TANGENT STIFFNESS MATRIX FOR SPACE FRAME MEMBER WITH BOTH MEMBER AND JOINT IMPERFECTION

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SYNOPSIS

Conventional procedure for study and analysis of space frame structures assumes that the members of these structures are geometrically perfect. Also the procedure assumes that the joints of space structures are pinned or completely rigid. Space frame members may suffer from geometric imperfections, also most joints of these structures are semi-rigid. A new space frame member tangent stiffness matrix which incorporates both the effects of member geometric imperfection and joint properties was developed.

INTRODUCTION

1. Space frame structures are widely used in practice. These structures can be used to cover large area without intermediate supports, while at the same time attain a very small weight of structure material per unit area. The two main components of any space structure are the members and the joints. Many new prefabricated joints have been developed. Most of these joints are not perfect pin nor perfect rigid but semi-rigid. Also space frame members may suffer from geometric imperfections along their length. These imperfections may be due to errors in fabrication, errors in erection and/or due to misuse of the structure after erection.
2. It has been realised that the pattern and the value of imperfections along the length of the compression member affect its behaviour significantly. Also the joint behaviour makes an important contribution to the behaviour of the whole structure, and the uncertain joint performance has been one of the critical factors in many structural collapses. The object of this research is to develop a tangent stiffness matrix for space frame member. This matrix incorporates both member and joint imperfections.

MEMBER IMPERFECTIONS

3. For space frame members, imperfections along the member length are initial curvature (bow or out of straightness) about the two principal axes of the member cross section. Also initial twist may be available. Twist imperfections have only significant effect in case of open cross section members. For closed section members, where most space structures consist of, the geometric imperfection that need to be considered is the initial bow of the members. Any arbitrary crooked shape of an imperfect member as in Figure (1) can be expressed as an appropriate summation of first, second and higher order sinusoidal curves (refs 1, 2) such as

$$y_o = \sum_{i=1}^{\infty} e_i \sin\frac{i\pi x}{L} \tag{1}$$

where y_o, e_i are as shown in Figure (1)

4. In practice members buckle in the shape of their first and/or second order modes. It is therefore considered sufficient for modeling the geometric imperfections that are likely to enhance the possible modes of failure in compression. So the member imperfections will be expressed using only the first two terms of Eqn. 1 to give

$$y_o = e_1 \sin\frac{\pi x}{L} + e_2 \sin\frac{2\pi x}{L} \tag{2}$$

This approach has been considered by Hatzis (ref. 2).

5. Referring to Figure (2) for prismatic imperfect member, and from the beam-column theory, the deformation y_1 of the member measured from its initial position is

$$y_1=\frac{M_1^c}{Q}\left(\frac{\sin\phi(L-x)}{\sin\phi L}-\frac{L-x}{L}\right)+\frac{M_2^c}{Q}\left(\frac{\sin\phi x}{\sin\phi L}-\frac{x}{L}\right) \quad (3)$$
$$+q\left(\frac{e_1}{1-q}\sin\frac{\pi x}{L}+\frac{e_2}{4-q}\sin\frac{2\pi x}{L}\right)$$

where

M_1^c, M_2^c are the end moments
Q is the axial force in the member
q is nondimensional axial force parameter $= Q/P_E$
e_1, e_2 are the amplitudes of the imperfections
ϕ $\sqrt{Q/EI}$
L is the member length
P_E is the member Euler load $= \pi^2 EI/L$
E is the modules of elasticity
I is the moment of inertia of the member cross section

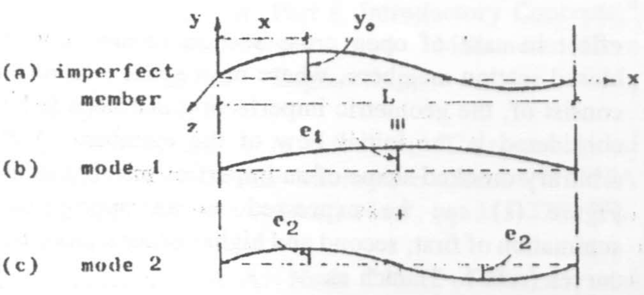

Figure 1. Member geometric imperfection.

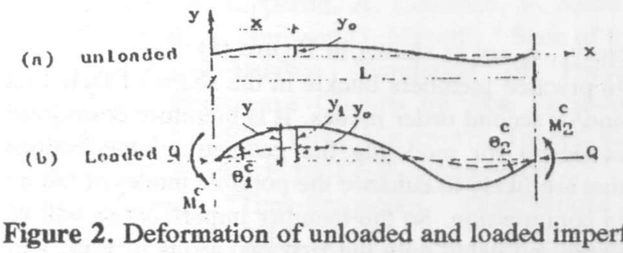

Figure 2. Deformation of unloaded and loaded imperfect member.

The total deformation y measured from x-axis can be obtained by adding Eqn. 2 to Eqn. 3 to give

$$y=y_1+y_o=\frac{M_1^c}{Q}\left(\frac{\sin\phi(L-x)}{\sin\phi L}-\frac{L-x}{L}\right)+\frac{M_2^c}{Q}\left(\frac{\sin\phi x}{\sin\phi L}-\frac{x}{L}\right) \quad (4)$$
$$+\frac{e_1}{1-q}\sin\frac{\pi x}{L}+\frac{e_2}{4-q}\sin\frac{2\pi x}{L}$$

The differential of Eqn. 4 with respect to x evaluated at x=0 and x=L for θ_1 and θ_2 respectively gives

$$\theta_1^c=\frac{L}{EI}(\nu_1 M_1^c+\nu_2 M_2^c)+\frac{\pi e_1}{L(1-q)}+\frac{8\pi e_2}{L(4-q)} \quad ,$$

(5

$$\theta_2^c=\frac{L}{EI}(\nu_2 M_1^c+\nu_1 M_2^c)-\frac{\pi e_1}{L(1-q)}+\frac{8\pi e_2}{L(4-q)}$$

where ν_1 and ν_2 are given in Table 1. Solving Eqns 5 for M_1 and M_2 leads to

$$M_1^c=\frac{EI}{L}(c_1\theta_1^c+c_2\theta_2^c-A_1e_1-A_2e_2) \quad , \quad (6$$

where

$$c_1=\frac{\nu_1}{\nu_1^2-\nu_2^2} \quad , \quad c_2=\frac{-\nu_2}{\nu_1^2-\nu_2^2} \quad (7$$

$$A_1=\frac{\pi(c_1-c_2)}{L(1-q)} \quad , \quad A_2=\frac{8\pi(c_1+c_2)}{L(4-q)} \quad (8$$

c_1 and c_2 are the ordinary stability functions given in Oran (ref. 3) and are listed in Table 1.

where

$$\varphi=\phi L=\pi\sqrt{q} \quad , \quad \psi=\varphi\sqrt{-1} \quad (9$$

JOINT IMPERFECTIONS

6. Joint imperfections considered are joint stiffness (neither pinned nor rigid but semi-rigid), and joint size. The joint stiffness is represented by the stiffness of a zero length spring at the ends of the member. The joint size is represented by a rigid arm with its length equal the size of the joints.

Joint bending stiffness

7. Figure (3) shows a geometrically imperfect member with springs at its ends. The slip rotations (the difference between the base member end rotations and the joint rotations), due to the bending flexibility of the joints are ϕ_1 and ϕ_2 at ends 1 and 2 respectively. The total joint rotations are

$$\theta_1^m=\theta_1^c+\phi_1 \quad , \quad \theta_2^m=\theta_2^c+\phi_2 \quad (10$$

Table 1. Stability function of frame member.

Q	Compression	Zero	Tension
ν_1	$\frac{1}{\varphi}(\frac{1}{\varphi}-\frac{1}{\tan\varphi})$	1/3	$\frac{1}{\psi}(\frac{1}{\tan\psi}-\frac{1}{\psi})$
ν_2	$\frac{1}{\varphi}(\frac{1}{\varphi}-\frac{1}{\sin\varphi})$	- 1/6	$\frac{1}{\psi}(\frac{1}{\sin\psi}-\frac{1}{\psi})$
c_1	$\frac{\varphi(\sin\varphi-\varphi\cos\varphi)}{2(1-\cos\varphi)-\varphi\sin\varphi}$	4	$\frac{\psi(\psi\cosh\psi-\sinh\psi)}{2(1-\cosh\psi)+\psi\sinh\psi}$
c_2	$\frac{\varphi(\varphi-\sin\varphi)}{2(1-\cos\varphi)-\varphi\sin\varphi}$	2	$\frac{\psi(\sinh\psi-\psi)}{2(1-\cosh\psi)+\psi\sinh\psi}$

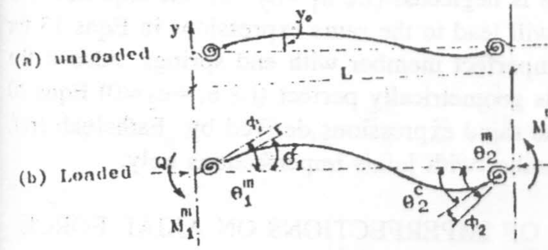

Figure 3. Imperfect member with end springs.

The slip rotations ϕ_1 and ϕ_2 depend on the bending stiffness k_{1b} and k_{2b} of joints 1 and 2. These stiffness can be related to the bending stiffness of the base member by

$$\phi_1=\frac{M_1^m}{k_{1b}}=\frac{M_1^m L}{EI}\varepsilon_1\,,\quad \phi_2=\frac{M_2^m}{k_{2b}}=\frac{M_2^m L}{EI}\varepsilon_2 \qquad (11)$$

where

$$\varepsilon_1=(\frac{EI}{L})/k_{1b}\,,\quad \varepsilon_2=(\frac{EI}{L})/k_{2b} \qquad (12)$$

Substitute the values of ϕ_1 and ϕ_2 from Eqn. 11 and the expression of θ_1^c and θ_2^c from Eqn. 5 into Eqn. 10 and solve for M_1^m and M_2^m to give

$$M_1^m=\frac{EI}{L}(c_{11}\theta_1^m+c_{12}\theta_2^m-A_{11}e_1-A_{12}e_2),$$

$$M_2^m=\frac{EI}{L}(c_{12}\theta_1^m+c_{22}\theta_2^m-A_{21}e_1-A_{22}e_2) \qquad (13)$$

where

$$c_{11}=\frac{\varepsilon_2(c_1^2-c_2^2)+c_1}{\sigma}\,,\ c_{12}=\frac{c_2}{\sigma}\,,\ c_{22}=\frac{\varepsilon_1(c_1^2-c_2^2)+c_1}{\sigma}$$

$$A_{11}=\frac{\pi(c_{11}-c_{12})}{L(1-q)}\,,\ A_{12}=\frac{8\pi(c_{11}+c_{12})}{L(4-q)}\,,$$

$$A_{21}=\frac{\pi(c_{22}-c_{12})}{L(1-q)}\,,\ A_{22}=\frac{8\pi(c_{22}+c_{12})}{L(4-q)} \qquad (14)$$

and

$$\sigma=\varepsilon_1\varepsilon_2(c_1^2-c_2^2)+(\varepsilon_1+\varepsilon_2)c_1+1 \qquad (15)$$

It is worth noting that any combination of end conditions (pinned, rigid, and flexible) can be obtained from Eqn. 13 by substituting the appropriate value of ε_1 and ε_2 (ε=0.0 for rigid joint and $\varepsilon=\infty$ for pinned joint).

Joint stiffness and size

8. Figure (4) shows an imperfect member with rigid parts at the ends, the length of which represents the size of the joints. The total length of the member between the centres of its joints is L, and the size of the joints ar $\lambda_1 L$ and $\lambda_2 L$ for joints 1 and 2 respectively, and the length of the base member is λL.

Consider equilibrium between the forces at the ends of the base member with springs and the forces at the centres of the joints as shown in Figure (4-c), then

$$M_1 = M_1^m + \lambda_1 L S - Q\lambda_1 L\theta_1 \ , \quad M_2 = M_2^m + \lambda_2 L S - Q\lambda_2 L\theta_2 \qquad (16)$$

where $M_1, M_2, \theta_1, \theta_2$, and S are as shown in Figure (4), and M_1^m, M_2^m are as given in Eqns 13 after replacing L by λL. The rotations at the ends of the springs θ_1^m and θ_2^m with respect to the chord of the base member are related to the rotations θ_1 and θ_2 at the centres of the joints 1 and 2 with respect to the chord of the whole member by

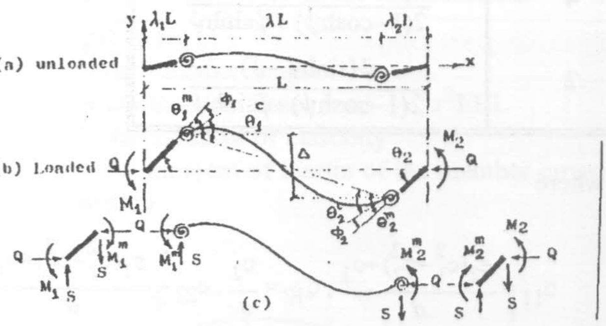

Figure 4. Imperfect member with springs and rigid arms.

$$\theta_1^m = \theta_1 - \frac{\Delta}{\lambda L} \quad , \quad \theta_2^m = \theta_2 - \frac{\Delta}{\lambda L} \qquad (17)$$

The shearing force S is related to M_1^m, M_2^m, and Q as

$$S = \frac{M_1^m + M_2^m + Q\Delta}{\lambda L} \qquad (18)$$

The value of Δ can be obtained from the condition

$$\Delta + \lambda_1 L\theta_1 + \lambda_2 L\theta_2 = 0 \qquad (19)$$

Substitute the values of M_1^m, M_2^m, S, and Δ in Eqns 16, then

$$M_1 = \frac{EI}{L}(\gamma_{11}\theta_1 + \gamma_{12}\theta_2 - \beta_{11}e_1 - \beta_{12}e_2),$$

$$M_2 = \frac{EI}{L}(\gamma_{12}\theta_1 + \gamma_{22}\theta_2 + \beta_{21}e_1 - \beta_{22}e_2) \qquad (20)$$

where

$$\gamma_{11} = \frac{1}{\lambda}[(h_1+1)^2C_{11} + 2h_1(h_1+1)c_{12} + h_1^2c_{22} - \phi^2h_1(h_1+1)],$$

$$\gamma_{12} = \frac{1}{\lambda}[h_2(h_1+1)C_{11} + (2h_1h_2+h_1+h_2+1)c_{12} + h_1(h_2+1)c_{22} - \phi^2h_1h_2],$$

$$\gamma_{22} = \frac{1}{\lambda}[h_2^2C_{11} + 2h_2(h_2+1)c_{12} + (h_2+1)^2c_{22} - \phi^2h_2(h_2+1)],$$

$$\beta_{11} = \frac{\pi}{\lambda L(1-q)}[h_1(c_{11}-c_{22}) + c_{11} - c_{12}],$$

$$\beta_{12} = \frac{8\pi}{\lambda L(4-q)}[h_1(c_{11}+2c_{12}+c_{22}) + c_{11} + c_{12}] \quad ,$$

$$\beta_{21} = \frac{\pi}{\lambda L(1-q)}[h_2(c_{22}-c_{11}) + c_{22} - c_{12}],$$

$$\beta_{22} = \frac{8\pi}{\lambda L(4-q)}[h_2(c_{11}+2c_{12}+c_{22}) + c_{22} + c_{12}] \qquad (21)$$

and $h_1 = \lambda_1/\lambda$, and $h_2 = \lambda_2/\lambda$. It can be seen that if the joints size is neglected (i.e $h_1 = h_2 = 0$), the expression in Eqns 20 will lead to the same expressions in Eqns 13 for case of imperfect member with end springs. Also if the member is geometrically perfect (i.e $e_1 = e_2 = 0$) Eqns 20 lead to the same expressions derived by Fathelbab (ref. 4) for member with joints imperfections only.

EFFECT OF IMPERFECTIONS ON AXIAL FORCE

9. For the space frame member shown in Fig. 1, and considering imperfections about the two principal axes y and z, the initial relative axial strain μ_o due to imperfection is

$$\mu_o = \frac{\pi^2}{L^2} \sum_{n=y,z} (e_{1n}^2 + e_{2n}^2) \qquad (22)$$

where e_1 and e_2 are as shown in Fig. 1, and n refers to axes y and z. For loaded member in Fig. 2, the total relative axial strain will be

$$\mu = \frac{QL}{EA} + \sum_{n=y,z} \mu_{bn} \qquad (23)$$

where μ_{bn} is the bowing axial strain (including the initial strain due to initial bowing) about the two principal axes, and A is the member cross sectional area. The value of μ_{bn} is

$$\mu_{bn} = b_{1n}(\theta_{1n}^c + \theta_{2n}^c)^2 + b_{2n}(\theta_{1n}^c - \theta_{2n}^c)^2 \qquad (24)$$

b_{1n} and b_{2n} are the so called bowing functions given by Saafan (ref. 5). the value of these functions are

$$b_{1n} = \frac{(c_{1n}+c_{2n})(c_{2n}-2)}{8\pi^2 q} \quad , \quad b_{2n} = \frac{c_{2n}}{8(c_{1n}+c_{2n})} \tag{25}$$

and c_{1n} and c_{2n} are the stability functions given in Table 1.

10. If the joints at the ends of the member are not completely rigid under the applied axial force Q, but have axial stiffness k_{1a} and k_{2a} at ends 1 and 2 respectively. Then the zero length spring represents the joint should have an axial stiffness equal the stiffness of the joint. The axial force in Fig. 3 is related to the net relative axial strain by

$$\frac{QL}{EA}(1+\varepsilon_{1a}+\varepsilon_{2a}) = \mu - \sum_{n=y,z} \mu_{bn} + \mu_o \tag{26}$$

or

$$Q = EA\varepsilon_a(\mu - \sum_{n=y,z} \mu_{bn} + \frac{\pi^2}{L^2} \sum_{n=y,z} (\frac{e_{1n}^2}{4} + e_{2n}^2) \tag{27}$$

where

$$\varepsilon_a = \frac{1}{(1+\varepsilon_{1a}+\varepsilon_{2a})} \; , \; \varepsilon_{1a} = \frac{EA}{L}/k_{1a} \; , \; \varepsilon_{2a} = \frac{EA}{L}/k_{2a} \tag{28}$$

11. If the joint size is considered as for the member in Fig. 4, Eqn. 27 should be modified to include the effect of joint size and L should be replaced by λL. The modification of Eqn. 27 was performed by Fathelbab (ref. 4) as follow

$$Q = EA\varepsilon_a(\mu - \sum_{n=y,z} c_{bn} + \sum_{n=y,z} c_{on}) \tag{29}$$

where

$$c_{bn} = \frac{1}{2}(h_1\theta_{1n}^2 + \theta_{2n}^2) + \lambda[b_{1n}\{(1+2h_1)\theta_{1n} + (1+2h_2)\theta_{2n} - \phi_{1n} - \phi_{2n}\}^2 + b_{2n}\{\theta_{1n} - \theta_{2n} - \phi_{1n} - \phi_{2n}\}^2 + \frac{(h_1\theta_{1n} + h_2\theta_{2n})}{2(1-B_n^m)}] \tag{30}$$

and

$$B_n^m = b_{1n}(\theta_{1n}^m + \theta_{2n}^m)^2 + b_{2n}(\theta_{1n}^m - \theta_{1n}^m)^2 \, , \, c_{on} = \frac{\pi^2}{(\lambda L)^2} \sum_{n=y,z} (\frac{e_{1n}^2}{4} + e_{2n}^2) \tag{31}$$

12. From Eqn. 29, the nondimensional axial force parameter q_n can be written in the form

$$q_n = \frac{Q}{P_{En}} = EA\varepsilon_a(\mu - c_{by} - c_{bz} + c_{oy} + c_{oz}) / \frac{\pi^2 EI_n}{(\lambda L)^2} = \frac{s_n^2 \varepsilon_a}{\pi^2}(\mu - c_{by} - c_{bz} + c_{oy} - c_{oz}) \tag{32}$$

where s_n is the slenderness ratio of the base member about n-axis $= \lambda L/\sqrt{I_n/A}$, and P_{En} is the Euler load about the same axis. If the value of q_n from Eqn. 32 is substituted into Eqn. 27, the latest can be rewritten in the form

$$QL = \frac{EI}{L}(\frac{\pi^2}{\lambda^2}q) \tag{33}$$

where I is a reference moment of inertia of the member cross section, and q is the axial nondimensional parameter w.r.t. this inertia.

EFFECT OF IMPERFECTION ON TORSIONAL MOMENT

13. As the twist imperfection effect is negligible for closed section members, so only joint imperfection is considered. If the joints have torsional stiffness k_{1t} and k_{2t} at ends 1 and 2 respectively, then the relation between the torsional moment M_t and the relative twisting angle θ_t will be

$$M_t = \frac{GJ}{L}\varepsilon_t \theta_t \tag{34}$$

where G is the shear modules, and J is the polar moment of inertia of the member cross section, and

$$\varepsilon_t = \frac{1}{\lambda(1+\varepsilon_{1t}+\varepsilon_{2t})} \; , \; \varepsilon_{1t} = \frac{GJ}{L}/k_{1t} \; , \; \varepsilon_{2t} = \frac{GJ}{L}/k_{2t} \tag{35}$$

MEMBER TANGENT STIFFNESS MATRIX

14. The expression in Eqn. 20, after generalized for the two axes y and z, along with Eqn. 34 and Eqn. 33 relate the independent member end forces to its relative end deformation. These expressions can be written in the form

$$\{S\} = \{f(u)\} \tag{36}$$

where {S} is the vector of the independent member end forces, (u) are the relative end deformations, and f are a set of nonlinear functions in (u). These functions include the effects of member and joint imperfections as well as the effect of axial force.

15. For any given displacement u, Eqn. 36 can be differentiated to give

$$\{\Delta S\} = [t]\{\Delta u\} \qquad (37)$$

where

$$\{\Delta S\}^T = \{\Delta M_{1z}, \Delta M_{2z}, \Delta M_{1y}, \Delta M_{2y}, \Delta M_t, \Delta GL\} \quad ,$$

$$\{\Delta u\}^T = \{\Delta\theta_{1z}, \Delta\theta_{2z}, \Delta\theta_{1y}, \Delta\theta_{2y}, \Delta\theta_t, \Delta\mu\} \qquad (38)$$

and [t] is the member tangent stiffness matrix w.r.t. its local current axes. The elements t_{ij} of [t] are given by

$$t_{ij} = \frac{\partial S_i}{\partial u_j} + \frac{\partial S_i}{\partial q}\cdot\frac{\partial q}{\partial u_j} \quad \text{for } i,j=1,2,\ldots,6 \qquad (39)$$

Here only one element (t_{11}), as an example, will be evaluated and the rest follow in similar way

$$t_{11} = \frac{\partial M_{1z}}{\partial\theta_{1z}} + \frac{\partial M_{1z}}{\partial q}\cdot\frac{\partial q}{\partial\theta_{1z}}$$

$$= \frac{EI_z}{L}[\gamma_{11z} + \gamma'_{11z}\theta_{1z} + \gamma'_{12z}\theta_{2z} - \beta'_{11z}e_{1y} - \beta'_{12z}e_{2y}]\frac{\partial q}{\partial\theta_{1z}} \qquad (40)$$

' denotes differentiation w.r.t. q, and $\gamma_{11z}, \gamma_{12z}, \beta_{11z}, \beta_{12z}$ can be evaluated from Eqns 21 in terms of c_1, c_2, c_1, and c_2. The value of c_1 and c_2 can be obtained in terms of b_1 and b_2 from the relation presented by Saafan (ref. 5), where

$$b_1 = \frac{c_1 + c_2}{4\pi^2} \quad , \quad b_2 = \frac{c_1 - c_2}{4\pi^2} \qquad (41)$$

The differentiation of q in Eqn. 39 can be evaluated from Eqn. 32.

15. After evaluating all elements of the tangent matrix [t], this matrix can be transferred from the member local current axes to the structure global axes. This transformation is done by considering geometry, equilibrium, and orientation of the member. Oran (ref. 3) presented a procedure for this transformation.

CONCLUSION

Expressions relate the independent space frame end forces to its end relative deformations were obtained. These expressions include the effects of member and joint imperfections as well as the effect of axial force and bowing functions. A procedures to obtain the member tangent stiffness matrix in its current local axes were presented.

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