NEW ALGORITHMS FOR JACOBIAN ELLIPTIC FUNCTIONS EMPLOYING THE MODIFIED POLYNOMIAL THEORY

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ABSTRACT

This paper presents new algorithms of the generalized power series technique for the analysis of Jacobian elliptic functions. The method uses the modified polynomial theory in the analysis of twelve nonlinear differential equations of the second order,on the basis of "Frobenius" technique. The technique operates entirely in the time domain and is ideally suited to computer-aided design of systems subjected to such nonlinear differential equations. The designed algorithms express the solutions in series forms. The recurrence relations which generate the coefficients have been obtained in special forms in order to facilitate the application of the special software designed to process such recurrence relations.

KEYWORDS

Algorithms of Nonlinear Differential Equations, CAD for Nonlinear Systems, and Software of Polynomial Theory.

INTRODUCTION

The analytical approach developed in this paper is useful for the calculation of steady state responses in nonlinear circuits and systems. In general, to deal with these systems, different analysis techniques, whose effectiveness are often strictly related to the particular characteristics of the system to be analyzed, have been proposed. These methods can be classified into three different types of approaches [1-5]:

a- Time domain analysis
b- Frequency domain analysis
c- Hybrid (mixed time frequency domain) analysis

In the first approach, which is quite efficient and probably the most widely used in low frequency circuit simulation, a discrete time integration of the differential circuit equations is carried out by using a numerical integration formula. Such analysis, consists of solving a system of nonlinear algebraic equations. Current methods for calculating the steady state system response can be classified into four categories:

i- Perturbation method [6],
ii- Brute force method [7],
iii- Shooting method [8], and
iv- Power series method [9].

In the second approach, the unknowns of the analyzed problem are the amplitudes and phases of the harmonics of the system response. Thus the basic problem in the frequency domain analysis is the large number of coupled equations which have to be solved in order to compute all those harmonics. Such technique is employed in high frequency system simulation. Current methods for calculating the steady state system response can be classified into four categories:

i- Volterra series method [10],
ii- Harmonic balance method [4],
iii- SPICE method and CAD technique [3], and
iv- Power series technique [11,12].

In the third approach, the fundamental ideas and methods of joint time-frequency distributions are to describe how the spectral content of a function (signal) is changing in time, and to develop the physical and mathematical ideas needed to understand what a time-varying spectrum is [13,14].

In general, the use of functional expansions in order to represent nonlinear systems has been energetically studied in the last forty years, in engineering and physics. Functional expansions have been applied in every branch of nonlinear system theory identification and modelization, realization, stability, optimal control, stochastic differential equations and filtering [9].

In the present paper, a new algorithm, based on the well known power series technique, is casted to handle a very attractive set of nonlinear differential equations which describe Jacobian elliptic functions [15] which have been derived under the forms in Table (I). This set has the general form

Table I. Coefficients and initial values of jacabian elliptic functions.

Function	α	β	Initial values	
			$y(0)$	$\dot{y}(0)$
sc	$-(2-m)^*$	$-2(1-m)$	0.0	1.0
cs	$-(2-m)$	-2	$y(k)^{**} = \{^{\infty}_{0.0}$	$\dot{y}(k)=\{^{\infty}_{-\sqrt{1-m}}$
nd	$-(2-m)$	$2(1-m)$	1.0	0.0
dn	$-(2-m)$	2	1.0	0.0
sn	$1+m$	$-2m$	0.0	1.0
cd	$1+m$	$-2m$	1.0	0.0
dc	$1+m$	-2	1.0	0.0
ns	$1+m$	-2	$y(k)=\{^{\infty}_{1.0}$	$\dot{y}(k)\{^{-\infty}_{0.0}$
nc	$1-2m$	$-2(1-m)$	1.0	0.0
ds	$1-2m$	-2	$y(k)=\{^{\infty}_{\sqrt{1-m}}$	$\dot{y}(k)\{^{-\infty}_{0.0}$
sd	$1-2m$	$2m(1-m)$	0.0	1.0
cn	$1-2m$	$2m$	1.0	0.0

* $0.0 \le m \le 1.0$ ** K ≡ a quarter period.

$$\ddot{y}+\alpha y+\beta y^3=0.0 \qquad (1)$$

In Table (I), the coefficients α and β and the initial conditions are tabulated. In fact, Eqn. (1) represents different applications in engineering and physics [16,17] and is known as Duffing equation. The nonlinearity in Eqn. (1) may take a general form as:

$$\ddot{y}+\alpha y+\beta y^2=0.0 \qquad (2)$$

The designed software handles the cases in which n belongs to real positive integers i.e. $n \in R^+$.

II BASIC MODEL AND ANALYSIS

II.1. Introduction

In the time domain, the numerical approach to solve Eqn. (2) is

$$y=\sum_{i=1}^{\infty} a_i t^{i-1}, \qquad (3)$$

with

$$\dot{y}=\sum_{i=1}^{\infty} i\, a_{i+1} t^{i-1}, \qquad (4)$$

$$\ddot{y}=\sum_{i=1}^{\infty} i\,(i+1) a_{i+2} t^{i-1}, \qquad (5)$$

and

$$y^n = (\sum_{i=1}^{\infty} a_i t^{i-1})^n = \sum_{i=1}^{\infty} b_i t^{i-1} \qquad (6)$$

The major contribution in this analysis is how to handle Eqn. (6) as it represents the generalization of the work of reference [18].

The use of Eqns.(3,5,6) into Eqn.(2) yields the general recurrence relation:

$$a_{i+2} = \frac{1}{i(i+1)}\{-\alpha a_i - \beta b_i\} \qquad (7)$$

$a_{i,s}$ are computed with the aid of the initial conditions a_1 and a_2.

Recalling Eqn. (6) under the form:

$$\sum_{i=1}^{\infty} b_i t^{i-1} = (a_1 + a_2 t + a_3 t^3 + \ldots + \ldots)^n \qquad (8)$$

we can now build the modified polynomial theory where it is easy to deduce that:

$$b_1 = a_1{}^n, \qquad (9)$$

$$b_1 = a_1{}^n, \qquad (9)$$

and

$$b_1 \, n \, a_1{}^{n-1} \, a_2 \qquad (10)$$

II.2. Derivation of the Modified Polynomial Theory

Starting from the following two equations:

$$y_1 = \sum_{i=1}^{\infty} a_i t^{i-1}, \qquad (11)$$

and

$$y_2 = \sum_{i=1}^{\infty} b_i t^{i-1}, \qquad (12)$$

we get:

$$y_1 \, y_2 = \sum_{i=1}^{\infty} c_i t^{i-1} \qquad (13)$$

where

$$c_i = \sum_{j=1}^{i} a_j b_{i-j+1} \qquad (14)$$

with

$$y_3 = \sum_{i=1}^{\infty} d_i t^{i-1} \qquad (15)$$

then we get:

$$y_1 \, y_2 \, y_3 = \sum_{i=1}^{\infty} e_i t^{i-1} \qquad (16)$$

where

$$e_i = \sum_{j=1}^{j=i} c_j d_{i-j+1} = \sum_{j=1}^{j=i} \left\{ \sum_{n=1}^{n=j} a_n b_{j-n+1} \right\} d_{i-j+1} \qquad (17)$$

with

$$y_4 = \sum_{i=1}^{\infty} f_i t^{i-1} \qquad (18)$$

then we get:

$$y_1 \, y_2 y_3 y_4 = \sum_{i=1}^{\infty} g_i t^{i-1}, \qquad (19)$$

where

$$g_i = \sum_{j=1}^{j=i} e_i f_{i-j+1} = \sum_{j=i}^{j=1} \left\{ \sum_{m=j}^{m=1} \left\{ \sum_{n=1}^{n=m} a_n b_{m-n+1} \right\} d_{j-m+1} f_{i-j+1} \right\} \qquad (20)$$

Now, if $a_i = b_i = d_i = f_i = A_i$, the following is obtained:

$$y^4 = \left(\sum_{i=1}^{\infty} A_i t^{i-1}\right)^4 = \sum_{i=1}^{\infty} G_i t^{i-1} \qquad (21)$$

where

$$G_i = \sum_{j=i}^{j=1} \left\{ \sum_{m=1}^{m=j} \left\{ \sum_{n=1}^{n=m} A_n A_{m-n+1} \right\} A_{j-m+1} A_{i-j+1} \right. \qquad (22)$$

Making the following substitutions:

$n = i_1, \quad m = i_2 \, , \, j = i_3,$

and

$i = i_4,$

Eqn. (22) reduces to:

$$\sum_{i_3=1}^{i_3=i_4} \left\{ \sum_{i_2=1}^{i_2=i_3} \left\{ \sum_{i_1=1}^{i_1=i_2} A_n A_{i_1} \right\} A_{i_3} - i_2 + 1 \right\} A_{i_4} - i_3 + 1 \qquad (23)$$

Thus the general case is

$$y^n = (\sum_{i=1}^{\infty} A_i t^{i-1})^n = \sum_{i=1}^{\infty} b_i t^{i-1}$$

where

$$b_i = \sum_{i_{n-1}}^{i_{n-1}=i_n} \left\{ \sum_{i_{n-2}=1}^{i_{n-2}=i_{n-1}} \left\{ \sum_{i_{n-3}=1}^{i_{n-3}=i_{n-4}} \{\sum \cdots \{\sum \cdots \{\sum \cdots \{\sum \cdots \right.\right.$$

$$\left\{ \sum_{i_1=1}^{i_1=i_2} A_{i_1} A_{i_2} - i_1 + 1 \right\} A_{i_3} - i_2 + 1 \} A_{i_5} - i_4 + 1 \} \ldots$$

$$\{\ldots 0\} A_{i_n} - i_{n-1} + 1 \tag{24}$$

Equation (24) contains (n-1) inner products. With the aid of the set $\{ \alpha, \beta, n, a_1, a_2\}$ and Eqns. (7,24), Eqn.(2) can be processed via the following algorithm.

II. 3. Description of the computer program

Starting from the set of controlling parameters and initial conditions $\{ \alpha, \beta, n, a_1, a_2\}$, and Eqn. (2), one can compute b_1 using Eqn. (24) then a_3 is computed using Eqn.(7). Then with the aid of the set $\{\alpha, \beta, n, a_1, a_2\ a_3\}$, b_2 is computed and then a_4. Such process can be handled to compute the desired coefficients $a_{i,s}$ for the desired level of the allowable truncation error. The following main program and the designed subroutine (POLY) are linked in order to find $a_{i,s}$.

II.4. The main program and the subroutine

The object of this program is the reading of the set $\{\alpha, \beta, n, a_1, a_2\}$ where it calls the subroutine "POLY" to compute the value of $b_{i,s}$ in successive operations. Eqn. (24) is simplified as follows:

$$b_i = \sum_{j=1}^{j=i} A_j A_{i-j+1}^{(n-1)}, \tag{25}$$

$$A_j^{(n-1)} = \sum_{j=1}^{j=i} A_j A_{i-j+1}^{(n-2)}, \tag{26}$$

$$A_j^{(n-2)} = \sum_{j=1}^{j=2} A_j A_{i-j+1}^{(n-3)}, \tag{27}$$

$$A_j^{(2)} = \sum_{j=1}^{j=i} A_j A_{i-j+1}^{(1)}, \tag{28}$$

and

$$A_j^{(1)} = A_j. \tag{29}$$

Thus the computation processes are starting by handling the set of equations {25,26,27,28,29} in a backward manner. These are done by empolying the subroutine "POLY" which is a novel one in this field. It is a simple-but-accurate design, and it can be processed for any set of controlling parameters without any tedious algebraic technique such as in Reference [15].

III. RESULTS AND DISCUSSION

In this section, three basic Jacobian elliptic functions (n=3 in Eqn.(2)) are processed, namely "sn", "cn", and "dn" functions for M = (0.1, 0.2, 0.3, 0.4, 0.5) and K = quarter period. These functions are shown in Table (II).

Table II. Coefficients and initial values of some jacabian elliptic functions.

Function	α	β	Initial Values	
			y(0)	y(0)
sn	1+m	-2m	0.0	1.0
cn	1-2m	2m	1.0	0.0
dn	-(20m)	2	1.0	0.0

At a given, the quarter-period K is calculated on the same basis as Reference [15]:

$$K=0.5\pi(1+(0.5)^2 m+(\frac{1}{2}.\frac{3}{4})^2 m^2+(\frac{1}{2}.\frac{3}{4}\frac{5}{6})^2 m^3+\ldots] \tag{30}$$

The functions are easily calculated and tabulated (Table III) without the need of any transformations or auxiliary functions.

Based on these figures, it is clear that, m affects only the period. There is a full agreement with the data published in Reference [15]. Sample of calculations of the three Jacobian elliptic functions "sn", "cn", and "dn" when processing at m =0.5 are given in Table (III), Appendix A. and Table (IV), Appendix B.

APPENDIX: A **Table III**. The coefficients $a_{i,s}$ for three Jacobian ellipt functions at m=0.5, and K=1.854075.

	" sn "	" cn "	" dn "
a_1	.00000000E+00	.10000000E+01	.10000000E+01
a_2	.10000000E+01	.00000000E+00	.00000000E+00
a_3	.00000000E+00	-.50000000E+00	-.25000000E+00
a_4	-.25000000E+00	.00000000E+00	.00000000E+00
a_5	.00000000E+00	.12500000E+00	.93750000E-01
a_6	.68750000E-01	.00000000E+00	.00000000E+00
a_7	.00000000E+00	-.37500000E-01	-.26562500E-01
a_8	-.20312500E-01	.00000000E+00	.00000000E+00
a_9	.00000000E+00	.10937500E-01	.77148430E-02
a_{10}	.58919270E-02	.00000000E+00	.00000000E+00
a_{11}	.00000000E+00	-.31770840E-02	-.22477210E-02
a_{12}	-.17138670E-02	.00000000E+00	.00000000E+00
a_{13}	.00000000E+00	.92447920E-03	.65368650E-03
a_{14}	.49862600E-03	.00000000E+00	.00000000E+00
a_{15}	.00000000E+00	-.26893030E-03	-.19015970E-03
a_{16}	-.14504690E-03	.00000000E+00	.00000000E+00
a_{17}	.00000000E+00	.78231430E-04	.55318230E-04
a_{18}	.42194400E-04	.00000000E+00	.00000000E+00
a_{19}	.00000000E+00	-.22757680E-04	-.16092100E-04
a_{20}	-.12274410E-04	.00000000E+00	.00000000E+00
a_{21}	.00000000E+00	.66202350E-05	.46812130E-05
a_{22}	.35706410E-05	.00000000E+00	.00000000E+00
a_{23}	.00000000E+00	-.19258340E-05	-.13617710E-05
a_{24}	-.10387040E-05	.00000000E+00	.00000000E+00
a_{25}	.00000000E+00	.56022760E-06	.39614070E-06
a_{26}	.30216020E-06	.00000000E+00	.00000000E+00
a_{27}	.00000000E+00	-.16297090E-06	-.11523780E-06
a_{28}	-.87898770E-07	.00000000E+00	.00000000E+00
a_{29}	.00000000E+00	.47408430E-07	.33522820E-07
a_{30}	.25569860E-07	.00000000E+00	.00000000E+00
a_{31}	.00000000E+00	-.13791170E-07	-.97518300E-08
a_{32}	-.74383040E-08	.00000000E+00	.00000000E+00

	" sn "	" cn "	" dn "
a_{33}	.00000000E+00	.40118690E-08	.28368190E-08
a_{34}	.21638120E-08	.00000000E+00	.00000000E+00
a_{35}	.00000000E+00	-.11670570E-08	-.82523430E-09
a_{36}	-.62945550E-09	.00000000E+00	.00000000E+00
a_{37}	.00000000E+00	.33949850E-09	.24006170E-09
a_{38}	.18310940E-09	.00000000E+00	.00000000E+00
a_{39}	.00000000E+00	-.98760510E-10	-.69834230E-10
a_{40}	-.53266750E-10	.00000000E+00	.00000000E+00
a_{41}	.00000000E+00	.28729560E-10	.20314860E-10
a_{42}	.15495360E-10	.00000000E+00	.00000000E+00
a_{43}	.00000000E+00	-.83574650E-11	-.59096190E-11
a_{44}	-.45076200E-11	.00000000E+00	.00000000E+00
a_{45}	.00000000E+00	.24311960E-11	.17191150E-11
a_{46}	.13112720E-11	.00000000E+00	.00000000E+00
a_{47}	.00000000E+00	-.70723800E-12	-.50009280E-12
a_{48}	-.38145070E-12	.00000000E+00	.00000000E+00
a_{49}	.00000000E+00	.20573640E-12	.14547760E-12
a_{50}	.11096450E-12	.00000000E+00	.00000000E+00
a_{51}	.00000000E+00	-.59848980E-13	-.42319610E-13
a_{52}	-.32279710E-13	.00000000E+00	.00000000E+00
a_{53}	.00000000E+00	.17410140E-13	.12310830E-13
a_{54}	.93902040E-14	.00000000E+00	.00000000E+00
a_{55}	.00000000E+00	-.50646310E-14	-.35812350E-14
a_{56}	-.27316220E-14	.00000000E+00	.00000000E+00
a_{57}	.00000000E+00	.14733070E-14	.10417860E-14
a_{58}	.79463230E-15	.00000000E+00	.00000000E+00
a_{59}	.00000000E+00	-.42858700E-15	-.30305680E-15
a_{60}	-.23115950E-15	.00000000E+00	.00000000E+00
a_{61}	.00000000E+00	.12467650E-15	.88159580E-16
a_{62}	.67244580E-16	.00000000E+00	.00000000E+00
a_{63}	.00000000E+00	-.36268530E-16	-.25645730E-16
a_{64}	-.19561530E-16	.00000000E+00	.00000000E+00
a_{65}	.00000000E+00	.10550570E-16	.74603750E-17
a_{66}	.56904730E-17	.00000000E+00	.00000000E+00
a_{67}	.00000000E+00	-.30691710E-17	-.21702320E-17
a_{68}	-.16553660E-17	.00000000E+00	.00000000E+00
a_{69}	.00000000E+00	.89282610E-18	.63132320E-18
a_{70}	.48154790E-18	.00000000E+00	.00000000E+00

Appendix B:

Table IV. The values of the major elliptic functions "sn", "cn", and "dn" over a quarter period for m=0.5, and k=1.854075.

t	"sn"	"cn"	"dn"
.00000000E+00	.00000000E+00	.10000000E+01	.10000000E+01
.92703730E-01	.92505020E-01	.99571220E+00	.99785840E+00
.18540750E+00	.18382900E+00	.98295820E+00	.99151580E+00
.27811120E+00	.27284530E+00	.96205790E+00	.98121240E+00
.37081490E+00	.35853100E+00	.93351780E+00	.96733020E+00
.46351860E+00	.44000510E+00	.89799540E+00	.95036720E+00
.55622240E+00	.51655420E+00	.85625450E+00	.93090590E+00
.64892610E+00	.58764370E+00	.80911970E+00	.90958090E+00
.74162980E+00	.65291560E+00	.75743060E+00	.88704600E+00
.83433350E+00	.71217470E+00	.70200230E+00	.86394660E+00
.92703730E+00	.76536680E+00	.64359440E+00	.84089640E+00
.10197410E+01	.81255290E+00	.58288730E+00	.81846140E+00
.11124450E+01	.85387970E+00	.52047020E+00	.79714780E+00
.12051480E+01	.88955230E+00	.45683340E+00	.77739850E+00
.12978520E+01	.91980770E+00	.39236930E+00	.75958990E+00
.13905560E+01	.94489300E+00	.32737950E+00	.74403540E+00
.14832600E+01	.96504580E+00	.26208190E+00	.73098800E+00
.15759630E+01	.98047900E+00	.19662500E+00	.72064580E+00
.16686670E+01	.99139420E+00	.13091040E+00	.71314170E+00
.17613710E+01	.99569710E+01	.92667610E-01	.71013640E+00
.18540750E+01	.10000000E+01	.00000000E+00	.70710680E+00

Another nonlinear differential equation which occurs in astronomy is the Lame-Emden equation:

$$\ddot{y} + \frac{2}{t}\dot{y} + y^{n} = 0.0 \qquad (31)$$

Solution of this equation such that $y=1$ and $\dot{y}=0.0$ when $t=0.0$ are known as Lame-Emden function of order n. By means of the substitutions:

$$y = =\frac{1}{\sqrt{2}} e^{x/2} Z,$$

$$t = e^{-x},$$

and

$$n = 5$$

Eqn. (31) reduces to:

$$\ddot{Z} - 0.25 + 0.25 Z^{5} = 0.0. \qquad (32)$$

Thus, it can be processed by the given software with a set of controlling parameters {1, 0.0, -0.25, 0.25, 5}.

IV. CONCLUSIONS

A novel subroutine is designed and employed to process second order nonlinear differential equations which exhibit a linear elastic term (αy) and a nonlinear one (βy^n). Such subroutine executes n inner products which facilitate the cast of solution under a simple series form. The process is done entirely in the time domain without any transformation or auxiliary functions. The elliptic Jacobian functions are specially processed. Other functions can be processed with the same software as Lame-Emden functions and others. Such technique is recommended when the system response is of low frequency.

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