EFFECT OF AXIAL FORCE ON THE BEHAVIOR OF BEAM COLUMN

Mahmoud A. Sharaki
Transportation Department Faculty of Engineering,
Alexandria University, Alexandria, Egypt.

ABSTRACT

In many application both axial and lateral forces exist in structural members, for example. harbor, costal, offshore construction and rail roads. Solution using exact stiffness matrix is already available. However, spacial variation of soil properties makes this approach out of question, or with limited applicability at most. In this paper a method is introduced to find the response, buckling load, and the stiffness constants of a beam column, the variation of soil properties along the members and the effect of the axial force. A comparison of the results obtained using the new proposed method and those introduced through other known methods is presented.

1. INTRODUCTION

The general equilibrium equation of a solid body is

$$[K]\{U\} = \{R\} \tag{1}$$

where,

[K] the stiffness matrix
{U} and {R} the nodal displacement and force vectors, respectively

The energy stored in any elastic deformable body is

$$E = 0.5\,\{U\}^T\,[K]\,\{U\} \tag{2}$$

For a beam column subjected to axial loading and embedded in soil, the energy stored in the system consists of three parts; flexural energy in the beam, potential energy due to the axial force and the strain energy in the soil mass.

2. OVERALL STIFFNESS MATRIX

As was mentioned before, the total energy stored in the system is the sum of three separate components. From (2) it seems natural to consider the overall stiffness matrix as the sum of three separate matrices that contribute to the flexural deformation, axial force potential and strain energy stored in the soil mass, respectively. In case of non linear soil, we use the tangential stiffness matrices and iterative procedures.

2.1 Flexural Energy

For a beam element, we assume the displacement to be given by

$$Y = \{H_f\}^T\,\{U\} \tag{3}$$

where,

{U} the nodal displacement
$\{H_f\}$ the shape function associated with flexural deformation, given by

$$\{H_f\} = \begin{Bmatrix} 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \\ x\left(1 - \frac{x}{L}\right)^2 \\ 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \\ \frac{x^2}{L}\left(\frac{x}{L} - 1\right) \end{Bmatrix} \tag{4}$$

L the length of the element,and
x a running coordinate along the element.

The degrees of freedom $U_1,\ldots,U_4$ are shown in figure (1)

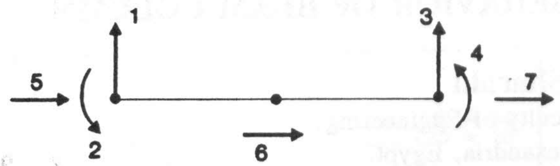

Figure 1. Positive sign convention.

The flexural stiffness matrix for a beam element can be obtained using a direct finite element formulation or using energy approach. In the later case, we have

$$E_f = \frac{EI}{2}\int_0^L \frac{d^2y}{dx^2}\,dx \qquad (5)$$

where,

E_f the flexural energy stored in the beam element, and
EI the flexural rigidity of the beam

In both cases, we get $[K_f]$ the beam stiffness matrix, for flexural deformation

$$[K_f] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{SYMMETRIC} & & & 4L^2 \end{bmatrix} \qquad (6)$$

2.2. *Potential Energy Due to Axial Force*

The potential energy due to axial force N is given by

$$E_a = -0.5\int_0^L N\left(\frac{dy}{dx}\right)^2 dx - 0.5\int_0^L N\frac{dy}{dx}\,dx \qquad (7)$$

The first integral on the right hand side of (7) represents the potential due to flexural deformation, and the second integral represents the potential due to axial deformation.

In case of allowing the axial force to vary linearly along the member, then we have shape functions of the form

$$\{H_a\} = \begin{Bmatrix} (1-\frac{x}{L})-0.5[1-(1-\frac{2x}{L})^2] \\ 1-(1-\frac{2x}{L})^2 \\ (\frac{x}{L})-0.5[1-(1-\frac{2x}{L})^2] \end{Bmatrix} \qquad (8)$$

The axial translation is given by (3) after replacing {H} and {U} by the appropriate shape functions and nodal displacements.

Allowing the normal force to vary linearly with the element, give a better approximation and hence enables one to use less elements.

Using (4),(8), after differentiation, and (3) with (7), and casting the result in a form similar to (2), we get

$$E_a = -0.5\{U\}^T[K_{G1}]\{U\} - \{U\}^T[K_{G2}]\{U\} \qquad (9)$$

where,

E_a the potential energy associated with the normal force due to flexural and axial deformation, and

$$[K_{G1}] = N_1\begin{bmatrix} \frac{6}{5}L & \frac{1}{10} & \frac{-6}{5}L & \frac{1}{10} \\ & \frac{2}{15}L & \frac{-1}{10} & \frac{-1}{30}L \\ & & \frac{6}{5}L & \frac{-1}{10} \\ & \text{SYMMETRIC} & & \frac{2}{15}L \end{bmatrix} \qquad (10)$$

and

$$[k_{G2}] = \frac{N_2 - N_1}{L}\begin{bmatrix} \frac{3}{5} & \frac{1}{10}L & \frac{-3}{5} & 0 \\ & \frac{1}{30}L^2 & \frac{-1}{10}L & \frac{-1}{60}L^2 \\ & & \frac{3}{5} & 0 \\ & \text{SYMMETRIC} & & \frac{1}{10}L^2 \end{bmatrix} \qquad (11)$$

where,

N_1 and N_2 the axial force at the first and second nodes of the element, respectively.

The matrices K_{G1} and K_{G2} are called geometrical stiffness matrices

The stiffness matrix K_a associated with axial deformation is given by

$$[K_a] = \frac{AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ & 16 & -8 \\ \text{SYMMETRIC} & & 7 \end{bmatrix} \qquad (12)$$

2.3. *Energy stored in the Soil Mass*

To find the stiffness matrix due to soil reaction, we process as follows. If the beam is deformed to take the first mode shape, see equation (4), given by

$$y = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

and the soil reaction F(x) is assumed to be given by

$$F(x) = f(x,y) \qquad (13)$$

where f(x,y) is the force per unit length of the beam, at distance x, for deflection y at the same distance x.

The nodal forces {F} associated with loading F(x) are given by [4]

$$\{F\} = \int_0^L \{H_f\}\, F(x)\, dx \qquad (14)$$

where

{F} a vector of th 4th order,

$\{H_f\}$ shpe function to be given by equation (4), and

F(x) the force per unit length of the beam, at distance x.

For a linear system, if the deflected shape y is given by $y = h_1$ where h_1 is the first mode shape, then the soil reaction F(x) is given by $F(x) = k\, h_1$, where k is a constant represening the subgrade reaction. Finally we get

$$\{F\} = k\int_0^L \{H_f\}\, h_1(x)\, dx \qquad (15)$$

In (13), (14) and (15), we assume a Winkler type foundation. However the method can be extended to cover the cases of continuum, assuming the existence of a Green function of the form

$$G(x) = \int_0^L f(y,x,\xi)\, d\xi \qquad (16)$$

where, y the deflection at point ξ.

The function G(x) is to replace F(x) in (14).

From the above and the definition of the stiffness constants, it may be concluded that if a beam element is deformed to take the shape given by h_1 and at the same time subjected to a lateral load defined by $k\, h_1$ then the nodal reaction (stiffness constants) for this case of loading and deformation are given by

$$K_{1,i} + F_i \qquad i = 1,2,3,4$$

Working as before for loading given by $k\, h_i$ $(i = 2,3,4)$ and deformation given by h_i $i=2,3,4$, respectively, we conclude that the stiffness matrix for the soil reaction is given by

$$[K_S] = k\int_0^L \{H_f\}\{H_f\}^T dx \qquad (17)$$

for the case of linear soil, and by

$$[K_S] = \int_0^L \{H_f\}\,\{F_1(x),\ldots,\ldots,F_4(x)\}\, dx \qquad (18)$$

for the case of non-linear soil.

where, $F_i(x)$ is the soil reaction due to deflection given by h_i, and is understood to be given by (13) or (16) according to the soil conditions.

The above applies equally for the case of soil stiffness constants due to displacements in the direction of the axis of the member, assuming that the appropriate shape functions and load vector are used in (17) and (18).

In case of linear soil of Winkler type, where the horizontal soil parameter k_h varies linearly with depth, the soil stiffness matrices $[K_{SH1}]$ and $[K_{SH2}]$, are given by

$$[K_{SH1}] = k_{h1}\begin{bmatrix} \frac{13}{35}L & \frac{11}{210}L^2 & \frac{9}{70}L & \frac{-13}{420}L^2 \\ & \frac{1}{105}L^3 & \frac{13}{420}L^2 & \frac{-1}{140}L^3 \\ & & \frac{13}{35}L & \frac{-11}{210}L^2 \\ & \text{SYMMETRIC} & & \frac{1}{105}L^3 \end{bmatrix} \qquad (19)$$

and

$$[K_{SH2}] = \frac{k_{h2}-k_{h1}}{L}\begin{bmatrix} \frac{3}{35}L^2 & \frac{1}{60}L^3 & \frac{9}{140}L^2 & \frac{-1}{70}L^3 \\ & \frac{1}{280}L^4 & \frac{1}{60}L^3 & \frac{-1}{280}L^4 \\ & & \frac{2}{7}L^2 & \frac{-1}{28}L^3 \\ & \text{SYMMETRIC} & & \frac{1}{168}L^4 \end{bmatrix} \qquad (20)$$

where k_{h1} and k_{h2} the soil parameters at the first and second nodal points of the element, respectively.

The soil stiffness matrix for the vertical direction , axial in case of piles and vertical members, is given by, assuming the axial force to vary linearly along the member,

$$[K_{SV1}] = k_{V1}\, L\begin{bmatrix} \frac{2}{15} & \frac{1}{15} & \frac{-1}{30} \\ & \frac{8}{15} & \frac{1}{15} \\ & \text{SYMMETRIC} & \frac{2}{15} \end{bmatrix} \qquad (21)$$

and

$$[K_{SV2}] = (k_{V2} - k_{V1})L \begin{bmatrix} \frac{1}{60} & 0 & \frac{1}{60} \\ & \frac{4}{15} & \frac{1}{15} \\ \text{SYMMETRIC} & & \frac{7}{60} \end{bmatrix} \quad (22)$$

where k_{V1} and k_{V2} are the soil parameters in the vertical direction at the first and second nodal points of the element

The method can be extended to cover the case where the variation of the soil parameters, along the member, is given in the form of a polynomial of any order, any analytical function, or tables.

The sum of $[K_{SV1}]$ and $[K_{SV2}]$ gives the total soil stiffness matrix $[K_{SV}]$ for vertical loading, loading in direction of the axis of the member.

$$[K_{SV}] = [K_{SV1}] + [K_{SV2}] \quad (23)$$

3 THE EQUATION OF MOTION

From equation (1) and parts 2.1 through 2.3, we find that the equation of motion for any element takes the form

$$\begin{bmatrix} K_B & | & 0 \\ --- & + & --- \\ 0 & | & K_N \end{bmatrix} \begin{Bmatrix} U_B \\ --- \\ U_N \end{Bmatrix} = \begin{Bmatrix} R_B \\ --- \\ R_N \end{Bmatrix} \quad (24)$$

where

$\{U_B\}$ = $\{U_1, \ldots, U_4\}$ is the displacement vector located in a plan perpendicular to the element.
$\{U_N\}$ = $\{U_5, \ldots, U_7\}$ is the displacement vector in the direction of the member axis.
$\{R_B\}$ the force vector associated with $\{U_B\}$
$\{R_N\}$ the force vector associated with $\{U_N\}$
$[K_B]$ the stiffness matrix relating $\{U_B\}$ and $\{R_B\}$, and is given by

$$[K_B] = [K_f] + [K_G] + [K_{SH}] \quad (25)$$

$[K_N]$ the stiffness matrix relating $\{U_N\}$ and $\{R_N\}$, and is given by

$$[K_N] = [K_a] + [K_{SV}] \quad (26)$$

We first assemble the global stiffness matrix for the complete structure (beam or pile), then solve for the displacements parallel to the structure's axes, $\{U_N\}$

Using the relation $N = EA \frac{dy}{dx}$, we find the normal force at the different nodal points, then we solve for the displacement vector $\{U_B\}$

4. THEORY VERFICATION

In what follows, we introduce the results of analysis obtained using the present method versus those obtained using analytical methods The results include different cases of loading, supporting conditions, and cases with and without surrounding soil.

4.1. *Axial Load*

For a simple beam loaded uniformly , and subjected to axial force P, the maximum deflection δ_{max} and the maximum moment M_{max} are given respectively by [1]

$$\delta_{max} = \frac{5qL^4}{384EI}\eta(u)$$

$$M_{max} = \frac{qL^2}{8}\lambda(u)$$

where

q the load intensity
L beam's length
EI flexural rigidity,and
$\eta(u)$ the deflection's amplification factor to be given by

$$\eta(u) = \frac{12(2\sec u - 2 - u^2)}{5u^4}$$

$\lambda(u)$ the moment's amplification factor to be given by

$$\lambda(u) = \frac{2(1-\cos u)}{u^2 \cos u}$$

and

$$u = \frac{L}{2}\sqrt{\frac{P}{EI}}$$

Table 1 shows the amplification factors for the two cases mentioned above. The factors are calculated once using Timoshenko's method given above, and once using the proposed method. The table shows a very good agreement between the two methods.

Table 1. Deflection and Moment amplification factors due to axial load for simple beam uniformly loaded.

2u	Deflection		Moment	
	Timoshenko	Present method	Timoshenko	Present method
0.0	1.000	1.000	1.000	1.000
0.2	1.004	1.005	1.004	1.011
0.4	1.016	1.017	1.016	1.024
0.6	1.037	1.039	1.038	1.046
0.8	1.070	1.070	1.073	1.079
1.0	1.114	1.114	1.117	1.124
1.2	1.173	1.172	1.176	1.184
1.4	1.250	1.25	1.255	1.264
1.6	1.354	1.352	1.361	1.370
1.8	1.494	1.492	1.504	1.514
2.0	1.690	1.685	1.704	1.715
2.2	1.962	1.968	1.989	2.007
2.4	2.400	2.410	2.441	2.464
2.6	3.181	3.187	3.240	3.269
2.8	4.822	4.888	4.938	5.030
2.9	6.790	6.807	6.940	7.018
3.0	11.490	11.455	11.670	11.833
π	∞	2021.14	∞	2093.79

4.2. *Buckling Load*

To check the validity of the proposed method to determine the buckling load of a beam column with different supporting conditions, we apply a small lateral load at any appropriate location and find the resulting lateral deformation associated with different levels of axial loading. The buckling load may be defined as the axial load that keep lateral deflection after removal of the lateral force that produced it, or, according to Timoshenko [1], the minimum axial load that causes the amplification factor to be reversed.

Figure (2) shows the relation between the axial load and the resulting lateral deflection for a cantilever. In this case a constant lateral load equal 1 ton is applied at the free end of the cantilever, and the axial load is applied at increments, each equal 0.1 of the buckling load. The buckling load calculated using the proposed method is found to be exactly the same as that calculated using the exact formula.

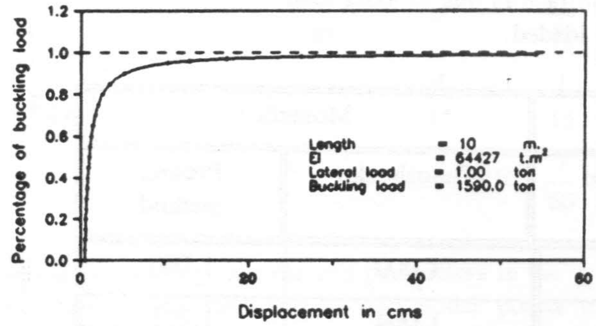

Figure 2. Effect of axial load on the displacement of the free end of a cantilever.

Figure (3) shows the relation between the axial load and the resulting lateral deflection for a hinged beam. In this case a constant lateral load equal 1 ton is applied at the mid span of the beam, and the axial load is applied in a manner similar to the one above. The buckling load calculated using the proposed method is found to be exactly the same as that calculated using the exact formula.

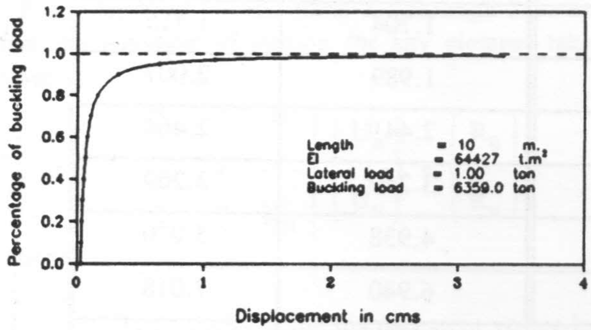

Figure 3. Effect of axial load on the displacement at mid span of hinged beam.

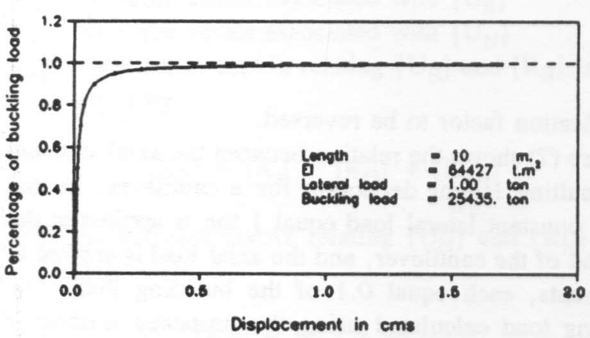

Figure 4. Effect of axial load on the deflection at mid span of fixed end beam.

Figure (4) shows the relation between the axial load and the resulting lateral deflection for a fixed end beam. This case is treated exactly as the one before. The buckling load calculated using the proposed method is found to be exactly the same as that calculated using the exact formula.

4.3 *Deflected Shape*

The deflected shape of a beam, with length L, on elastic foundation, with subgrade reaction k, with both of its ends hinged, is given, for the case of concentrated moment M acting at one of its ends, by [2]

$$y = \frac{2MB^2}{k(\cosh^2 BL - \cos^2 BL)} *$$

$$* [\cosh BL \sin Bx \sinh B(L-x) \cos BL \sinh Bx \sin B(L-x)]$$

where

$$B = \sqrt[4]{\frac{k}{4EI}}$$

Figure (5) shows the deflected shape for the case of long beam, where $BL = 5.936 > 5$ as calculated using the exact method and the proposed method using 2,5, and 10 elements. Figure (6) shows the deflected shape for the case of short beam, where $BL = 0.469 < 0.6$ as calculated using the exact method and the proposed method using 2,5,and 10 elements.

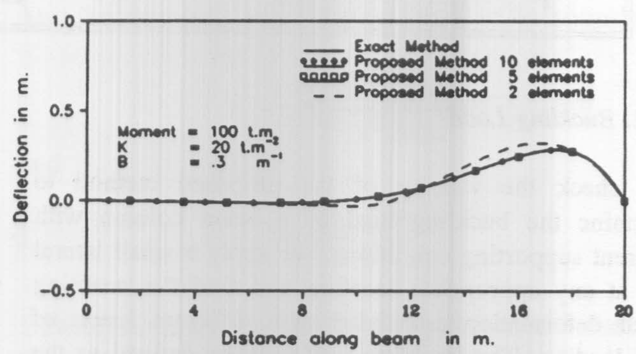

Figure 5. Deflected shape of a hinged-hinged long beam due to a moment at one of its ends.

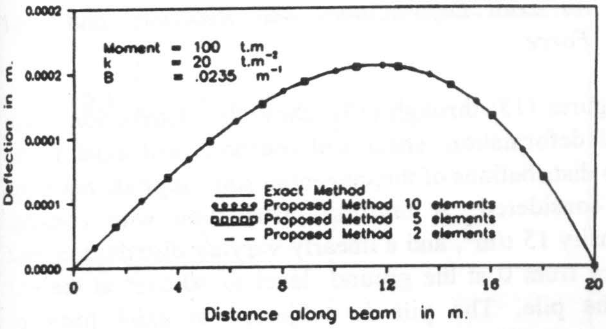

Figure 6. Deflected shape of hinged-hinged short beam due to a moment at one of the ends.

Figures (5) and (6) show that the proposed method predicts the deflected shape very accurately, even with a small number of elements.

4.4. *Effect of Variation in Subgrade Reaction on the Deflected Shape, Bending Moment and Soil Reaction*

Figures (7) and (8) show the variation in the deflection, for the cases considered above, due to variation in the distribution of the subgrade reaction. Three cases are considered;

case a:- the subgrade reaction is constant along the length of the beam, with value equal 20 $t.m^{-2}$.

case b:- the subgrade reaction varies linearly from 0 at the right end of the beam , point of application of the load, to 40 $t.m^{-2}$ at the left end of the beam.

case c:- the same as case b, with the values of the subgrade reaction change places.

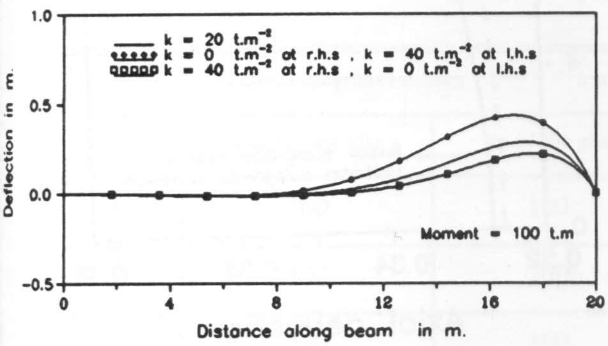

Figure 7. Variation in displacement due to variation in subgrade reaction, hinged-hinged beam with moment at one of its ends.

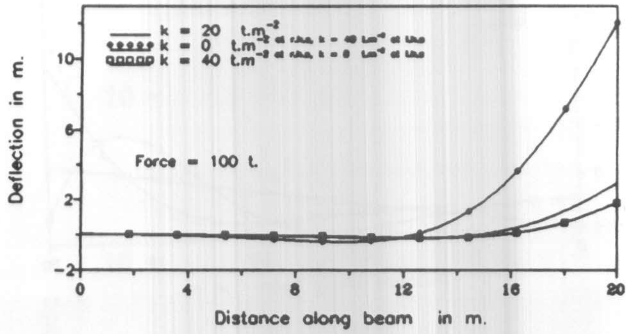

Figure 8. Variation in displacement due to variation in subgrade reaction, free-free beam with force at one of its ends.

Figures (9) and (10) show the variation in soil reaction, while figures (11) and (12) show the variation in bending moment along the beam. Table 2 illustrates the symbols used in the figures (9) through (12)

Table 2. Lateral subgrade reaction and factor B used to produce Figures (9) through (12)

Case	Lateral subgrade reaction k t/m^2		LB m^{-1}	
	at L.H.S	at R.H.S	at L.H.S	at R.H.S
A	20	20	5.936	5.936
B	20	20	0.469	0.469
C	40	0	7.059	0
D	0	40	0	7.059

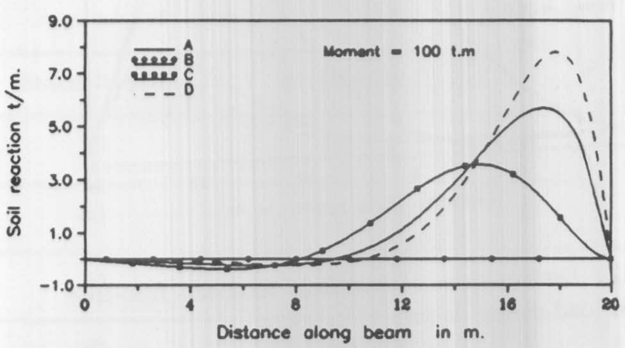

Figure 9. Distribution of soil reaction, hinged-hinged beam with moment acting at one of its ends.

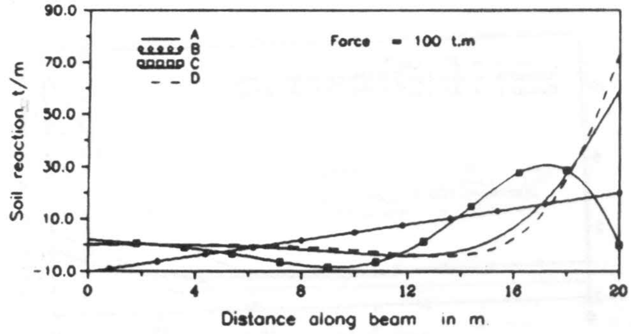

Figure 10. Distribution of soil reaction free-free beam with load acting at one of its ends.

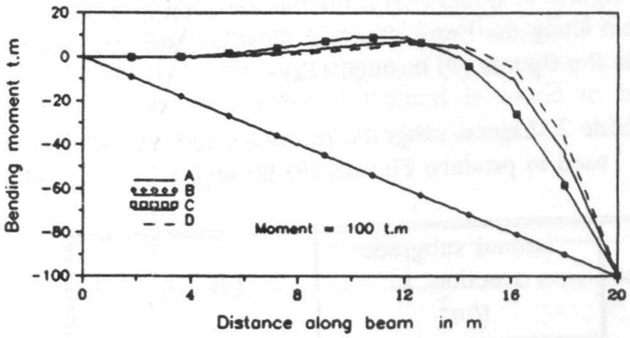

Figure 11. Distribution of bending moment hinged-hinged beam with moment acting at one of its ends.

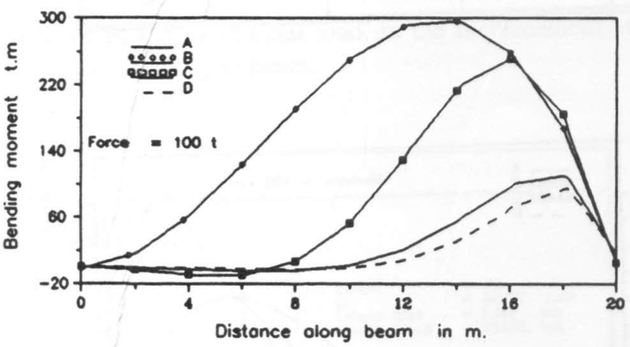

Figure 12. Distribution of bending moment free-free beam with load acting at one of its ends.

4.5. *Effect of Variation of Tangetial Subgrade Reaction on axial Deformation, Soil Reaction, and Axial Force*

Figures (13) through (15) show the distribution of the axial deformation, shear soil reaction, and axial force. Two distributions of the tangential soil subgrade reactions are considered. A uniform distribution with constant intensity 15 t/m^2, and a linearly varying distribution; that varies from 0 at the ground level to 30 t/m^2 at the end of the pile. The pile is subjected to axial force of magnitude 100 ton at the top , EA of the pile is 41233.4 ton, the over all length of the pile is 22 m. with 2 m. prolonged out of the ground.

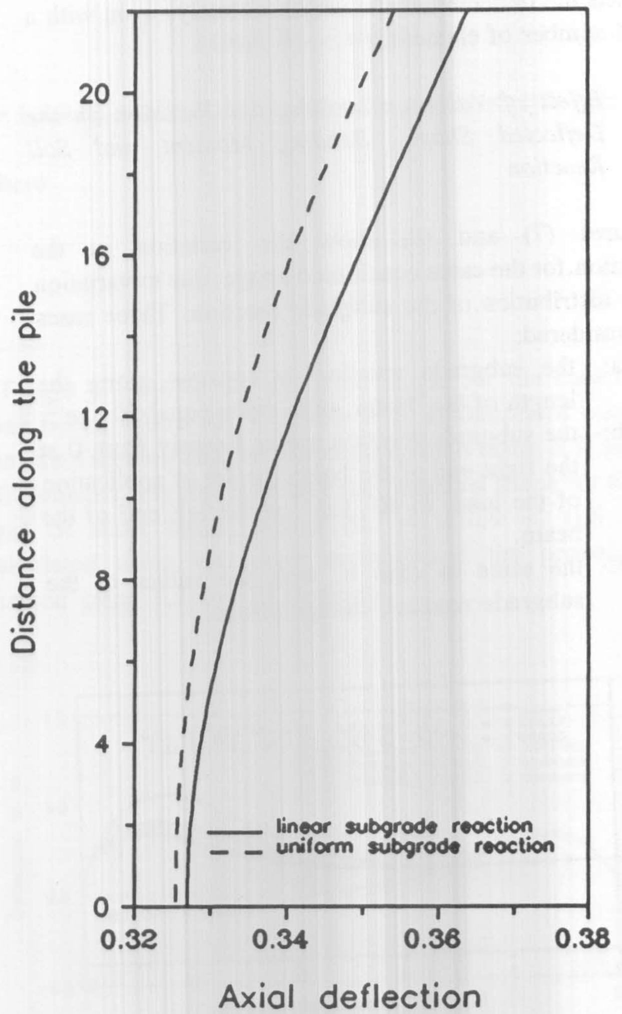

Figure 13. Axial deformation.

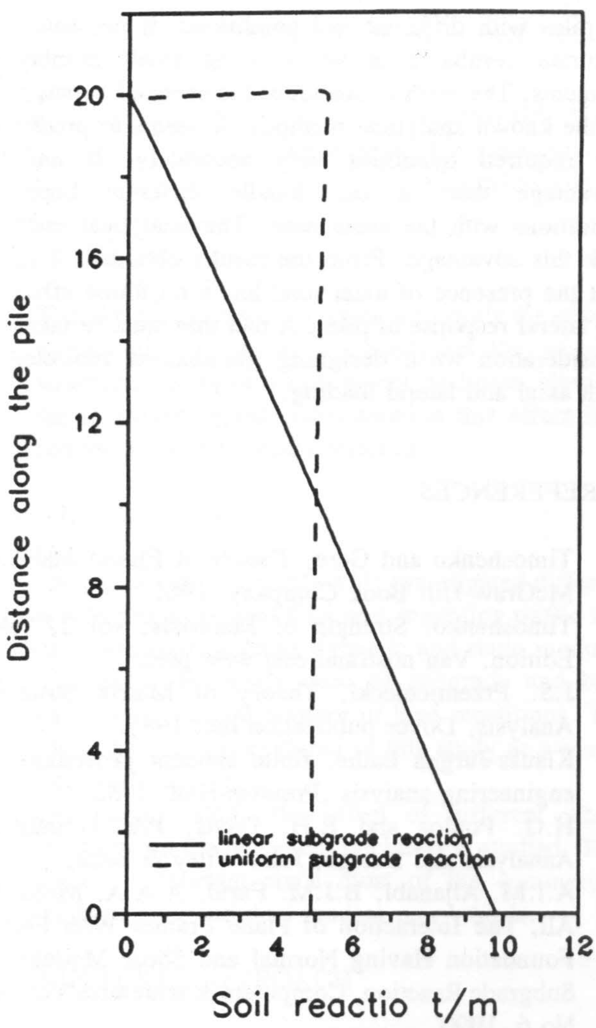

Figure 14. Tangential soil reaction.

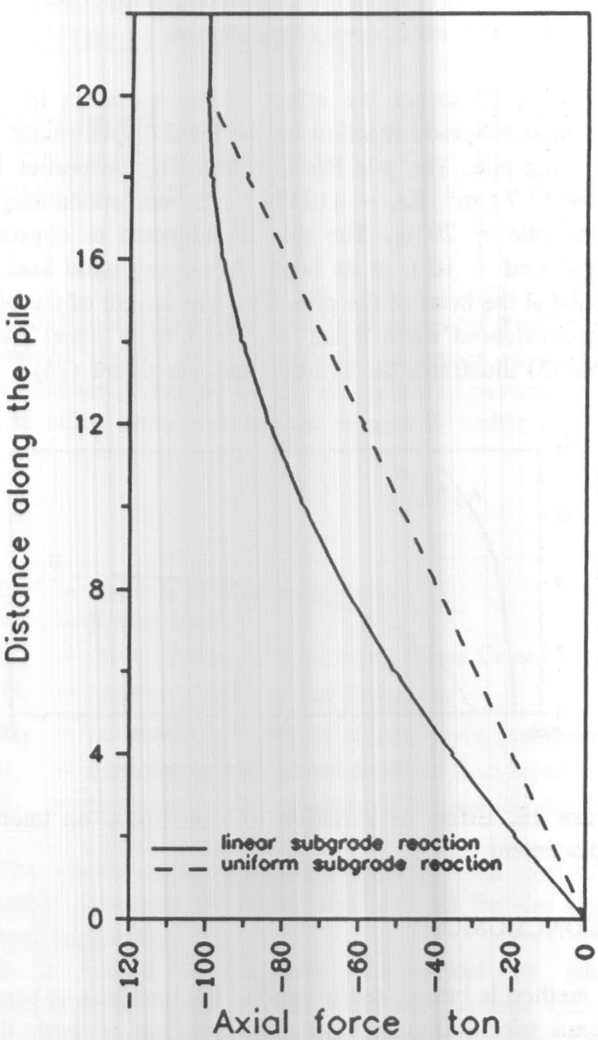

Figure 15. Axial force.

Table 3. Symbols used in Figure (16).

Case	Lateral subgrade reaction t/m^2		Axial subgrade reaction t/m^2		free length m.
	head of the pile	tip of the pile	head of the pile	tip of the pile	
A	100	100	30	30	0
B	100	100	60	0	0
C	100	100	30	30	2
D	100	100	60	60	2

4.6. *Effect of Varaition of Tangential Subgrade Reaction on Lateral Displacement*

Figure (16) shows the effect of the variation in the tangential subgrade reaction on the head displacement of a floating pile. The pile has the following properties EI = 64427.7 t.m^2, EA = 412334 t., the embedded length of the pile = 20 m. The pile is subjected to constant lateral load = 10 t. at its head. A varying axial load is applied at the head of the pile. The free length of the pile was considered once equal zero and once equal two. Table (3) illustrates the symbols used in figure (16)

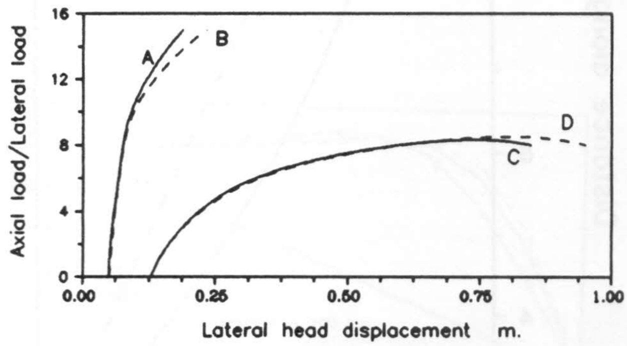

Figure 16. Effect of variation of axial force on lateral displacement.

5. CONCLUSION

A method is introduced to predict the response of beam column taking into account the variation in both the lateral and the tangential subgrade reactions. The method considers uniform and linearly varying subgrade reactions. However, it can be extended directly to cover the case of variation of order higher than the first. The method is capable of predicting the buckling load and the response of piles with different end conditions. It presents very accurate results even when using small number of elements. The method introduced is checked versus some of the known analytical methods. It seems to predict all the required quantities very accurately. It has the advantage that; it can handle different boundary conditions with the same ease. The analytical methods lack this advantage. From the results obtained, it seems that the presence of axial load has a profound effect on the lateral response of piles. A fact that must be taken int consideration when designing foundations subjected to both axial and lateral loading.

6. REFERENCES

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