ON THE INVERSE EIGENVALUE PROBLEM: A SOLUTION USING ORTHOGONAL POLYNOMIALS

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ABSTRACT

A new method, for the solution of the inverse eigenvalue problem, is presented. The proposed technique is designed to generate matrices with prespecified eigenvalues. This method is based on the well-known properties of orthogonal polynomials together with the Gaussian quadrature integration.

Key Words: Inverse Eigenproblem, Gaussian quadrature integration
MSC1991 Categories: 15A18,33 C45, 65F25

1.INTRODUCTION

Inverse eigenvalue problems arise often in Applied Mathematics, Applied Physics and Control Theory. They are treated extensively by many authors, e.g. cf. [3-7,10-13]. In this paper, we propose a fundamentally different approach based on sampling orthogonal functions over their interval of orthogonality. It can be completely automated to generate a wide range of matrices with selectable eigenvalue spectrum.

Consider the eigenvalue decomposition of an n-by-n real matrix *A* given in [14] by

$$A = U \Lambda U^T \tag{1}$$

where Λ is an n-by-n diagonal matrix of real eigenvalues and **U** is an n-by-n matrix of normalized eigenvectors of the matrix **A**. The matrix **U** is orthogonal since $U^T U = I$ where the superscript "T" denotes "transposition" and I denotes the identity matrix.

Equation (1) shows that if the orthogonal matrix **U** can be generated, then the matrix **A** with prescribed eigenvalues Λ is simple to create. It should be noticed that the equation

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i u_i u_i^T, \tag{2}$$

where $\mathbf{u}_i$ is the i-th column vector of **U**, allows the matrix **A** to be created without having to devote extra storage to the orthogonal matrix **U**.

2. CREATION OF ACCURATE ORTHOGONAL MATRICES

Consider the Gaussian quadrature integration [1,8]

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=0}^{p-1} w_i\, f(x_i), \tag{3}$$

where w_i and x_i are the weights and abscissa locations for the p-point Gaussian integration, respectively. This integration formula for p points is theoretically exact for all polynomials of degree $2p+1$ or less [4]. Using the orthogonal properties of some special functions, we can find $f_n(x)$ and $f_m(x)$ such that

$$\int_{-1}^{1} f_n(x) f_m(x)\,dx = 1 \quad m=n, \qquad =0 \quad m \neq n. \tag{4}$$

This integration could be computed, close to machine accuracy, using Gaussian quadrature by

$$\sum_{i=0}^{p-1} w_i\, f_n(x_i)\, f_m(x_i) = 1, \quad m=n; \qquad =0, \quad m \neq n. \tag{5}$$

Consequently, the orthogonal matrix U can be simply created in the form:

$$U = \begin{bmatrix} \sqrt{w_o} f_o(x_o) & \sqrt{w_o} f_1(x_o) & \dots & \sqrt{w_o} f_n(x_o) \\ \sqrt{w_1} f_o(x_1) & \sqrt{w_1} f_1(x_1) & \dots & \sqrt{w_1} f_n(x_1) \\ & & & \\ \sqrt{w_{p-1}}\, f_o(x_{p-1}) & \sqrt{w_{p-1}} f_1(x_{p-1}) & \dots & \sqrt{w_{p-1}} f_n(x_{p-1}) \end{bmatrix} \quad (6)$$

Taking the dot product of any two column vectors of U is equivalent to numerically integrating Equation (4) using the Gaussian quadrature form (5).
The high accuracy of the p-point Gaussian quadrature with polynomials of degree $\leq 2\,p+1$ implies that U is orthogonal to a high degree of accuracy.

2.1 *U-Creation Using Legendre Polynomials:*

Legendre polynomials L_n (x) are well-defined in [1] by the generating function

$$(1-2\,h\,x + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n L_n(x) \quad (7)$$

from which

$$L_o(x) = 1$$

$$L_1(x) = x$$

$$L_2(x) = \frac{1}{2}(3x^2-1)$$

$$L_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\dots\dots\dots\dots \quad (8)$$

with the orthogonality property

$$\int_{-1}^{1} L_n(x) L_m(x)\,dx = 0, \qquad n \neq m;$$

$$= \frac{2}{2n+1}, \qquad n = m. \quad (9)$$

Then, taking

$$f_n(x) = \sqrt{\frac{2n+1}{2}}\; L_n(x), \quad (10)$$

we can set

$$f_o(x) = \sqrt{1/2}$$

$$f_1(x) = \sqrt{3/2}\; x,$$

$$f_2(x) = \sqrt{5/8}\;(3x^2-1),$$

$$f_3(x) = \sqrt{7/8}\;(5x^3-3x),$$

$$\dots\dots\dots\dots \quad (11)$$

and

$$\int_{-1}^{1} f_n^2(x)\,dx = \sum_{i=0}^{p-1} w_i\, f_n^2(x_i) = 1,$$

$$\int_{-1}^{1} f_n(x)\, f_m(x)\,dx = \sum_{i=0}^{p-1} w_i\, f_n(x_i) f_m(x_i) = 0. \quad (12)$$

Considering a 4-by-4 matrix example, the abscissae and weight values for a 4-point Gaussian integration are given in [1] by

$$x_0 = -x_2 = 0.3399810436,$$

$$x_1 = -x_3 = 0.8611363116,$$

and

$$w_0 = w_2 = 0.6521451549,$$

$$w_1 = w_3 = 0.3478548451.$$

The resulting U -matrix, defined in (6) and computed with double precision arithmetic, will be

$$U = \begin{bmatrix} 0.571027649 & 0.336257876 & -0.417046068 & -0.622037488 \\ 0.417046067 & 0.622037489 & 0.571027647 & 0.336257874 \\ 0.571027649 & -0.336257876 & -0.417046068 & 0.622037488 \\ 0.417046067 & -0.622037489 & 0.571027647 & -0.336257874 \end{bmatrix}$$

It can be verified that this matrix is orthogonal to within an error of 6×10^{-9}. A matrix with the selected eigenvalues λ_1, λ_2, λ_3, and λ_4 is then computed by forming the product

$$A = U \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} U^T \quad (13)$$

The right eigenvectors are simply the column vectors of the matrix U. A complete spectrum of U-matrices can be automatically generated by varying the integration order corresponding to the matrix size and the number of prespecified eigenvalues.

2.2 *U -Creation Using Tchebyshev Polynomials*

The Chebyshev polynomial of degree n in x is defined in [2] by

$$T_n(x) = \cos(n \cos^{-1} x) \quad (14)$$

from which

$$T_0(x) = 1,$$

$$T_1(x) = x,$$

$$T_2(x) = 2x^2 - 1,$$

$$T_3(x) = 4x^3 - 3x,$$

$$\ldots\ldots\ldots \quad (15)$$

with the orthogonality property

$$\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & m \neq n; \\ \frac{\pi}{2}, & m = n \neq 0; \\ \pi, & m = n = 0. \end{cases} \quad (16)$$

Then taking

$$f_n(x) = \sqrt{\frac{2}{\pi\sqrt{1-x^2}}}\, T_n(x), n \neq 0$$

we can set

$$f_0(x) = \frac{1}{\sqrt{2}} \phi(x),$$

$$f_1(x) = x\phi(x),$$

$$f_2(x) = (2x^2 - 1)\phi(x),$$

$$f_3(x) = (4x^3 - 3x)\phi(x).$$

$$\ldots\ldots\ldots \quad (17)$$

with

$$\phi(x) = \sqrt{\frac{2}{\pi\sqrt{1-x^2}}}, \quad (18)$$

and

$$\int_{-1}^{1} f_n^2(x)\, dx = \sum_{i=0}^{p-1} w_i\, f_n^2(x_i) = 1,$$

$$\int_{-1}^{1} f_n(x) f_m(x)\, dx = \sum_{i=0}^{p-1} w_i\, f_n(x_i) f_m(x_i) = 0. \quad (19)$$

Considering a 4-by-4 matrix example with the same abscissae given in Section 2.1. The weights in this case are $w_0 = w_2 = 0.570125788$, $w_1 = w_3 = 0.490391524$. The resulting U- matrix, defined in (6) and computed with double precision arithmetic, will be

$$U = \begin{bmatrix} 0.540355373 & 0.460439775 & -0.477627776 & -0.535979687 \\ 0.456087739 & 0.534991526 & 0.521413182 & 0.461222045 \\ 0.540355373 & -0.460439775 & -0.477627776 & 0.535979687 \\ 0.456087739 & -0.534991526 & 0.521413182 & 0.461222045 \end{bmatrix}$$

It can also be verified that this matrix is orthogonal to within an error of order 10^{-9}.

3. CONCLUSION

A simple method to generate matrices with prespecified eigenvalues has been presented. The method is based on the orthogonal properties of orthogonal polynomials and Gaussian quadrature integration. Legendre and Chebyshev polynomials are used as illustrative examples for the proposed technique.

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