AN EDGE-CRACKED PLATE WITH FREE AND CONSTRAINED BOUNDARIES SUBJECTED TO SUDDEN CONVECTIVE COOLING

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ABSTRACT

The transient thermal stress edge crack problem for an elastic strip with free and fully constrained boundaries is considered. The plate is suddenly subjected to convective cooling on the face containing the edge crack while the other face is insulated. The solution of the problem is obtained by using the superposition technique results in a singular integral equation which is solved numerically. The results of the transient temperature and thermal stress distributions in the uncracked strip are presented. Also, numerical results are obtained for the stress intensity factor in terms of Fourier number, crack length, and different values of Biot number.

INTRODUCTION

The study of cracking in brittle solids under thermal stresses is important in many engineering applications. When a surface subjected to sudden change in temperature, specially in the presence of cracks, large thermal stresses arise around the crack tips. These stresses may propagate the crack resulting in serious damage. The important mode of mechanical failure is the subcritical crack growth, which needs the determination of the stress intensity factor as a function of time and crack length.

There are many studies of a cracked plate subjected to transient thermal stresses. For example, Sih [1] considered the singular character of the thermal stresses at the crack tips of a line crack in an infinite medium. Rizk and Radwan [2] studied the cracked semi-infinite medium subjected to sudden cooling in the form of ramp function. Nied [3-4] discussed the problem of an edge cracked plate under convective cooling and heating respectively with free boundaries. Also, Rizk and Radwan [5] analyzed the case of an edge cracked plate under thermal shock due to sudden cooling in terms of ramp function with free boundaries.

In this paper, we are interested in the problem of an edge cracked plate subjected to convective cooling in the face containing the crack with free boundary in the face x = 0, and fully constrained in the other face x = H, which is shown in Figure (1-a).

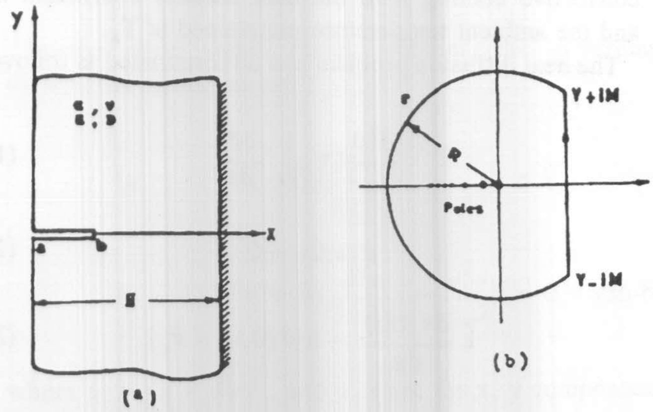

Figure 1-a. The edge crack geometry with free boundary at x = 0 and fully constrained boundary at x = H.
Figure 1-b. Contour Γ used to evaluate the integral in Eq. (11).

The problem is solved under the assumption that the inertia effects are negligible. The previous work on dynamic thermoelasticity seems to justify this assumption [6-7]. Also, in this study the thermoelastic coupling effects and the dependence of thermoelastic coefficients are neglected. By assuming the material is linear, the principle of superposition can be used to formulate the problem in terms of a singular integral equation which is solved numerically. The main results of this work are the stress intensity factor as a function of nondimensional time (Fourier number), crack length, and various values of Biot number.

MATHEMATICAL FORMULATION

The formulation of the crack problem depicted in Figure (1-a), utilized the transient thermal stress distribution from the uncracked strip. The procedure is first to obtain the transient temperature distribution by solving the diffusion equation, and to use it in the uncracked strip with free and fully constrained boundaries to determine the transient thermal stress distribution. Once we obtain the thermal stress distribution for the uncracked strip, the solution of the crack problem can be obtained by applying the equal and opposite of these stresses to the crack surfaces.

Temperature Distribution

Consider an elastic strip of thickness H is at an inial temperature T_o and insulated along the plane x = H. At t = 0, the surface x = 0 is suddenly subjected to convective cooling with the heat transfer coefficient h, and the ambient temperature maintained at T_a.

The heat diffusion problem can be formulated as follows

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{D}\frac{\partial\theta}{\partial t} \tag{1}$$

$$\theta(x,0) = 0 \tag{2}$$

$$\acute{k}\,\frac{\partial\theta(0,t)}{\partial x} = h[\theta(0,t) - \theta_o] \tag{3}$$

$$\frac{\partial\theta(H,t)}{\partial x} = 0 \tag{4}$$

where

$$\theta(x,t) = T(x,t) - T_o \tag{5}$$

$$\theta_o = T_a - T_o \tag{6}$$

$\acute{k}$ and D are the material thermal conductivity and thermal diffusivity, respectively. By applying Laplace transform we can have [8]

$$\frac{d^2\bar{\theta}(x,p)}{dx^2} = q^2\,\bar{\theta}(x,p) \tag{7}$$

$$\acute{k}\,\frac{d\bar{\theta}(0,p)}{dx} = h[\bar{\theta}(0,p) - \frac{\theta_o}{p}] \tag{8}$$

$$\frac{d\bar{\theta}(H,p)}{dx} = 0 \tag{9}$$

where $\bar{\theta}(x,p)$ is the Laplace transform of $\theta(x,t)$ and $q^2 = p/D$. The solution of equation (7) under the boundary conditions (8) and (9) can be written as

$$\frac{\bar{\theta}(x,p)}{\theta_o} = \frac{\cosh qH\ \cosh qx - \sinh qH\ \sinh qx}{p(\cosh qH + \frac{\acute{k}}{h}q\ \sinh qH)} \tag{10}$$

Applying the inversion theorem of Laplace transform we get

$$\frac{\theta(x,t)}{\theta_o} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{tz}[\cosh qH \cosh qx - \sinh qH \sinh qx]}{z[\cosh qH + \frac{\acute{k}}{h} q \sinh qH]}\,dz \tag{11}$$

The integrand in equation (11) is a single-valued function of z with simple poles at z = 0 and at the roots of the equation

$$\cosh qH + \frac{\acute{k}}{h}q\ \sinh qH = 0 \tag{12}$$

By putting $q = \sqrt{(z/D)} = i\eta$, then we can obtain the transcendental equation

$$\lambda_n \tan\lambda_n = Bi \tag{13}$$

where Bi is the Biot number defining by $h H/\acute{k}$, and $\lambda_n = H\eta_n$. So, the location of the simple poles are

$$z_n = -D\,\eta_n^2 = -\frac{D}{H^2}\lambda_n^2 \quad ,n = 1,2,\ldots\infty \tag{14}$$

These poles are distributed in negative direction of real axis. By applying the Residue theorem around the contour Γ shown in Figure (1-b), and defining the dimensionless quantities

$$\tau = \frac{tD}{H^2} \quad , \quad x^* = \frac{x}{H} \tag{15}$$

The temperature distribution can be written as

$$\frac{T(x,t) - T_o}{T_a - T_o} = 1 - 2\sum_{n=1}^{\infty} \frac{\sin\lambda_n \cos\lambda_n(x^* - 1)}{\lambda_n + \frac{1}{2}\sin 2\lambda_n}\, e^{-\tau\lambda_n^2} \tag{16}$$

which can be put in the form

$$\frac{T(x,t)-T_a}{T_o-T_a}=2\sum_{n=1}^{\infty}\frac{\sin\lambda_n\cos\lambda_n(x^*-1)}{\lambda_n+\frac{1}{2}\sin 2\lambda_n}e^{-\tau\lambda_n^2} \quad (17)$$

where the eigenvalues λ_n are determined from equation (13)

Thermal Stresses in The Uncracked Strip:

The elastic strip is assumed to be free at the boundary x = 0, and fully constrained in the other boundary x = H. So,it would remain flat under self-equilibrating transient thermal stresses, i.e. we can assume that the strip undergo uniform strain $\varepsilon_o(t)$ over the thickness H.The thermal stresses and strains would satisfy the following relations (see for example [9])

$$\sigma_{xx}^T = 0 \quad , \quad \sigma_{yy}^T = \sigma_{zz}^T \quad (18)$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_o(t) \quad (19)$$

Also, in the absence of external load, the thermal stresses would satisfy the condition of no resultant force in y and z directions, i.e.

$$\int_0^H \sigma_{yy}^T dx = 0 , \quad \int_0^H \sigma_{zz}^T dx = 0 \quad (20)$$

Applying the Hook's law we obtain

$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy}^T - \nu\,\sigma_{zz}^T) + \alpha\,T(x,t) \quad (21)$$

where E , ν and α are Young's modulus, Poisson's ratio and coefficient of thermal expansion. Substituting equations (18) and (19) into equation (21) we get

$$\sigma_{yy}^T(x,t) = \frac{E}{1-\nu}\,[\varepsilon_o(t) - \alpha\,T(x,t)] \quad (22)$$

Integrating equation (22) over the cross-section and using the condition (20) we can have

$$\varepsilon_o(t) = \frac{\alpha}{H}\int_0^H T(x,t)\,dx \quad (23)$$

Then the thermal stresses $\sigma_{yy}^T(x,t)$ can be found as

$$\sigma_{yy}^T(x,t) = \frac{E\,\alpha}{1-\nu}\,[\int_0^H T(x,t)dx - T(x,t)] \quad (24)$$

Substituting equation (17) into equation (24), the thermal stresses become

$$\frac{\sigma_{yy}^T(x^*,\tau)(1-\nu)}{\alpha E(T_o-T_a)} = 2\sum_{n=1}^{\infty}\frac{\sin^2\lambda_n}{\lambda_n^2+\frac{\lambda_n}{2}\sin 2\lambda_n}e^{-\tau\lambda_n^2}$$

$$-2\sum_{n=1}^{\infty}\frac{\sin\lambda_n\cos\lambda_n(x^*-1)}{\lambda_n+\frac{1}{2}\sin 2\lambda_n}e^{-\tau\lambda_n^2} \quad (25)$$

The Crack Problem

In the plane strain crack problem, the governing differential equations are

$$(\chi - 1)\nabla^2 u + 2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\partial y}\right) = 0 \quad (26\text{-a})$$

$$(\chi - 1)\nabla^2 v + 2\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial y^2}\right) = 0 \quad (26\text{-b})$$

where $\chi = (3 - 4\nu)$, and u, v are the x, y components of the displacement vector. Because of symmetry the problem is considered for $y > 0$. The boundary and mixed conditions of the problem are

$$\sigma_{xx}(0,y) = 0 \quad , \quad \sigma_{xy}(0,y) = 0 \quad (27)$$

$$\sigma_{xy}(H,y) = 0 \quad , \quad u(H,y) = 0 \quad (28)$$

$$u \to 0 \quad , \quad v \to 0 \quad \text{as} \quad y \to \infty \quad (29)$$

$$\sigma_{xy}(x,0) = 0 \quad (30)$$

$$v(x,0) = 0 \quad , \; 0 < x < a \; , \; b < x < H \quad (31\text{-a})$$

$$\sigma_{yy}(x,0) = p(x) = -\,\sigma_{yy}^T(x,t) \; , \; a < x < b \quad (31\text{-b})$$

The solution of the mixed boundary value problem can be obtained by expressing the displacement components u, v in terms of Fourier integral,i.e.

$$u(x,y) = \frac{2}{\pi}\int_0^{\infty} F(x,\omega)\cos y\omega\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(y,\beta)e^{ix\beta}d\beta \quad (32\text{-a})$$

$$v(x,y) = \frac{2}{\pi}\int_0^{\infty} G(x,\omega)\sin y\omega\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(y,\beta)e^{ix\beta}d\beta \quad (32\text{-b})$$

where F, R, G, Q are unknown functions which can be determined by substituting equations (32) into equations (26). By using the stress-displacement relations, the conditions (27-31), and defining the density function

$$\phi(x) = \frac{\partial v(x,0)}{\partial x}, \quad 0<x<H \qquad (33)$$

The crack problem can be reduced, after lengthy manipulations, into the following singular integral equation

$$\int_a^b \frac{\phi(s)}{s-x}ds + \int_a^b k(x,s)\phi(s)ds = \frac{\pi(\chi+1)}{4\mu}p(x), \quad a<x<b \qquad (34)$$

where the kernel k(x,s) is in the form

$$k(x,s) = \int_0^{\infty} G(x,s,\omega)\,d\omega \qquad (35)$$

where G can be found in appendix A. It can be seen that the kernel k(x,s) is bounded as long as we have embedded crack ($a>0$, $b<H$), but when the crack is terminated at the boundary ($a = 0$ or $b = H$), some terms in the kernel would be unbounded. So, we can rewrite the kernel in the following form

$$k(x,s) = k^f(x,s) + k_1^s(x,s) + k_2^s(x,s) \qquad (36)$$

where $k^f(x,s)$ is the bounded terms as $a \to 0$ and $b \to H$, $k_1^s(x,s)$ is unbounded as $a \to 0$, $b<H$, and $k_2^s(x,s)$ is unbounded as $a>0$, $b = H$ i.e.

$$k_1^s(x,s) = -\frac{1}{s+x} + \frac{6x}{(s+x)^2} - \frac{4x^2}{(s+x)^3} \qquad (37)$$

$$k_2^s = -\frac{1}{(2H-s-x)} \qquad (38)$$

The singular behavior of the solution at the end points can be obtained by following Muskhelishvili technique [10], who assumed the unknown function ϕ as

$$\phi(s) = \frac{g(s)}{(s-a)^{\gamma_1}(b-s)^{\gamma_2}} \qquad (39)$$

where the unknown function g(s) is bounded with $g(a) \neq 0$ and $g(b) \neq 0$, and the powers of singularity γ_1 , γ_2 at the end points should satisfy $0 < \mathrm{Re}(\gamma_1, \gamma_2) < 1$ .defining the sectionally holomorphic function

$$F(z) = \frac{1}{\pi}\int_a^b \frac{\phi(s)}{s-z}ds \qquad (40)$$

and following the function-theoretic method [11,12], the characteristic equations for the case of an edge crack ($a = 0$, $b < H$) are

$$\cos\pi\gamma_1 - 2(\gamma_1 - 1)^2 + 1 = 0 \qquad (41)$$

$$\cot\pi\gamma_2 = 0 \qquad (42)$$

The only acceptable root of equation (41) is $\gamma_1 = 0$, that is the function $\phi(s)$ is bounded at $s = 0$. The root of equation (42) is $\gamma_2 = 1/2$. So, the density function can be written as

$$\phi(s) = \frac{g(s)}{(b-s)^{1/2}} \qquad (43)$$

Then the stresses are bounded at $x = 0$, and have singular behavior at $x = b$. Defining the stress intensity factor at $x = b$ as

$$K(b) = \lim_{x \to b} \sqrt{2(x-b)}\;\sigma_{yy}(x,0) \qquad (44)$$

where $\sigma_{yy}(x,0)$ is the stress outside the crack. Observing that the stress in equation (34) outside and as well as inside the crack, then substituting equation (34) into equation (44) we get

$$K(b) = \frac{4\mu}{\pi(\chi+1)}\lim_{x\to b}\sqrt{2(x-b)}\left[\int_0^b \frac{\phi(s)}{s-x}ds + \int_0^b k(x,s)\phi(s)ds\right] \qquad (45)$$

It is clear that k(x,s) is bounded as $x \to b$ and $s \to b$. By using the definition of sectionally holomorphic function defined in equation (40) and the density function given by (43), the asymptotic analysis will reduce the Cauchy integral into the form

$$\frac{1}{\pi}\int_0^b \frac{\phi(s)}{s-x}\,ds = \frac{g(0)}{b^{1/2}} - \frac{g(b)}{(x-b)^{1/2}} + \text{B.T.} \qquad (46)$$

where B.T. is bounded terms. Substituting equation (46) into equation (45),the stress intensity factor at the end x = b is given by

$$K(b) = -\frac{4\mu}{\chi+1}\sqrt{2}\,g(b) \qquad (47)$$

NUMERICAL PROCEDURE

To calculate the stress intensity factor given in equation (47), we should obtain the unknown function g(s) from the singular integral equation (34) which is solved numerically. Normalized the interval (a,b) by the following change in variables

$$r = \frac{2}{b-a}\left(x - \frac{b+a}{2}\right), \rho = \frac{2}{b-a}\left(s - \frac{b+a}{2}\right) \qquad (48)$$

the singular integral equation (34) may be reduced to

$$\int_{-1}^{+1} \frac{\psi(\rho)}{(\rho-r)(1-\rho)^{1/2}}\,d\rho + \int_{-1}^{+1} \bar{k}(r,\rho)\,\psi(\rho)\,d\rho = \frac{\pi(\chi+1)}{4\mu}q(r) \qquad (49)$$

where

$$\psi(\rho) = \left(\frac{2}{b-a}\right)^{1/2} g(s)$$

$$\bar{k}(r,\rho) = \left(\frac{b-a}{2}\right) k(x,s), \quad p(x) = q(r)$$

Expressing the unknown function $\psi(\rho)$ in terms of polynomial of finite degree, and follow the procedure technique developed in [2,13], the unknown coefficients of the polynomial can be determined. Then the stress intensity factor can be calculated from equation (47).

RESULTS AND CONCLUSION

The normalized transient temperature and thermal stress distributions, which are determined from equations (17) and (25), are shown in Figures (2), (7) for different values of Biot number (Bi = ∞, 20, 1). When Bi = ∞, the solution corresponds to the case of unit step change in temperature at the strip boundary (x = 0). The results are plotted against the nondimensional distance (x^* = x/H) for different values of the dimensionless time (Fourier number) $\tau = tD/H^2$. As expected, when the Biot number decreases, the temperature gradient through the plate thickness decreases and the maximum thermal stress decreases accordingly. So, the most severe case corresponds to Bi = ∞ . Also, It can be seen that, at any instant in time the thermal stress is tensile in the region near the cooled surface, while it is compressive in the other region near the insulated surface. This behavior is different than that obtained for the thermal stress edge cracked plate with free boundaries at which the thermal stress is compressive in the interior of the plate and tensile on the cooled and insulated surfaces [3,5], while it is similar to the case of transient thermal stress hollow cylinder problem with Bi = ∞ [14].

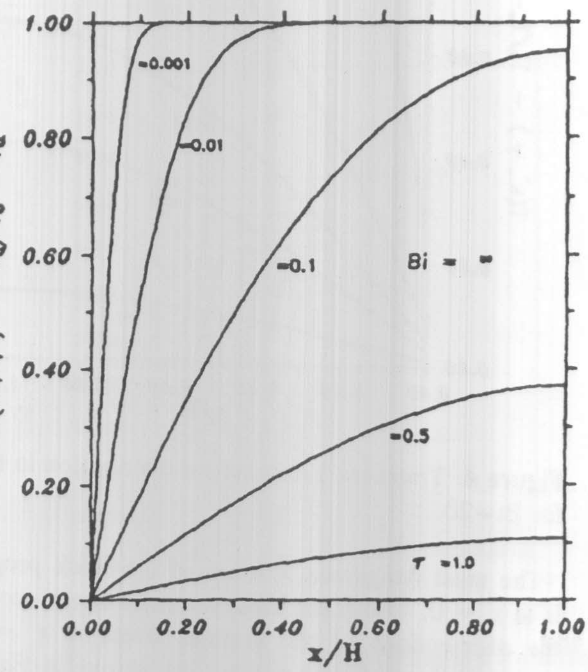

Figure 2. Transient temperature distribution in the strip for Bi = ∞.

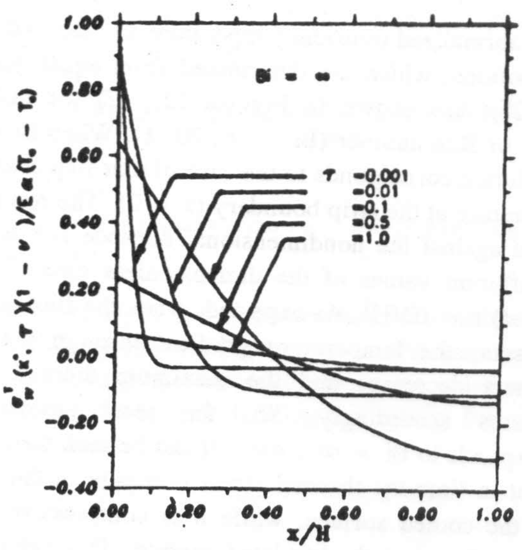

Figure 3. Transient thermal stresses in the strip for Bi = ∞.

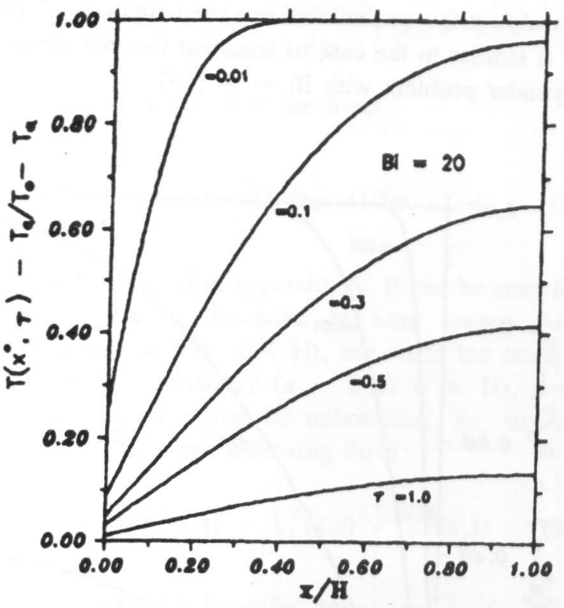

Figure 4. Transient Temperature distribution in the strip for Bi=20.

The most dangerous location of the crack propagation is at x = 0, where the highest tensile stress occurs. So, the distributions of the thermal stress at x = 0, for various values of Biot number are shown in Figure (8). It is clear that, the variation of the thermal stresses is strongly controlled by Biot number. The maximum thermal stress does not occur at t = 0, but it is delayed with certain amount of time depending on the Biot number.

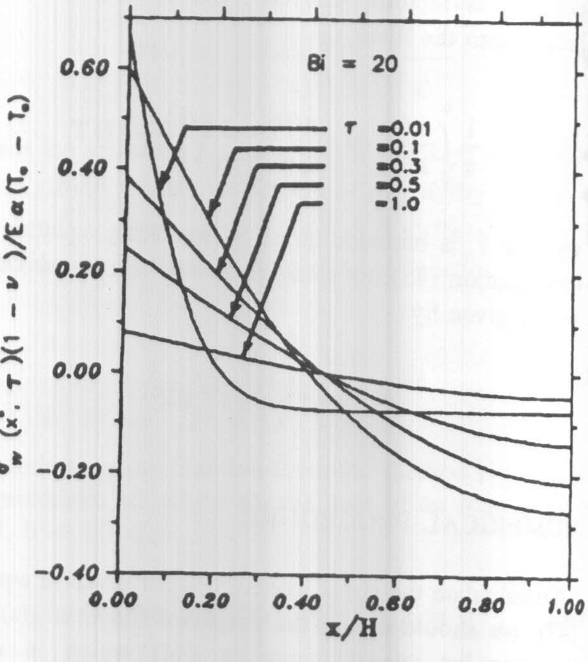

Figure 5. Transient thermal stresses in the strip for Bi=20.

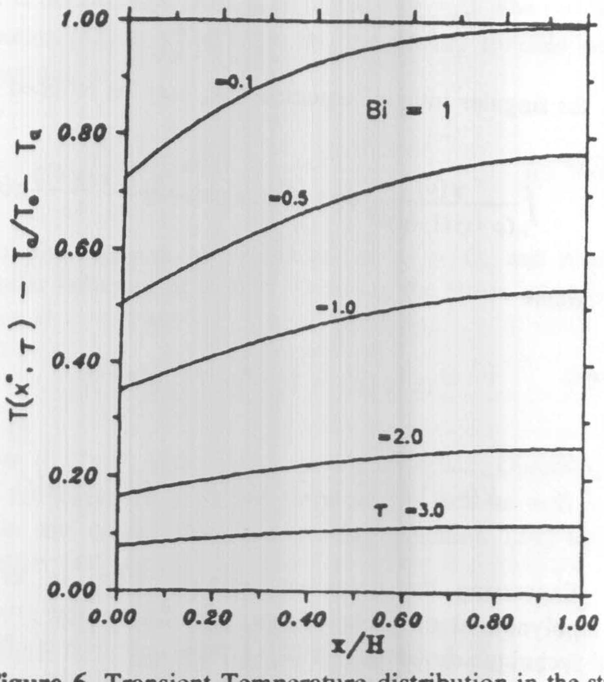

Figure 6. Transient Temperature distribution in the strip for Bi=1.

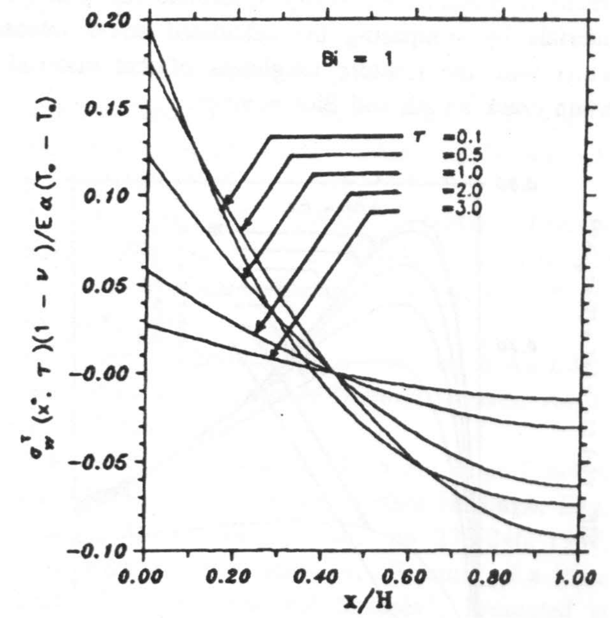

Figure 7. Transient thermal stresses in the strip for Bi = 1.

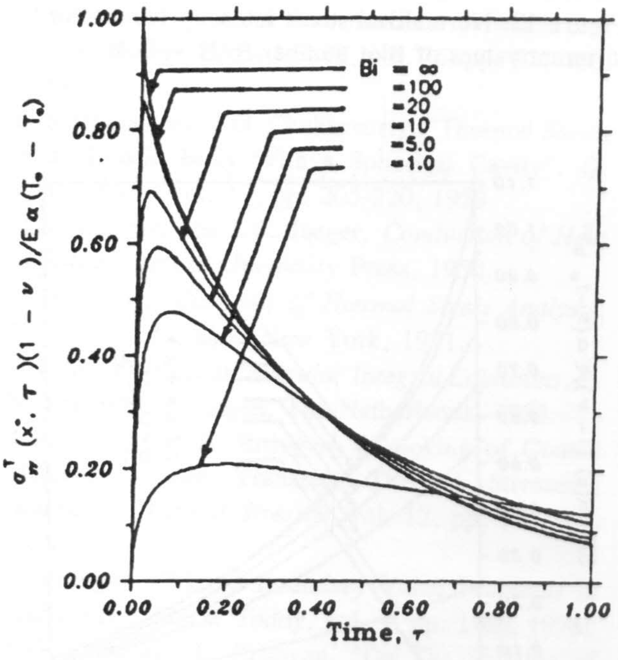

Figure 8. Transient thermal stresses on the surface x/H = 0 for various values of Biot number (Bi).

The normalized stress intensity factors defined by

$$K(b)(1-\nu)/E\alpha(T_o - T_a)\sqrt{b} \qquad (50)$$

are plotted in Figures (9) - (12). Figure (9) corresponds to the step function change in temperature at the boundary (Bi = ∞), while Figure (10) represents an example for different values of Biot number (Bi = 20). It can be observed from the figures that the normalized stress intensity factor reaches a peak value after a significant period of time and then decreases as the nondimensional time increases. Also, the larger the crack length (b/H), the smaller the normalized stress intensity factor. In Figures (11) - (12), the normalized stress intensity factors are plotted versus τ , for various values of Biot number and for two different crack lengths (b/H = 0.1, 0.4). The influence of Biot number and the crack length on the stress intensity factors is obvious.

Because the maximum stress intensity factor is an important parameter in the mechanical failure, it is plotted versus the crack length (b/H) for different values of Biot number Figure (13). It can be seen from the figure that the normalized stress intensity factor decreases as the crack length increases. This is because the transient thermal stresses decrease away from the cooled surface, as well as the compressive stresses which act on the crack tips of sufficiently large cracks.

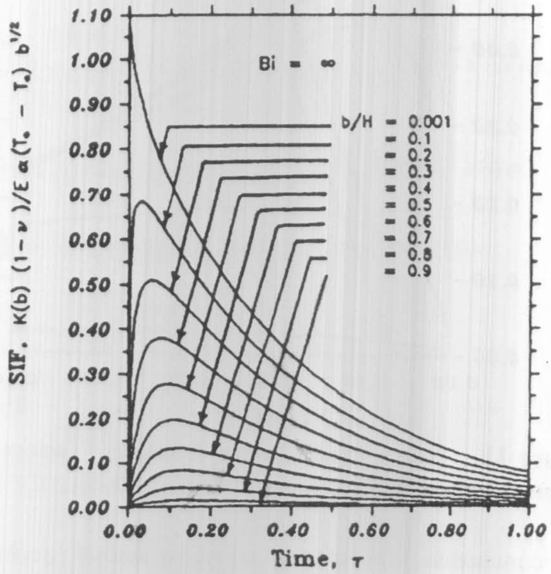

Figure 9. Normalized stress intensity factors for different values of the crack length (b/H), (Bi = ∞).

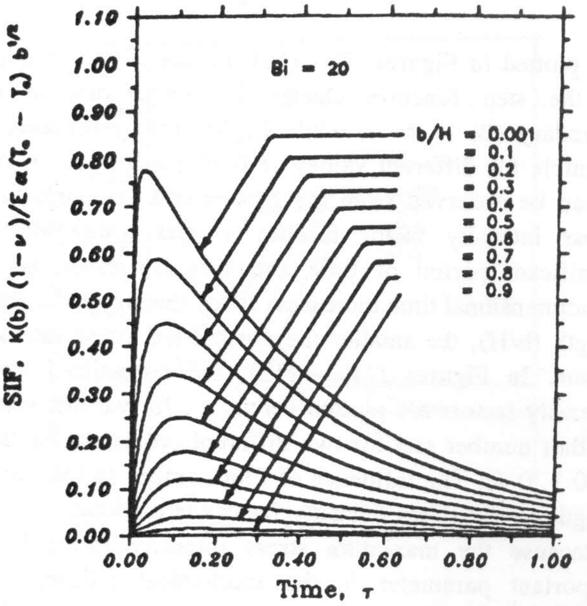

Figure 10. Normalized stress intensity factors for different values of the crack length (b/H), (Bi=20).

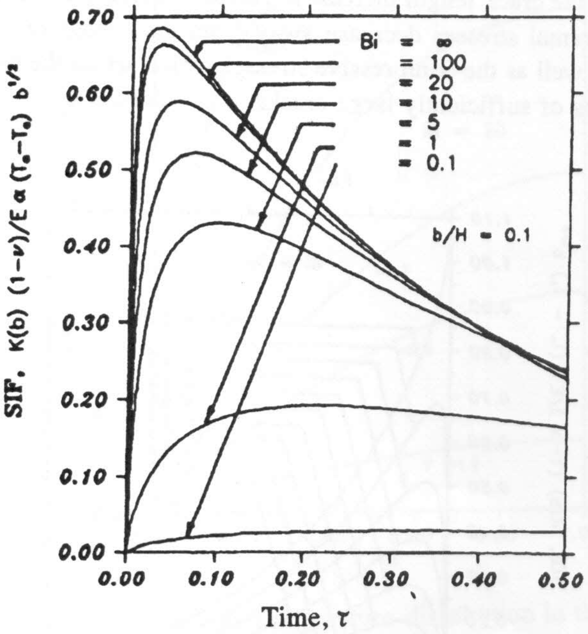

Figure 11. Normalized stress intensity factors for different values of Biot number (b/H = 0.1).

In conclusion, the integral equation analytic technique used to determine the transient stress intensity factors is simple and quite general. It enable us to calculate the stress intensity factors as a function of the crack length, Biot number, at any instant of time with no restrictions on the crack length. Also, Figure (13) can be used directly to predict the severity of thermal shock in brittle materials by comparing the calculated stress intensity factors with the fracture toughness of that material at certain crack length and Biot number.

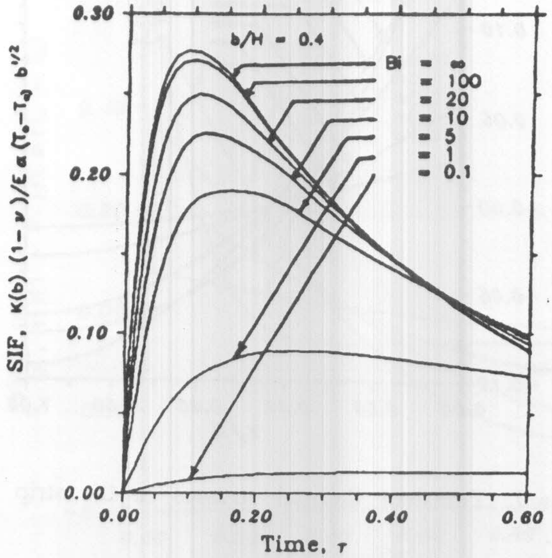

Figure 12. Normalized stress intensity factors for different values of Biot number (b/H = 0.4).

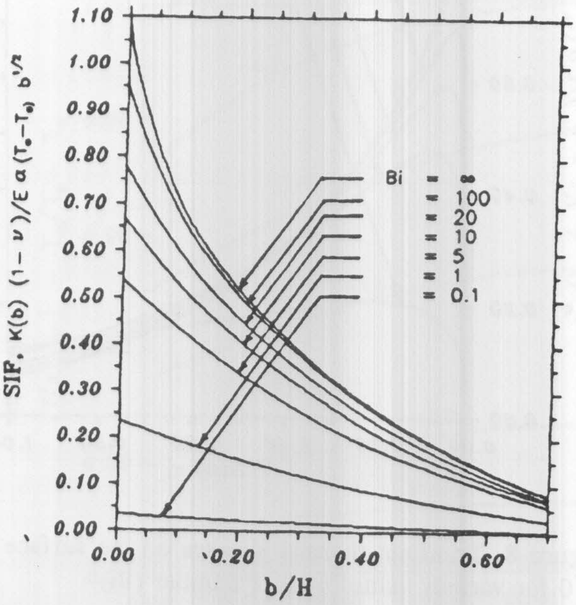

Figure 13. Maximum stress intensity factors versus crack length for different values of Biot number.

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NOMENCLATURE

Bi	Biot number ($h\,H/\acute{k}$)
D	Thermal diffusivity
E	Young modulus of elasticity
h	Heat transfer coefficient
H	Thickness of the plate
K	Stress intensity factor
$\acute{k}$	Coefficient of heat conduction
T	Temperature at any time
T_a	Ambient temperature
T_o	Initial temperature
u,v	Components of the displacement vector
x,y,z	cartesian coordinates
α	Coefficient of thermal expansion
μ , χ	Elastic constants
ν	Poisson's ratio
σ	stress
ϵ	strain
τ	Normalized time (Fourier number tD/H^2)

Appendix A

The function appeared in equation (35) takes the following expression

$$G(x,s,\omega) = -\frac{1}{2M}[(4-4\omega H+2\omega(s+x))\,e^{(s+x-2H)\omega}$$

$$+((1-2s\omega)(4\omega H-2\omega x-3)-1)\,e^{-(s-x+2H)\omega}$$

$$+((1+2s\omega)(2\omega x+3)+1)\,e^{(s+x-4H)\omega}$$

$$+(-4+2\omega(s-x))\,e^{-(s-x+4H)\omega}$$

$$-((-1+2\omega s)(-3+2\omega x)+1)\,e^{-(s+x)\omega}$$

$$-((1+2\omega s-4\omega H)(-3+2\omega x)-1)\,e^{(s-x-2H)\omega}$$

$$-(4+4\omega H-2\omega(s+x))\,e^{-(s+x+2H)\omega}$$

$$-(-4-2\omega(s-x))\,e^{(s-x-4H)\omega}]$$

where

$$M = 1 + 4\omega H e^{-2H\omega} - e^{-4H\omega}$$