2\alpha_2 + 2\alpha_3$$

Solving the four equations, we get:

$$\alpha_0 = e^{-t}(4 - 2\cos t) - e^{-2t}$$
$$\alpha_1 = e^{-t}(6 - 4\cos t - \sin t) - 2e^{-2t}$$
$$\alpha_2 = e^{-t}(4 - 2.5\cos t - 1.5\sin t) - 1.5e^{-2t}$$
$$\alpha_3 = e^{-t}(1 - 0.5\cos t - 0.5\sin t) - 0.5e^{-2t}$$

The transition matrix from (2.15) is:

$$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \alpha_3 A^3$$

where:
$$A = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -8 & 6 & -5 & \\ 2 & -2 & & \end{bmatrix},$$

$$A^2 = \begin{bmatrix} -8 & 6 & -5 & \\ 2 & -2 & & \\ 40 & -30 & 17 & 6 \\ & & 2 & -2 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 40 & -30 & 17 & 6 \\ & & 2 & -2 \\ -124 & 90 & -45 & -30 \\ -20 & 16 & -10 & \end{bmatrix}$$

Substituting for A's and α-coefficients, hence:

$$e^{At} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix}$$

where:

$$s_{11} = e^{-t}(12 - 2\cos t - 8\sin t) - 9e^{-2t}$$
$$s_{12} = e^{-t}(-6 + 6\sin t) + 6e^{-2t}$$
$$s_{13} = e^{-t}(3 - 2\sin t) - 3e^{-2t}$$
$$s_{14} = e^{-t}(6 - 3\cos t - 3\sin t) - 3e^{-2t}$$
$$s_{21} = e^{-t}(8 - 5\cos t - 3\sin t) - 3e^{-2t}$$
$$s_{22} = e^{-t}(-4 + 3\cos t + 3\sin t) + 2e^{-2t}$$
$$s_{23} = e^{-t}(2 - \cos t - \sin t) - e^{-2t}$$
$$s_{24} = e^{-t}(4 - 3\cos t) - e^{-2t}$$
$$s_{31} = e^{-t}(-12 - 6\cos t + 10\sin t) + 18e^{-2t}$$
$$s_{32} = e^{-t}(6 + 6\cos t - 6\sin t) - 12e^{-2t}$$
$$s_{33} = e^{-t}(-3 - 2\cos t + 2\sin t) + 6e^{-2t}$$
$$s_{34} = e^{-t}(-6 + 6\sin t) + 6e^{-2t}$$
$$s_{41} = e^{-t}(-8 + 2\cos t + 8\sin t) + 6e^{-2t}$$
$$s_{42} = e^{-t}(4 - 6\sin t) - 4e^{-2t}$$
$$s_{43} = e^{-t}(-2 + 2\sin t) + 2e^{-2t}$$
$$s_{44} = e^{-t}(-4 + 3\cos t + 3\sin t) + 2e^{-2t}$$

Substituting into (4.1), we get:

$$y_1 = x_1 = 3s_{11} + 4s_{12} = e^{-t}(12 - 6\cos t) - 3e^{-2t}$$
$$y_2 = x_2 = e^{-t}(8 - 3\cos t + 3\sin t) - e^{-2t}$$
$$y_3 = e^{-t}(-12 + 6\cos t + 6\sin t) + 6e^{-2t}$$
$$y_4 = dx_2/dt = e^{-t}(-8 + 6\cos t) + 2e^{-2t}$$

This solution is traced in Fig(2).

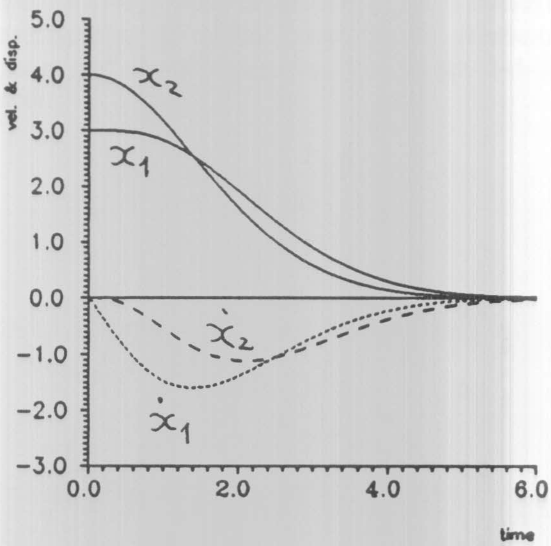

Figure 2. 2-stage free oscillation

Example 4.2. Fig(3) represents the results of a single stage mass/spring system of the following parameters: $m_{11}=5$, $k_{01}=10$, $x_1=0.5$, $k_{11}=d_{01}=d_{11}=0$, $F_1=\sin t$.

The A-matrix is $\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ and the eigenvalues are $\pm i\sqrt{2}$.

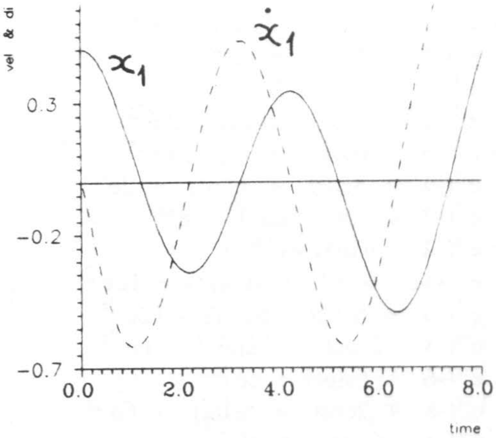

Figure 3. Single-stage pure oscillation

Example 4.3. Fig(4) represents the results of a 3-stage forced oscillation from equilibrium, each force is a cos function of a frequency equal the eigenvalue (frequency of oscillation), where $F_1 = \cos(1.8019t)$, $F_2 = \cos(0.445t)$, $F_3 = \cos(1.247t)$. The parameters are $m_{11} = m_{22} = m_{33} = k_{01} = k_{12} = k_{23} = 1$, all other parameters are zero. The eigenvalues are $\pm i1.8019$, $\pm i0.445$, $\pm i1.247$ and A-matrix is

$$\begin{bmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -1 & & & \end{bmatrix}$$

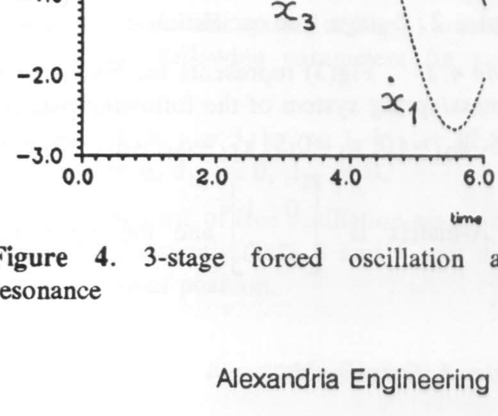

Figure 4. 3-stage forced oscillation at resonance

Example 4.4. Fig(5) represents the results of a 2-stage feedback control oscillation, where the forces applied to the masses are sin waves and functions of displacement, where $F_1 = \sin 3t$, $F_2 = (x_2/t)^2$. The parameters are: $m_{11} = 5$, $m_{22} = 8$, $d_{11} = 2$, $d_{22} = 4$, $d_{01} = 0.7$, $d_{11} = 7$, $k_{01} = 3$, $k_{11} = 0.2$, $k_{12} = 12$, $k_{22} = 0$, $x_1 = x_2 = 0$ (from equilibrium). The eigenvalues are: $-1.393 \pm 1.531i$, $-0.264 \pm 0.393i$ and the A-matrix is

$$\begin{bmatrix} & & 1 & \\ & & & 1 \\ -3.04 & 2.4 & -1.94 & 1.4 \\ 1.5 & -1.5 & 0.875 & -1.375 \end{bmatrix}$$

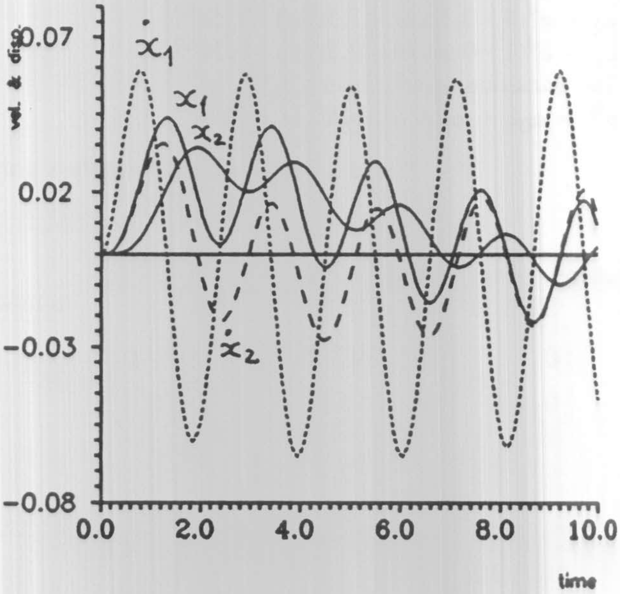

Figure 5. 2-stage random oscillation control from equilibrium

Example 4.5. Consider a heavily damped random control system of 3-stages, the time response of each variable is decaying, as Fig(6) illustrates. The applied forces are $F_1 = 0$, $F_2 = e^{-\dot{x}_3^2 t}$, $F_3 = \cos t$. The parameters are $m_{11} = 6$, $m_{22} = 3$, $m_{33} = 8$, $d_{11} = 1$, $d_{12} = 0.1$, $d_{23} = 2$, $k_{33} = 0.5$, $k_{01} = 2$, $k_{12} = 3$, $k_{23} = 2$. The other parameters are zero. The displacements are $x_1 = 0$, $x_2 = 8$, $x_3 = 3$. The eigenvalues are $-0.352 \pm 1.354i$, $-0.197 \pm 0.752i$, $-0.017 \pm 0.342i$, and the A-matrix is:

$$\begin{bmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ -.83 & .5 & & -.18 & .017 & \\ 1 & -1.67 & .67 & .03 & -.7 & .67 \\ & .25 & -.31 & & .25 & -.25 \end{bmatrix}$$

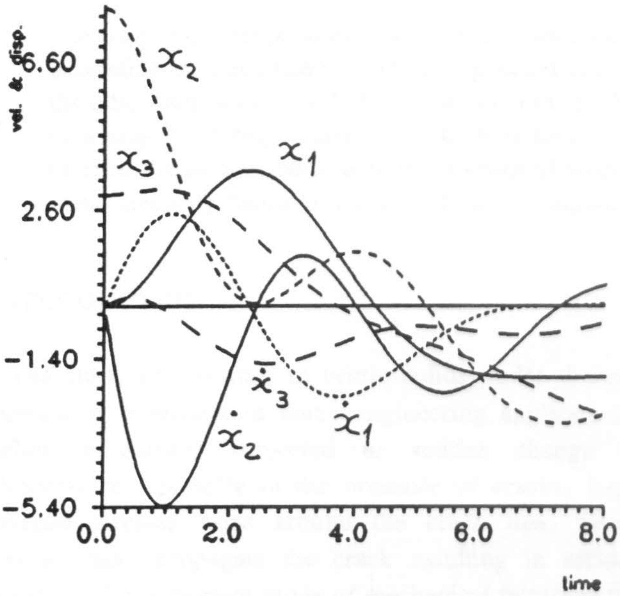

Figure 6. 3-stage random control system

5. CONCLUSION

The solution of a vector-matrix differential equation, representing the equation of motion of n-degrees of freedom non-conservative mechanical system, has been considered. General results for the associated eigenvalue problem have been established in section 2. The equation has been represented in state space form and a solution, based on the Cayley-Hamilton theorem, has been presented which has the following features

1. The computations to be carried out are moderate. The software package, **DYN.EXE**, requires only 30 kilobytes of the memory space.
2. There is no need to similarity transformation.
3. There is no need to compute the associated eigenvectors. Simulation results show that the computation time in CPU seconds is too small compared with the classical matrix method.
4. The method can be extended to systems with random inputs. This will be considered elsewhere.
5. The method can be used for the simulation of vibrating systems for the purpose of identification and control. This will be discussed in a separate paper.

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