ON THE SOLUTION OF THE INVERSE EIGENPROBLEM

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ABSTRACT

In this paper, we propose an new algorithm for the solution of the inverse eigenproblem. The algorithm falls in the class of eigenvector methods. The main idea is based on the selection of a well-conditioned set of vectors from given subspaces and the solution aims to minimize the sensitivity of the assigned eigenvalues to perturbation in the matrices. The problem is formulated as an unconstrained minimization problem. The derivation of the gradient expression is then obtained. Based on this expression and using some powerful Quasi-Newton methods, the new algorithm proved to be efficient and numerically stable. A comparison in performance with existing methods is given.

1. INTRODUCTION

Given a controllable pair (A, B), the problem of obtaining the matrix F such that the combined matrix $(A + B F)$ has a specified spectrum Λ is studied in the context of state space-based synthesis methods in linear control theory [1]. Several methods are already proposed in the literature. There are four different classes of methods, namely; (i) Traditional techniques [1-4], which proved to be inefficient and numerically unstable; (ii) Matrix equation methods [5,6]; (iii) Direct methods [7-9], which do not, in general, lead to robust solutions; and (iv) The eigenvector methods [10-13], which are iterative techniques proposed to improve robustness. The most powerful of all are the methods that are based on the latter techniques. In this paper, we propose an algorithm for the solution of the problem. The algorithm falls in the fourth class and is iterative in nature. The matrix F is obtained by assigning linearly independent eigenvectors corresponding to the required eigenvalues such that the matrix of eigenvectors is as well-conditioned as possible. The assigned eigenvalues are then as insensitive to perturbations as possible. The resulting matrix F is the reasonably bounded as may be expected. The sensitivity of the matrix $(A+BF)$ depends on the magnitude of the condition numbers, then every reasonable measure of the condition number is an indication of the robustness of the eigenproblem. In this paper, we give a choice of such a measure. This will lead to a function of several variables that needs to be minimized. The gradient vector of this function is derived and used together with a powerful Quasi-Newton routine to obtain the limit point of the sequence. The proposed method has been applied to several test problems found in the literature and the results are compared with known techniques.

2. PROBLEM STATEMENT AND SOLUTION

Given a real $n\times n$ matrix A, a real full-rank $n\times m$ matrix B $(m<n)$ and a set L of n complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ closed under complex conjugation; it is required to find a non-singular $n\times n$ matrix X and a real $m\times n$ matrix F satisfying

$$(A + B F) X = X \Lambda, \Lambda = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \quad (1)$$

such that

(i) The eigenvalues of ($A + B F$) are $\lambda_1, \lambda_2, \ldots, \lambda_n$;
(ii) The eigenvalues of ($A + B F$) are as insensitive to perturbations in this matrix as possible.

It has been established by Wonham [1] that the inverse problem has a solution if and only if the pair (A, B) is controllable.

In [10] and [12], numerical algorithms for the computation of robust solutions to the inverse eigenproblem have been proposed. The technique described consists of three basic steps.

Step 1. Decompose the matrix B as

$$B = \begin{bmatrix} U_o & . & U_1 \end{bmatrix} \begin{bmatrix} Z \\ \\ 0 \end{bmatrix} \quad (2)$$

where $\begin{bmatrix} U_o & . & U_1 \end{bmatrix}$ is a real orthogonal matrix and Z is nonsingular. The assumption that B is of full rank implies the existence of this decomposition [10]. In order to determine U_o, U_1 and Z; construct orthonormal bases comprized by the columns of the matrices S_j and $\hat{S}_j$ for the space

$$\Sigma_j = \text{N}\{ U_1^*, (A - \lambda_j I)\} \quad (3)$$

and its complement $\hat{\Sigma}_j$ for $\lambda_j \in L$, $j = 1, 2, \ldots, n$; $\text{N}\{.\}$

denotes the null space; and the superscript * denotes transposition.

The required decomposition of B can be taken as the singular value decomposition (SVD) in which case $Z = \Sigma V^*$, where $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a positive matrix and V is orthogonal [14]. Alternatively, the QR decomposition could be used, in which case Z is an upper triangular matrix [15]. It is to be noticed that the QR decomposition is computationally less expensive, but the SVD gives useful information on the singular values of B. The construction of the bases for Σ_j and $\hat{\Sigma}_j$ is also achieved by QR decomposition of $[U_1^* (A - \lambda_j I)]^*$ as

$$[U_1^* (A - \lambda_j I)]^* = [\hat{S}_j, S_j] \begin{bmatrix} R_j \\ 0 \end{bmatrix} \tag{4}$$

then S_j and $\hat{S}_j$ are the required matrices.

Alternatively, we determine the SVD of $[U_1^* (A - \lambda_j I)]$ in the partitioned form

$$[U_1^* (A - \lambda_j I)] = T[\Gamma_j, 0][\hat{S}_j, S_j]^* \tag{5}$$

where Γ_j is the diagonal matrix of singular values. Then the columns of S_j and $\hat{S}_j$ give the required orthonormal bases.

Step 2. Select vectors $x_j = S_j w_j \in \Gamma_j$ with $\|x_j\|_2 = 1$, $j = 1, 2, \ldots, n$ such that $X = [x_1, x_2, \ldots, x_n]$ is well-conditioned.

Step 3. Find the matrix $M = A + BF$ by solving the matrix equation $MX = X\Lambda$ and compute F explicitly from

$$F = Z^1 U_o^* (M - A) \tag{6}$$

The matrix $M = X\Lambda X^{-1}$ is constructed in this step by solving the equation $X^* M^* = (X\Lambda)^*$ for M^* using the direct LU method [15]. This process is stable for a well-conditioned matrix X. The computation of F id then achieved by straightforward matrix multiplication in the case where Z is given by the SVD process, or by using back substitution to solve the equation $ZF = U_o^* (M - A)$ in the case where Z is given by the QR process.

The key step in this procedure is Step 2. In [10], four methods for accomplishing this step were described, while in [12], a more efficient method was presented and discussed. The methods are all iterative, common in steps 1 and 3; and all aim to minimize different measure of the conditioning of the matrix X.

In step 1, we note that the decomposition can be carried out most efficiently if the matrix to be decomposed is first reduced to a staircase form [16]. This requires less than $n^2(3n+m)$ operations. The number of operations needed to find each subspace is then $m(n-m)(2n-m)$ or a total of $O(n^3 m)$. The computation of M in step 3 requires $O(n^3)$ operations and the computation of F needs $O(nm^2)$ operations. We note that the total amount of work required in steps 1 and 3 is comparable to the number of operations needed for one iteration in step 2, and is not a significant factor in the total operation count. Standard library software for obtaining QR and LU decompositions of matrices and for solving systems of linear equations [17] are used to accomplish these two steps.

2.1. *Measure of Robustness*

In this subsection, measures of robustness of the eigenproblem (1) are presented. Let $X = [x_1, x_2, \ldots, x_n]$ and $Y = [y_1, y_2, \ldots, y_n]$ where $Y = (X^{-1})^*$. It is well-known (see [18]) that the sensitivity of the matrix $(A + BF)$ depends on the magnitude of the condition number c_j where

$$c_j = \|x_j\|_2 \|y_j\|_2 \geq 1. \tag{7}$$

Hence every reasonable measure for the magnitude of the vector $C = [c_1, c_2, \ldots, c_n]^*$ is a reflection of the robustness of the eigenproblem (1). A number of different measures of the robustness of the eigenproblem (1) are considered in [12]. In this paper, we consider the following measures.

$$\nu_c = \|C\|_2 / \sqrt{n}, \tag{8}$$

$$\nu_c(D) = \|DC\|_2 / \|D\|_2 \tag{9}$$

where D is a real diagonal scaling matrix equals to diag $(d_1, d_2, \ldots, d_n)$, with $d_j \geq 0$ for all j. which is applied to the matrix Y^* of left eigenvectors. More details on measures of robustness can be found in [10].

The aim of this work is to propose a new method for accomplishing Step 2 by iteratively constructing a well-conditioned set of eigenvectors from given subspaces. Our proposed scheme aims, at each step of the iteration, to reduce the value of the measure of robustness.

3. THE PROPOSED ALGORITHM

In this section, we present the suggested algorithm for iteratively constructing a well-conditioned set of eigenvectors from given subspaces. The objective here is to choose vectors $x_j \in \Sigma_j$, $j = 1, 2, \ldots, n$, so as to minimize the measure of conditioning $\nu_c(D)$ defined by equation (9).

3.1. *Main Procedure*

Assume that Π_j is the column space of the set of real $n \times m$ matrices S_j, $S_j^* S_j = 1$, $j = 1, 2, \ldots, n$. Let

$$X = [x_1, x_2, \ldots, x_n],$$
$$X_j = [x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n]$$

$$Y = [X^{-1}]^* = [y_1, y_2, \ldots, y_n], \text{ and}$$

$$P_j = \begin{bmatrix} 0 & I_{j-1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{n-j} \end{bmatrix} \tag{10}$$

where I_j is the $n \times n$ identity matrix, and

$$x_j = S_j w_j, \ \hat{w}_j = [w_1^*, \ldots, w_{j-1}^*, w_{j+1}^*, \ldots, w_n^*] \tag{11}$$

where w_j are $m \times 1$ real vectors. Then

$$X P_j = (x_j, X_j) \tag{12}$$

Using the SVD, we get

$$X_j = [U_j, u_j] \begin{bmatrix} \Sigma_j \\ 0 \end{bmatrix} V_j^*$$

where u_j are $n \times 1$ real vectors,

$$\Sigma_j = \text{diag}\,(\sigma_1^{(j)}, \ldots, \sigma_{n-1}^{(j)}),$$

$[U_j, u_j]$ and V_j are orthogonal matrices and

$$X_j = X_j(\hat{w}_j), \ U_j = U_j(\hat{w}_j), \ u_j = u_j(\hat{w}_j),$$
$$V_j = V_j(\hat{w}_j) \text{ and } \Sigma_j = \Sigma_j(\hat{w}_j) \tag{13}$$

Therefore

$$Y^* = P_j [x_j, X_j]^{-1} =$$

$$P_j \begin{bmatrix} 0 & 1/u_j^* x_j \\ V_j \Sigma_j^{-1} & * \end{bmatrix} \begin{bmatrix} U_j^* \\ u_j^* \end{bmatrix} = P_j \begin{bmatrix} u_j^*/u_j^* x_j \\ * \end{bmatrix}$$

Thus

$$y_j = u_j(\hat{w}_j) / u_j(\hat{w}_j)^* S_j w_j, \ j = 1, 2, \ldots, n \tag{14}$$

Hence, for all j, we have

$$\nu_c^2 = \sum_{j=1}^{n} (d_j^2 c_j^2) / \sum_{j=1}^{n} d_j^2 = \sum_{j=1}^{n} \delta_j \| x_j \|_2^2 \| y_j \|_2^2$$
$$= \sum_{j=1}^{n} \frac{\delta_j w_j^* w_j}{[u_j(w_j)^* S_j w_j]^2} \equiv f(w) \tag{15}$$

where

$$w = [w_1^*, w_2^*, \ldots, w_n^*]^*,$$
$$w_j = (w_{1j}, w_{2j}, \ldots, w_{mj})^*, j = 1, 2, \ldots, n$$

and

$$\delta_j = d_j^2 / \sum_{j=1}^{n} d_j^2 > 0 \ \ \forall j \ \text{ and } \ \sum_{j=1}^{n} \delta_j = 1. \tag{16}$$

From this analysis, the idea is now to solve the unconstrained minimization problem: Minimize $f(w)$ defined by equations (15) and (16), where w is a real $nm \times 1$ vector. Let

$$A_j(\hat{w}_j) = X_j(\hat{w}_j) X_j(\hat{w}_j)^* \tag{17}$$

Utilizing equation (13), we have

$$A_j(\hat{w}_j) = [U_j(\hat{w}_j), u_j(\hat{w}_j)] = \begin{bmatrix} \Sigma_j(\hat{w}_j) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_j(\hat{w}_j)^* \\ u_j(\hat{w}_j)^* \end{bmatrix} \tag{18}$$

Assuming that the matrix $X_j(\hat{w}_j)$ is of full column rank, then the diagonal matrix $\Sigma_j(\hat{w}_j)$ is nonsingular and $u_j(\hat{w}_j)$ is a unit eigenvector of $A_j(\hat{w}_j)$ corresponding to the simple zero eigenvalue.

It is well-known that the eigenvector $u_j(\hat{w}_j)$ may be defined as a real analytic function in some neighborhood of $\hat{w}_j$ [19] and the partial derivatives of $u_j(\hat{w}_j)$ with respect to w_{pq} can be represented as follows,

$$\frac{\partial u_j(\hat{w}_j)}{\partial w_{pq}} = U_j(\hat{w}_j) \Sigma_j(\hat{w}_j)^{-2} U_j(\hat{w}_j)^* \frac{\partial A_j(\hat{w}_j)}{\partial w_{pq}} u_j(\hat{w}_j),$$
$$q \neq j, \ \ p = 1, 2, \ldots, m \tag{19}$$

Since $A_j(\hat{w}_j) = \sum_{k=1, k \neq j} S_k w_k w_k^* S_k^*$, then

$$\frac{\partial A_j(\hat{w}_j)}{\partial w_{pq}} = S_q (w_q e_p^* + e_p w_q^*) S_q^*, \ q \neq j \tag{20}$$

where e_p is the p -th column of I_m. Combining (19) and (20), we get

$$\frac{\partial(u_j(\hat{w}_j)^*)}{\partial w_{pq}} = -u_j(w_j)^* S_q(w_q e_p^* + e_p w_q^*) S_q^*$$

$$U_j(\hat{w}_j)\Sigma_j(\hat{w}_j)^{-2} U_j(\hat{w}_j)^* \qquad (21)$$

and

$$\frac{\partial(u_j(\hat{w}_j)^*)}{\partial w_q} = -[u_j(\hat{w}_j)^* S_q w_q I_m + S_q^* u_j(\hat{w}_j) w_q^*] S_q^*$$

$$U_j(\hat{w}_j)\Sigma_j(\hat{w}_j)^{-2} U_j(\hat{w}_j)^* \qquad (22)$$

Let $\nabla f(w)$ be the gradient vector and let

$$\frac{\partial f(w)}{\partial w_q} = \left[\frac{\partial f(w)}{\partial w_{1q}}, \frac{\partial f(w)}{\partial w_{2q}}, \ldots, \frac{\partial f(w)}{\partial w_{nq}}\right]^*, q = 1, 2, \ldots, n \qquad (23)$$

then

$$\nabla f(w) = \left[\left(\frac{\partial f(w)}{\partial w_1}\right)^*, \left(\frac{\partial f(w)}{\partial w_2}\right)^*, \ldots, \left(\frac{\partial f(w)}{\partial w_n}\right)^*\right]^* \qquad (24)$$

Combining (15), (23) and (24), we get

$$\frac{\partial f(w)}{\partial w_q} = 2\delta_q \frac{w_q - \dfrac{w_q^* w_q S_q^* u_q(\hat{w}_q)}{u_q(\hat{w}_q)^* S_q w_q}}{[u_q(\hat{w}_q)^* S_q w_q]^*} + 2 \sum_{j=1, j\neq q} \frac{\delta_j w_j^* w_j}{u_j(\hat{w}_j)^* S_j w_j} Y_{(j,q)}(w) \frac{S_q^* Z_j(\hat{w}_j) S_j w_j}{[u_j(w_j)^* S_j w_j]^2},$$

$$q = 1, 2, \ldots, n \qquad (25)$$

where

$$Y_{(j,q)}(w) = u_j(\hat{w}_j)^* S_q w_q I_{mq} + S_q^* u_j(\hat{w}_j) w_q^*, \qquad (26)$$

and

$$Z_j(\hat{w}_j) = U_j(\hat{w}_j)\Sigma_j(\hat{w}_j)^{-2} U_j(\hat{w}_j)^*. \qquad (27)$$

From the previous analysis, the following theorem holds.

Theorem

Suppose that $X_j(\hat{w}_j)$, $U_j(\hat{w}_j)$, $u_j(\hat{w}_j)$, $\Sigma_j(\hat{w}_j)$ and $f(w)$ are defined as in (10) to (13). Assume that the matrix $X_j(\hat{w}_j)$ for $j = 1, 2, \ldots, n$ are of full column rank. Then the formulae (24) to (27) give the expression for the gradient vector $\nabla f(w)$.

3.2. *The Optimization Procedure*

Several powerful gradient methods are available in the literature. Amongst the most powerful are the Quasi Newton Methods [20,22]. The Quasi-Newton technique o Davidon-Fletcher-Powell [20] is now applied to minimize th objective function $f(w)$, using the gradient expressio derived in the previous subsection. We present here the basi steps in the new procedure.

Initialization Step. Let $\varepsilon > 0$ be the termination criterion Choose an initial real vector $(mn \times 1)$

$$w^{(0)} = [w_1^{(0)*}, w_2^{(0)*}, \ldots, w_n^{(0)*}]^* \; \forall j$$

and an $nm \times nm$ positive definite matrix H_o.

Main Step

1. Let $k := 0$.
2. Compute $g_k = \nabla f(w^{(k)})$, $d_k = -H_k g_k$.
3. Determine $w^{(k+1)} = w^{(k)} + \lambda_k d_k$ using line searc along the direction d_k [20], where
$$f(w^{(k)} + \lambda_k d_k) = \min_{\lambda > 0} f(w^{(k)} + \lambda d_k)$$
4. Set $\Delta w^{(k)} = w^{(k+1)} - w^{(k)}$. If $\| \Delta w^{(k)} \|_2 \leq \varepsilon$ STOP, $w = w^{(k)}$ is an approximate optimal solution If $\| \Delta w^{(k)} \|_2 > \varepsilon$, GO TO Step 5.
5. Compute $g_{k+1} = \nabla f(w^{(k+1)})$, $\gamma_k = g_{k+1} - g_k$,
$$H_{k+1} = H_k + \frac{\Delta w^{(k)} (\Delta w^{(k)})^*}{(\Delta w^{(k)})^* \gamma_k} - \frac{H_k \gamma_k \gamma_k^* H_k^*}{\gamma_k^* H_k \gamma_k},$$
$$\text{and } d_{k+1} = -H_{k+1} g_{k+1}.$$
6. Replace k by $k + 1$ and GO TO Step 3.

Some remarks are now due.

Remark 1. At the initialization step, if the initial vector $w^{(0)}$ is chosen such that for some j of indices 1, 2, ..., n; th matrix $X_j(\hat{w}_j^{(0)})$ has a very small singular value, then w must choose another initial vector.

Remark 2. At step 2, we compute $\nabla f(w^{(k)})$ by the formula (24) to (27), in which the SVD's $X_1(\hat{w}_1^{(k)})$, $X_2(\hat{w}_2^{(k)})$, ..., $X_n(\hat{w}_n^{(k)})$ are obtained by standard routines. The matrix $X_j(\hat{w}_j^{(k)})$ i obtained by rank-one update of the matrix $X_{j-1}(\hat{w}_{j-1}^{(k)})$ hence it is necessary to develop techniques for updating th SVD [21].

Remark 3. At step 3, the line search technique is based on curve fitting procedures, such as quadratic or cubic interpolations [20].

4. NUMERICAL EXAMPLES

To illustrate the effectiveness of the new procedure, several numerical examples were solved. Among them the problem given in [12]. A comparison between the new procedure and existing techniques are also presented. In the test example, $n=3$, $m=2$ and

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The eigenvalues of A are $\{1,2,3\}$, thus the pair (A,B) is controllable but unstable. We assign the set of eigenvalues $L = \{-0.2, -0.2, -10\}$, as in [12], and we compute the matrices X and F such that $(A+BF)X = X\Lambda$, $\Lambda = \text{diag}(-0.2,-0.2,-10)$. Step 1 as described in the previous section gives,

$$S_1 = S_2 = \begin{bmatrix} 0.842844 & -0.359245 \\ 0.492834 & 0.216164 \\ 0.216169 & 0.907862 \end{bmatrix},$$

$$S_3 = \begin{bmatrix} -0.817060 & 0.557086 \\ 0.418760 & 0.396300 \\ 0.396300 & 0.729796 \end{bmatrix}.$$

The condition number of $S = [S_1, S_2, S_3]$ is $C_2(S) = 3.65057$, and a matrix F with good conditioning is expected. The proposed method is applied on this test example. Floating point numbers are represented by a double precision 56-bit mantissa and a 7 bit exponent to radix 2, whence the machine precision is 7.22E-16, tolerance 0.1E-37. When choosing $w^{(0)} = [w_1^{(0)*}, w_2^{(0)*}, w_3^{(0)*}]^*$ with

$$w_1^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w_2^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad w_3^{(0)} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},$$

as an initial vector with weights $\delta_1=\delta_2=\delta_3=1/3$, after 4 iterations, we obtain

$$X = \begin{bmatrix} 0.88322 & -0.50356 & -1.12971 \\ 0.49861 & 0.12788 & 0.35651 \\ 0.11429 & 0.86170 & 0.22162 \end{bmatrix},$$

$$F = \begin{bmatrix} 2.44276 & -3.78554 \\ -6.22520 & 7.44796 \\ 2.24393 & -4.27650 \end{bmatrix},$$

where $\| X_1 \|_2 = 1.00635$, $\| X_2 \|_2 = 1.00620$ and $\| X_3 \|_2 = 1.20518$.

In Table 1, the results obtained by applying the new procedure are given together with the results of the procedure in [10] and [12].

Computational practice showed that the choice of an approximate initial vector $w^{(0)}$ is important (see remark 1, in the previous section). It is worthwhile to point out that we can use the procedure in [12] to produce a reasonable initial vector before applying the proposed scheme. For example, if we choose $w_1^{(0)} = w_2^{(0)} = w_3^{(0)} = [1/\sqrt{2}, 1/\sqrt{2}]^*$ as an initial vector, then we find that the 3×2 matrix $X_3^{(0)} = [S_1 w_1^{(0)}, S_2 w_2^{(0)}]$ is not of full column rank. In fact, $X_3^{(0)}$ has singular values $\{2,0\}$. Consequently, in this case, it is impossible to compute the gradient vector $\nabla f(w^{(0)})$ and to proceed with the new method. But when using the above mentioned initial vector, with the method in [12] and taking $\gamma_{12}=\gamma_{13}=\gamma_{23}=1/\sqrt{3}$, $\alpha^2=0.7$ and $\omega=2.75$, then after two iterations we get $w^{(2)} = [w_1^{(2)*}\ w_2^{(2)*}\ w_3^{(2)*}]^*$ in which

$$w_1^{(2)} = \begin{bmatrix} 0.664007 \\ -0.747726 \end{bmatrix}, \quad w_2^{(2)} = \begin{bmatrix} 0.900583 \\ 0.434683 \end{bmatrix},$$

$$w_3^{(2)} = \begin{bmatrix} 0.912127 \\ 0.409907 \end{bmatrix}$$

Then taking $w^{(2)}$ as a new initial vector with the new algorithm, and after 7 iterations we obtain, with $\delta_1=\delta_2=\delta_3=1/3$, a good solution

$$X = \begin{bmatrix} 0.54844 & 1.11890 & -1.28961 \\ -0.13971 & 0.59312 & 0.40691 \\ -0.93949 & 0.14354 & 0.25292 \end{bmatrix},$$

$$F = \begin{bmatrix} -4.49346 & -3.65376 \\ 8.00917 & 7.73998 \\ -3.83710 & -4.18704 \end{bmatrix},$$

where $\| X_1 \|_2 = 1.08778$, $\| X_2 \|_2 = 1.27450$

and $\| X_3 \|_2 = 1.37573$. The corresponding condition numbers are $c_1=1.40408$, $c_2=1.40426$ and $c_3=1.78887$. The computed matrix F has

$$\| . \|_2 = 13.7656.$$

5. CONCLUSION

The inverse eigenproblem is transformed to the simple problem of function minimization. The measures of robustness form the parameters of the objective function to be minimized. The gradient expression of this function is then derived and used together with an existing Quasi-Newton method to obtain the solution of the problem.

Numerical results have shown that this new technique is stable in terms of computation behaviour and the convergence is achieved at the limit point of the iterative process. A comparison between some existing techniques and the new procedure showed that the proposed method is superior in terms of computation requirement and effectiveness.

Table 1. Results for the methods in [10] and [12] and the proposed method. (N is the Number of iterations).

Method	$[c_1\ c_2\ c_3]^*$	$\|c\|_2$	$\|X\|_2 \|X^{-1}\|_2$	$\|F\|_2$	N
Method 0 in [10]	$\begin{bmatrix}1.34\\1.47\\1.79\end{bmatrix}$	2.7221	3.27320	16.4600	-
Method 1 in [10]	$\begin{bmatrix}1.473\\1.425\\1.789\end{bmatrix}$	2.72095	3.27316	16.4617	-
Method 2/3 in [10]	$\begin{bmatrix}1.59\\1.41\\1.79\end{bmatrix}$	2.77850	3.28270	16.4600	-
Method in [12] with $\alpha^2=0.7$, $\omega=2.75$, $\gamma_{12}=\gamma_{13}=\gamma_{23}=1/\sqrt{3}$	$\begin{bmatrix}1.677\\1.677\\1.789\end{bmatrix}$	2.97091	3.36093	17.5298	15
Method in [12] with $\alpha^2=0.9$, $\omega=2.75$, $\gamma_{12}=\gamma_{13}=\sqrt{0.45}$, $\gamma_{23}=\sqrt{0.1}$	$\begin{bmatrix}1.000\\1.789\\1.789\end{bmatrix}$	2.72093	3.27317	12.6006	22
New method with $\delta_1=\delta_2=\delta_3=1/3$	$\begin{bmatrix}1.39456\\1.39478\\1.78934\end{bmatrix}$	2.66307	3.27932	11.7000	4
New method with $\delta_1=\delta_2=0.1$, $\delta_3=0.8$	$\begin{bmatrix}1.39470\\1.39464\\1.78934\end{bmatrix}$	2.66307	3.27371	11.8064	4

REFERENCES

[1] W.M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 2nd. ed., Springer Verlag, N.Y., 1979.

[2] C.T. Chen, *Linear System Theory and Design*, 2nd. ed., Holt, Rinehart and Winston, N.Y. 1984.

[3] T. Kailath, *Linear Systems*, Prentice Hall, Englewood Kliffs, N.J., 1980.

[4] R.V. Patel, "Algorithm for Eigenvalue Assignment in Multivariable Systems", in *Advances in Computer-Aided Control System Engineering*, M. Jamshidi and C.J. Herget, eds., Amsterdam, North Holland, 1985.

[5] S.P. Bhattacharyya and E. de Souza, "Pole Assignment via Sylvester's Equation", *Syst. Cont. Lett.*, vol. 1, no. 4, pp. 261-263, 1982.

[6] M.A. Shalaby, "A Numerically Stable Algorithm for the Inverse Eigenproblem", to be published.

[7] C.S. Miminis and C.C. Paige, "An Algorithm for Pole Assignment of Time-Invariant Linear Systems", *Int. J. Contr.*, vol. 35, pp. 341-354, 1982.

[8] A. Varga, "Numerically Stable Algorithm for Standard Controllability Form Determination", *Electronic Lett.*, vol. 17, pp. 74-75, 1985.

[9] M. Arnold and B.N. Datta, "An Algorithm for the Multi-Input Eigenvalue Assignment Problem", *IEEE Trans. Aut. Contr.*, Vol. AC-35, pp. 1149-1152, 1990.

[10] J. Kautsky, N.K. Nichols and P. Van Dooren, "Robust Pole Assignment in Linear State Feedback", *Int. J. Contr.*, vol. 41, pp. 1129-1155, 1985.

[11] R.K. Cavin and S.P. Bhattacharyya, "Robust and Well-Conditioned Eigenstructure Assignment Via Sylvester Equation", *Opt. Contr. Appl. Methods*, vol. 4, pp. 205-212, 1983.

[12] M.A. Shalaby and R.A. El-Attar, "A Computational Procedure for a Robust Solution to the State Feedback Pole Assignment Problem", *Proc. 3rd Conf. on AME*, MTC Cairo, Egypt, CA-5, pp. 41-55, 1988.

[13] K.E. Chu, "A Pole Assignment Algorithm for Linear State Feedback", *Syst. Contr. Lett.*, pp. 288-300, 1986.

[14] M.A. Shalaby, "Singular Value Decomposition of a Matrix: Its Computation and some Applications related to Linear Control Systems", *Ph.D. Thesis*, Alexandria University, Egypt, 1985.

[15] B.T. Smith et al., *EISPACK Guide*, vol. 6, 2nd ed., Springer Verlag, N.Y., 1977.

[16] P. Van Dooren, "The Generalized Eigenstructure Problem in Linear System Theory", *IEEE Trans. Aut. Contr.*, vol. AC-26, pp. 111-130, 1981.

[17] J. Dongerra et al., *LINPACK User's Guide*, SIAM, Philadelphia, 1979.

[18] J.H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford Press, 1956.

[19] J.G. Sun, "Eigenvalues and Eigenvectors of Matrix Dependent on Several Parameters", *J. Comp. Math.*, vol 3, pp. 351-364, 1985.

[20] E.M.L. Beale, *Introduction to Optimization*, John Wiley, N.Y., 1988.

[21] J.R. Bunch and C.P. Neilsen, "Updating the Singular Value Decomposition", *Numer. Math.*, vol. 31, pp. 111-128, 1987.

[22] J.E. Dennis and J.J. Moré, "Quasi-Newton Methods, Motivation and Theory", *SIAM Review*, Vol. 19, pp. 46-89, 1977.