VARIOUS TYPES OF STRONGLY G-GRADED RINGS

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ABSTRACT

Let G be a group and $R = \bigoplus_{g \in G} R_g$ be a G-graded ring, i.e., $R_g R_h \subseteq R_{gh}$ for all g, h $\in$ G. Many studies in group graded rings assume R to be a strongly graded ring, i.e., $R_g R_h = R_{gh}$ for all g, h $\in$ G. But this strong condition is hard to statisfy. In this paper we look for other weaker and useful conditions. We define three successively stronger properties that a grading may have. Then we investiagte the relationship between these new strong gradings and the stronger nodegenerate and faithful properties which are motivated by the work of Choen and Rowen.

INTROCUTION

Throughout this paper R is a ring with unity 1, graded by a group G, i.e., $R = \bigoplus_{g \in G} R_g$ for additive subgroups R_g, satisfying $R_g R_h \subseteq R_{gh}$ for all g, h$\in$ G. Obviously, R_e is a subring of R and $1 \in R_e$, where e is the identity of G.

Methods of graded rings have proven to be successful tools in the structure theory for commutative and non commutative rings. For a ring R and a group G one can write R as a G-Graded ring with $R_e = R$ and $R_g = 0$ otherwise (trivial graduation of R by G). But this graduation does not help us to study R or its graded modules. So, it is very important to give new conditions on the graduation of R. Many studies in group graded rings assume R to be a strongly graded ring, i.e., $R_g R_h = R_{gh}$ for all g, h$\in$ G. These rings have been studied by E.C. Dade in [3], where they were called clifford systems. However, this strong condition is hard to satisfy. So, one should look for other weaker and useful condition.

In this paper we define three sucessively stronger properties that a grading may have. Then we investigate the the relationship between these new strong gradings and the stronger nondegenerate and faithful properties which are motivated by the work of Cohen and Rowen in [2].

1. NONDEGENERATE AND FAITHFUL GRADED RINGS

Let R be an associative ring with unity 1 and G be a group with identity e. We say that R is a G-graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all g,h$\in$ G. We consider (R,G) to be the G-graded ring R and supp (R,G) = $\{g \in G: R_g \neq 0\}$. The elements of R_g are called homogeneous of dimension g. For $x \in R$, we write x_g for the compoent of x in R_g, so that x can be written uniquely as $\sum_{g \in G} x_g$.

Definition 1.1

Let R be a G-graded ring. The map

(,): $R \times R \to R_e$ defined by $(x,y) = (xy)_e$ is called the inner product map on R.

Proposition 1.2

Let R be a G-graded ring and (,) be the inner product map on R.

(1) If (,) is R_e-bilinear then R_e is commutative subring of R.

(2) If each element of R_e commutes with each element of R then (,) is R_e-bilinear.

Proof: (1) suppose (,) is R_e-bilinear. Let r, s $\in$ R_e, then $(r,r+s) = (r(r+s))_e = (r^2)_e + (rs)_e = r^2 + rs$. By assumption $(r,r+s) = (r,r) + s\,(r,1) = r^2+sr$. Therefore, rs = sr and hence R_e is commutative.

(2) Let x,y,z $\in$ R and r $\in$ R_e. Clearly, $(x+ry,z) = (x,z) + r\,(y,z)$. On the other hand, $(x,y+rz) = (x(y+rz))_e = (xy)_e + (xrz)_e$. But xr= rx implies $(x,y+rz) = (xy)_e + r\,(xz)_e = (x,y) + r\,(x,z)$. Hence (,) is R_e-bilnear.

However, if R_e is commutative then this doesn't mean (,) is R_e-bilinear as we see in the following example.

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Example 1.3

Let $G = Z_2$ and $R = M_2(Z_2)$. Then R is a G-graded ring with: $R_0 = \begin{bmatrix} Z_2 & 0 \\ 0 & Z_2 \end{bmatrix}$ and $R_1 = \begin{bmatrix} 0 & Z_2 \\ Z_2 & 0 \end{bmatrix}$. Clearly, R_0 is commutative. Let $r = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in R_0, x = z = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and

$$y = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}. \text{Then } (x, y + rz) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \text{and } (x,y) + r(x,z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So, (,) is not R_e-bilinear.

Definition 1.4

Let R be a G-graded ring and (,) be the inner product map on R. Then (R,G) is said to be left (resp. right) nondegenerate if (r,R) = 0 (resp. (R,r) = 0) implies r = 0. If (R,G) is both left and right nondegenerate then we say (R,G) s nondegenerate.

Theorem 1.5 ([1]).

Let R be a G-graed ring. Then (R,G) is nondegenerate iff for any $a_g \in R_g$-0, $a_g R_{g^{-1}} \neq 0$ and $R_{g^{-1}} a_g \neq 0$.

Definition 1.6

Let R be a G-graded ring and (,) be the inner product map on R. Then (R,G) is said to be left (resp. right) faithful if for any $a_g \in R_g - 0$, $a_g R_h \neq 0$ (resp. $R_h a_g \neq 0$) for all $h \in G$. If (R,G) is both left and right faithful then we say (R,G) is faithful.

Definition 1.7

For a G-graded ring R we way (R,G) is semiprime if R has non-zero nilpotent graded ideal. Also, (R,G) is said to be prime if the product of any two non-zero graded ideals of R is non-zero.

Theorem 1.8 ([2]).

Suppose R_e is semiprime. Then (R,G) is left nondegenerate iff (R,G) is right nondegenerate.

Proposition 1.9

Suppose R_e is prime. Then (R,G) is left faithful iff (R,G) is right faithful.

Proof: Suppose R_e is prime and (R,G) is left faithful. Let $a_g \in R_g$-0 and $h \in G$. Then $R_{h^{-1}} R_h$, $a_g R_{g^{-1}}$ are two non-zero right ideals of R_e. So, $R_{h^{-1}} R_h a_g R_{g^{-1}} \neq 0$ and hence $R_h a_g \neq 0$, i.e., (R,G) is right faithful. The other part is similar.

In the following proposition we give necessary and sufficient conditions for a G-graded ring R to be faithful.

Proposition 1.10

Let R be a G-graded ring. Then (R,G) is faithful iff for any non-zero graded ideal I of R, $I_g \neq 0$ for all $g \in G$.

Proof: Suppose (R,G) is faithful. Let I be a right graded ideal of R such that $I_h = 0$ for some $h \in G$ (a similar argument works for left ideals).

Let $g \in G$. Then $(I \cap R_{hg}) R_{g^{-1}} \subseteq (I R_{g^{-1}}) \cap (R_{hg} R_{g^{-1}}) \subseteq$

$I \cap R_h = I_h = 0$. So, by faithfulness of R, $I \cap R_{gh} = I_{hg} = 0$ and hence $I_g = 0$ for all $g \in G$, i.e., I=0.

Conversely, let $a_g \in R_g - 0$. Then $I = R a_g$ is a non-zero left graed ideal of R and hence $I_{hg} \neq 0$ for all $h \in G$. But $I_{hg} = R a_g \cap R_{hg} = R_h a_g$. So, $R_h a_g \neq 0$ for all $h \in G$, i.e., (R,G) is right faithful. Similarly we show (R,G) is left faithful by choosing $I = a_g R$.

2. VARIOUS STRONGLY G-GRADED RINGS

The stronger nondegenerate and faithful properties are motivated by the work of Cohen and Rowen in [2]. We now define three successively stronger properties that a grading may have. The first is due to Dade [4] and Fell [5].

Definition 2.1

For a G-graded ring R we say:

(1). (R,G) is strong if $R_g R_h = R_{gh}$ for all $g, h \in G$. But this definition is equivalent to $1 \in R_g R_{g^{-1}}$ for all $g \in G$ (see proposition 1.6 of [4]).

(2) (R,G) is first strong if $1 \in R_g R_{g^{-1}}$ for all $g \in$ supp (R,G).

(3) (R,G) is second strong if $R_g R_h = R_{gh}$ for all $g,h \in$ supp (R,G) and supp (R,G) is a monoid in G.

Clearly, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). To see that the three definitions are distinct, we consider some examples.

Example 2.2

(A first strong grading which is not strong). Let $G = Z_4$ and $R = M_2(Z_2)$. Then R is first strong with:

$$R_0 = \begin{bmatrix} Z_2 & 0 \\ 0 & Z_2 \end{bmatrix}, R_2 = \begin{bmatrix} 0 & Z_2 \\ Z_2 & 0 \end{bmatrix}$$

and $R_1 = R_3 = 0$. But clearly R cannot be written as a strongly G-graded ring.

Example 2.3

(A second strong grading which is not fist strong). Let K be a field and $R = K[x]$ with the usual graduation by Z. Then supp $(R, Z) = N \cup \{0\}$ is a monoid in Z, and $R_i R_j = R_{i+j}$ for all i, j ϵ supp (R, Z), i.e., (R, Z) is second strong. Since 2 ϵ supp (R, Z) and $R_2 R_{-2} = 0 \neq R_0$, (R, Z) is not first strong.

Lemma 2.4 ([12]).

If (R,G) is first strong then supp (R,G) $\leq$ G.

Facts 2.5

(1) If (R,G) is second strong and supp (R,G) $\leq$ G then by definition (R,G) is first strong.
(2) IF G is a finite group and (R,G) is second strong then for each g ϵ supp (R,G), $g^{-1} = g^n$ for some n ϵ N. So, g^{-1} ϵ supp (R,G) and hence supp (R,G) $\leq$ G. Therefore, (R,G) is first strong.
(3) IF G is a finite group and (R,G) is first strong then this doesnot mean that (R,G) is strong (see Example 2.2).

Now, we study the relationship between (nondegenerate, faithful) gradings and these new strong gradings.

One can easily see that: (R,G) strong $\Rightarrow$ (R,G) faithful $\Rightarrow$ (R,G) nondegenerate. See [1] for the distinction of these dejuncions,

The relationship between nondegenerate and first strong is given in the following proposition.

Proposition 2.6

Let R be a G-graded ring. Then (R,G) is first strong iff (R,G) is nondegenerate and second strong.

Proof: Suppose (R,G) is first strong. By Lemma 2.4, supp (R,G) $\leq$ G, and then supp (R,G) is a monoid in G. Let g,h ϵ supp (R,G). Then $R_{gh} = R_e R_{gh} = R_g R_{g^{-1}} R_{gh} \subseteq R_g R_h$. So, $R_g R_h = R_{gh}$ for all g,h ϵ supp (R,G), i.e., (R,G) is second strong. To show (R,G) is nondegenerate, let a_g ϵ $R_g - 0$. If $a_g R_{g^{-1}} = 0$ then $a_g R_{g^{-1}} R_g = 0$ and hence $a_g = 0$, a contradiction. Therefore, (R,G) is left nondegenerate. Similarly, (R,G) is right nondegenerate.

Suppose (R,G) is second strong and nondegenerate. Let g ϵ supp (R,G). Then $R_g \neq 0$ and hence $R_g R_{g^{-1}} \neq 0$. So, g^{-1} ϵ supp (R,G) and $R_g R_{g^{-1}} = R_e$ for all g ϵ supp (R,G), i.e., R_g is first strong.

Now, if (R,G) is first strong then (R,G) is nondegenerate. However, the converse is not true as we see in the following example.

Example 2.7

(A nondegenerate grading which is not first strong). Let $G = Z_3$ and $R = K[x]$, where K is a field. Then R is a G-graded ring with:

R_0 $<1, x^3, x^6, \ldots>$, $R_1 = <x, x^4, x^7, \ldots>$ and $R_2 = <x^2, x^5, x^8, \ldots>$. Clearly (R,G) is nondegenerate. But 1,2 ϵ supp (R,G) and $R_1 R_2 \neq R_0$ implies (R,G) is not second strong and hence (R,G) is not first strong.

Example 2.8.

(A first strong grading which is not faithful). Let K be a field, $R = M_2(K)$ and $G = Z_4$. Then R is G-graded ring with: $R_0 = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$ and $R_2 = \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$ and $R_1 = R_3 = 0$.

Clearly, (R,G) is first strong but not faithful.

Example 2.9.

(A faithful grading which is not first strong). Let K be a field, $R = K[x]$ and $G = Z_3 =$. Then R is a G-graded ring with:

$$R_0 = \left\{ \sum_{i=0}^{n} \alpha_i x^{3i} : n \epsilon N \cup \{0\}, \alpha_i \epsilon K \right\},$$

$$R_1 = \left\{ \sum_{i=0}^{n} \alpha_i x^{3i+1} : n \epsilon N \cup \{0\}, \alpha_i \epsilon K \right\},$$

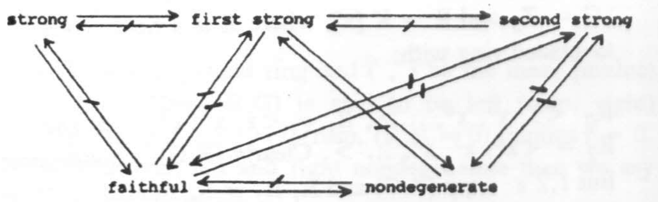
$$R_2 = \left\{ \sum_{i=0}^{n} \alpha_i x^{3i+2} : n \in N \cup \{0\}, \alpha_i \in K \right\},$$

Clearly, (R,G) is faithful and supp (R,G) = G. Since $R_1 R_2 \neq R_0$, (R,G) is not second strong and then (R,G) is not first strong.

Now, if (R,G) is nondegenerate then (R,G) need not be second strong (Example 2.7). Also, if (R,G) is faithful then (R,G) need not be second strong (Example 2.9).

In Example 2.3, (R,G) is second strong. So by proposition 2.6, (R,G) is an example of second strong which is not nondegenerate and hence not faithful.

So, we have the following diagram:



In the following proposition we give the relationship between faithful and strong radings.

Proposition 2.10

Let R be a G-graded ring. Then (R,G) is strong iff (R,G) is second strong and faithful.

Proof: Clearly, if(R,G) is strong then (R,G) is second strong and faithful. Conversely, if (R,G) is second strong and faithful then supp (R,G) G. So, $R_g R_h = R_{gh}$ for all $g,h \in G$, i.e., (R,G) is strong.

Definition 2.11

Let R be a ring with two graduations (R,G) and (R,H). Then (R,G) is almost equivalent to (R,H) if there exists a function f: R → R satisfying the following two conditions:

(i) f is a ring isomorphism from R to R, and

(ii) for each $h \in H$, there exists $g \in G$ such that $f(R_g) = R_h$.

In [13] we proved that the nondegenerate property is preserved between almost equivalent graduations, but the faithfulness and strong properties are not. Now, we show that the first and second strong properties are preserved between almost equivalent graduations.

Proposition 2.12

Suppose (R,G) is almost equivalent to (R,H). Then (R,G) is first strong iff (R,H) is first strong.

Proof: Assume (R,G) is almost equivalent to (R,H) by f. Suppose (R,G) is first strong and $h \in$ supp (R,H). Then there exists $g \in$ supp (R,G) such that $f(R_g) = R_h$. Since $R_g R_{g^{-1}} \neq 0$, $R_{h^{-1}} = f(R_{g^{-1}})$. So, $R_h R_{h^{-1}} = f(R_g R_{g^{-1}}) = f(R_{e_G}) = R_{e_H}$. Conversely, suppose (R,H) is first strong and $g \in$ supp (R,G). Then $f(R_g) = R_h$ for some $h \in$ supp (R,H). Hence $f(R_g R_{g^{-1}}) = R_h R_{g^{-1}} = R_{e_H} = f(R_{e_G})$. But f is 1-1 implies $R_g R_{g^{-1}} = R_{e_G}$.

Proposition 2.13

Suppose (R,G) is almost equivalent to (R,H). Then (R,G) is second strong iff (R,H) is second strong.

Proof: Assume (R,G) is almost equivalent to (R,H) by f. Suppose (R,G) is second strong and $h_1, h_2 \in$ supp (R,H). Then there exists $g_1, g_2 \in$ supp (R,G) such that $f(R_{g_i}) = R_{h_i}$ for i = 1,2. Since supp (R,G) is a monoid we have $g_1 g_2 \in$ supp (R,G) and hence $R_{h_1}, R_{h_2} = f(R_{g_1} R_{g_2}) = f(R_{g_1 g_2}) \neq 0$. So, $h_1 h_2 \in$ supp (R,H), i.e., supp (R,H) is a monoid in H. Now, $0 \neq f(R_{g_1 g_2}) \subseteq R_{h_1 h_2}$ implies $R_{h_1 h_2} = f(R_{g_1 g_2}) = R_{h_1 h_2}$. Thus (R,H) is second strong. Conversely, assume (R,H) is second strong and $g_1, g_2 \in$ supp (R,G). Then $f((R_{g_i}) = R_{h_i}$ for i = 1,2 and some $h_1, h_2 \in$ supp (R,H). So, $0 \neq R_{h_1 h_2} = R_{h_1} R_{h_2} = f(R_{g_1} R_{g_2})$ implies $R_{g_1 g_2} \neq 0$ and hence supp (R,G) is a monoid in G. Since f is 1-1 and $f(R_{g_1 g_2}) = R_{h_1 h_2} = f(R_{g_1} R_{g_2})$ we have $R_{g_1 g_2} = R_{g_1} R_{g_2}$ for all $g_1, g_2 \in$ supp (R,G).

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