CUBIC SPLINE ON QUINTIC SPLINE METHOD FOR NON-LINEAR BOUNDARY VALUE PROBLEMS

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ABSTRACT

Cubic spline on quintic spline interpolations are used to approximate the derivative terms in the numerical scheme used for the solution of non-linear boundary value problems in one dimension. The accuracy is improved to the order $O(h^8)$ while in a previous paper [1], using cubic spline on spline interpolations, it was shown to be of order $O(h^6)$.

INTRODUCTION

Consider the non-linear differential equation:

$$\nabla^2 c = f(c, \bar{x}) \tag{1}$$

which represents a general class of problems in engineering, as mentioned in [2], where ∇^2 represents the Laplacian operator, c the dependent variable, $f(c,\bar{x})$ the non-linear rate model, and $\bar{x}$ the position vector.

In a one dimensional geometry the Laplacian operator takes the form

$$\nabla^2 c = \frac{d^2 c}{dx^2}$$

where x is the distance parameter for the chosen geometry. Thus

$$\frac{d^2 c}{dx^2} = f(c,x) \tag{2}$$

This second order differential equation is reduced to a system of two simultaneous first order differential equations which can be written in the general form

$$\frac{dc_p}{dx} = f_p(x, c_1, c_2), \quad p = 1, 2. \tag{3}$$

where c_1 represents the dependent variable, and c_2 its first derivative.

From now on the subscript p will be dropped. Subdividing the x range [0, 1] into n equal parts such that $h = 1/n$ and $x_i = ih$, i=0, 1, ..., n, and integrating an equation of the system (3) for the interval $[x_{i-1}, x_i]$, denoting the resulting quantity by Q_i (c) which clearly equals 0, we get

$$Q_i(c) = c_i - c_{i-1} - \int_{x_{i-1}}^{x_i} f(x, c(x))\,dx = 0, \qquad c_i = c(x_i), \quad i = 1, 2, \dots, n \tag{4}$$

The above integral can be evaluated in many different ways. Using Euler-Maclaurin summation formula [3], we obtain

$$Q_i(c) = c_i - c_{i-1} - \frac{h}{2}(f_i - f_{i-1}) + \frac{h^2}{12}(f_i^1 - f_{i-1}^1) - \frac{h^4}{720}(f_i^3 - f_{i-1}^3) + \frac{h^6}{30240}(f_i^5 - f_{i-1}^5) - \frac{h^8}{1209600} f^7(\xi_i) \tag{5}$$

where the superscripts on f denote the order of derivatives and $x_{i-1} \le \xi_i \le x_i$, $f_i = f(x, c(x_i))$.

To determine f^1 and f^3 from f, we use a quintic spline, then using f^3 we can determine f^5 by cubic spline.

QUINTIC SPLINES

The quintic spline formula that interpolates f(x) at the knots and uses the first and third derivatives in its formulation may, in the i^{th} interval $[x_{i-1}, x_i]$, be written as, [3],

$$\begin{aligned} S_i = {} & \tfrac{1}{2}[(2w^5 - 5w^4 + 5w^2) f_i + (2\bar{w}^5 - 5\bar{w}^4 + 5\bar{w}^2) f_{i-1}] \\ & - \tfrac{h}{4}[(2w^5 - 5w^4 + 3w^2) F_i - (2\bar{w}^5 - 5\bar{w}^4 + 3\bar{w}^2) F_{i-1}] \\ & + \tfrac{h^3}{48}[(2w^5 - 3w^4 + w^2) T_i - (2\bar{w}^5 - 3\bar{w}^4 + \bar{w}^2) T_{i-1}] \end{aligned} \tag{6}$$

where

$$w = (x_i - x_{i-1})/h, \quad \bar{w} = 1 - w,$$

$$w' = \frac{1}{h} \quad \text{and} \quad \overline{w}' = -\frac{1}{h}$$

It can be easily shown that

$$S_i(x_i) = f_i, \quad S_i^1(x_i) = F_i \text{ and } S_i^3(x_i) = T_i,$$

$$S_i(x_{i-1}) = f_{i-1}, S_i^1(x_{i-1}) = F_{i-1} \text{ and } S_i^3(x_{i-1}) = T_{i-1},$$

where the superscripts on S_i denote the order of derivatives.

If we let $S_i^2(x_i) = S_{i+1}^2(x_i)$, we get

$$-T_{i+1} + 6T_i - T_{i-1} + \frac{12}{h^2}(3F_{i+1} + 14F_i + 3F_{i-1}) = \frac{120}{h^3}(f_{i+1} - f_{i-1}) \quad (7)$$

and

$$-3T_{i+1} - 14T_i - 3T_{i-1} + \frac{60}{h^2}(F_{i+1} + 2F_i + F_{i-1}) = \frac{120}{h^3}(f_{i+1} - f_{i-1}) \quad (8)$$

From equations (7) and (8), we have

$$T_i = -\frac{3}{2h^2}(F_{i+1} + 8F_i + F_{i-1}) + \frac{15}{2h^3}(f_{i+1} - f_{i-1}) \quad (9)$$

From equations (9) and (7), and using the fact that $S_i^1 = F_i$, we obtain

$$S_{i+2}^1 + 26S_{i+1}^1 + 66S_i^1 + 26S_{i-1}^1 + S_{i-2}^1 = \frac{5}{h}(f_{i+2} + 10f_{i+1} - 10f_{i-1} - f_{i-2}) \quad (10)$$

Also, since $S_i^3 = T_i$, equation (9) can be written as

$$S_i^3 = -\frac{3}{2h^2}(S_{i+1}^1 + 8S_i^1 + S_{i-1}^1) + \frac{15}{2h^3}(f_{i+1} - f_{i-1}) \quad (11)$$

ERROR BOUNDS

To determine the error bounds on $|f_i^1 - S_i^1|$ and $|f_i^3 - S_i^3|$, we have the following theorem.

Theorem

Let S be a quintic interpolating spline on f such that equation (10) holds with

$$S_0^1 = f_0^1, \ S_1^1 = f_1^1 \text{ and } S_{n-1}^1 = f_{n-1}^1, \ S_n^1 = f_n^1,$$

and

$$S_i^3 = -\frac{3}{2h^2}(S_{i+1}^1 + 8S_i^1 + S_{i-1}^1) + \frac{15}{2h^3}(f_{i+1} - f_{i-1})$$

with $S_0^3 = f_0^3$, $S_n^3 = f_n^3$. Then

$$|f_i^1 - S_i^1| = O(h^6) \text{ and } |f_i^3 - S_i^3| = O(h^4)$$

Proof

Subtracting $f_{i+2}^1 + 26f_{i+1}^1 + 66f_i^1 + 26f_{i-1}^1 + f_{i-2}^1$ from both sides of equation (10), and let $f_i^1 - S_i^1 = e_i^1$, we get

$$e_{i+2}^1 + 26e_{i+1}^1 + 66e_i^1 + 26e_{i-1}^1 + e_{i-2}^1$$

$$= -\frac{5}{h}(f_{i+2} + 10f_{i+1} - 10f_{i-1} - f_{i-2}) + f_{i+2}^1 + 26f_{i+1}^1 + 66f_i^1 + 26f_{i-1}^1 + f_{i-2}^1$$

Using Taylor's expansion for the right hand side (RHS), we obtain

$$\text{RHS} = -\frac{300}{7!}h^6 f^7 + \ldots = -\frac{5}{84}h^6 f^7 + \ldots$$

Hence

$$|f_i^1 - S_i^1| = O(h^6)$$

Also, since

$$S_i^3 = -\frac{3}{2h^2}(S_{i+1}^1 + 8S_i^1 + S_{i-1}^1) + \frac{15}{2h^3}(f_{i+1} - f_{i-1})$$

then

$$|f_i^3 - S_i^3| = e_i^3 = f_i^3 - \frac{15}{2h^3}[f_{i+1} - f_{i-1}] + \frac{3}{2h^2}[(S_{i+1}^1 - f_{i+1}^1)$$

$$+ 8(S_i^1 - f_i^1) + (S_{i-1}^1 - f_{i-1}^1)] + \frac{3}{2h^2}[f_{i+1}^1 + 8f_i^1 + f_{i-1}^1]$$

Using Taylor's expansion for the RHS, we get

$$|f_i^3 - S_i^3| = O(h^4).$$

To make use of equation (10), we need values for S_0^1, S_1^1, S_{n-1}^1 and S_n^1. The following approximations of order h^6 for f_0^1, f_1^1, f_{n-1}^1 and f_n^1 may be used.

$$f_0^1 = \frac{1}{60h}[-147f_0 + 360f_1 - 450f_2 + 400f_3 - 225f_4 + 72f_5 - 10f_6] + O(h^6) \quad (12)$$

$$f_1^1 = \frac{1}{60h}[-10f_0 - 77f_1 + 150f_2 - 100f_3 + 50f_4 - 15f_5 + 2f_6] + O(h^6) \quad (13)$$

f_n^1 and f_{n-1}^1 are skew symmetric to f_0^1 and f_1^1.

Also for equation (11), we need values for S_0^3 and S_n^3. The following approximations of order h^4 for f_0^3 and f_n^3 may be easily derived and used.

$$f_0^3=\frac{1}{8h^3}[-49f_0+232f_1-461f_2+496f_3-307f_4+104f_5-15f_6]+O(h^4) \quad (14)$$

f_n^3 is skew symmetric to f_0^3.

CUBIC SPLINES ON QUINTIC SPLINES

Cubic splines $\hat{S}_i$ are used on the quintic splines S_i^3 (computed from equations (11) and (7)) using the well known second derivative equation for cubic splines [4] as follows.

$$\hat{S}_{i+1}^5+4\hat{S}_i^5+\hat{S}_{i-1}^5=\frac{6}{h^2}(S_{i+1}^3-2S_i^3+S_{i-1}^3),$$

$$i = 1, 2, \ldots, n-1 \quad (15)$$

where $\hat{S}_i^5$ denotes the second derivative of S_i^3.

Lemma

The error bounds for equation (15) is given by

$$|\hat{S}_i^5-f_i^5|=O(h^2)$$

Proof

Subtracting $f_{i+1}^5+4f_i^5+f_{i-1}^5$ from both sides of equation (15) and let $E_i^5=f_i^5-\hat{S}_i^5$, we get

$$E_{i+1}^5+4E_i^5+E_{i-1}^5=-\frac{6}{h^2}(S_{i+1}^3-2S_i^3+S_{i-1}^3)+f_{i+1}^5+4f_i^5+f_{i-1}^5$$

From the previous analysis, it is easily seen that

$$S_{i+1}^3-2S_i^3+S_{i-1}^3=f_{i+1}^3-2f_i^3+f_{i-1}^3+O(h^4)$$

Hence,

$$E_{i+1}^5+4E_i^5+E_{i-1}^5=-\frac{6}{h^2}(f_{i+1}^3-2f_i^3+f_{i-1}^3)+O(h^2)+f_{i+1}^5+4f_i^5+f_{i-1}^5$$

Using Taylor's expansion on the RHS, we get

$$\text{RHS} = O(h^2)$$

Therefore

$$|\hat{S}_i^5-f_i^5|=O(h^2)$$

The following approximations for $\hat{S}_0^5$ and $\hat{S}_n^5$, see [5], are needed when equation (15) is used,

$$\hat{S}_0^5=\frac{1}{h^2}(2S_0^3-5S_1^3+4S_2^3-S_3^3)+O(h^2)$$

while $\hat{S}_n^5$ is symmetric to $\hat{S}_0^5$.

It has been shown that the approximations used for f_i^1, f_i^3 and f_i^5 when using S_i^1, S_i^3 and $\hat{S}_i^5$ are respectively of order $O(h^6)$, $O(h^4)$ and $O(h^2)$.

Substituting in the Euler-Maclaurin formula (5), we get

$$Q_i(c)=c_i-c_{i-1}-\frac{h}{2}(f_i+f_{i-1})+\frac{h^2}{12}(S_i^1-S_{i-1}^1)$$

$$-\frac{h^4}{720}(S_i^3-S_{i-1}^3)+\frac{h^6}{30240}(\hat{S}_i^5-\hat{S}_{i-1}^5)+O(h^8) \quad (16)$$

This shows that the method is of order $O(h^8)$.

To solve the system of equations (16), as in [6], we write $Q_i(c)$ in the form,

$$Q_i(c)=u_i(c)+v_i(c) \quad (17)$$

where

$$u_i(c)=c_i-c_{i-1}-\frac{h}{2}(f_i+f_{i-1}) \quad (18)$$

$$v_i(c)=\frac{h^2}{12}(S_i^1-S_{i-1}^1)-\frac{h^4}{720}(S_i^3-S_{i-1}^3)+\frac{h^6}{30240}(\hat{S}_i^5-\hat{S}_{i-1}^5) \quad (19)$$

Using the modified Newton's nonlinear iterative technique, we get

$$c^{(k+1)}=c^{(k)}-J^{-1}(u)Q(c^{(k)}),\quad k=0,1,2,\ldots \quad (20)$$

where J(u) is the Jacobian of the values of u given by equation (18). The convergence of this modified Newton technique is proved in [7]. As it can be evidently seen J(u) has only few elements per row, i.e., it has a sparse structure. For the solution of equation (20) various sparse matrix techniques can be used [8].

NUMERICAL RESULTS

To illustrate the effectiveness of the method described in this paper, two of the cases studied in [1] were solved. The first is a linear case while the second is nonlinear. The programs were run on a MicroVax 3500 at the Faculty of Engineering, Alexandria University. The boundary conditions used in the examples are at $x=0$, $dc/dx=0$; and at $x=1$, $c=1$.

Linear Case

$$f = - e^{2x}c$$

where f is dependent on x.

The exact solution for this case as shown in [1] is obtained in terms of Bessel functions of the first and second kinds $J_n(x)$ and $Y_n(x)$.

The numerical results shown in Table I were obtained in two iterations with a tolerance of 10^{-4} and $h=0.01$. The maximum error encountered at all points between the numerical solution and the exact solution is of the order 10^{-5} for c and 10^{-4} for dc/dx.

Table I

x	c	dc/dx
0.0	7.6484	0.0000
0.2	7.4735	- 1.8655
0.4	6.8505	- 4.5015
0.6	5.6196	- 7.9227
0.8	3.6552	-11.7072
1.0	1.0000	-14.4918

Non-linear Case

$$f = Mc^2$$

For $M=8$, 100 subdivisions an a tolerance of 10^{-4}, numerical results are shown in Table II.

Table II

x	c	dc/dx
0.0	0.3171	0.0000
0.2	0.3335	0.1665
0.4	0.3861	0.3670
0.6	0.4876	0.6696
0.8	0.6682	1.1921
1.0	1.0000	2.2722

CONCLUSION

In this paper a method based on cubic on quintic splines interpolations for the solution of one dimensional boundary value problem has been described and the corresponding error bounds have been derived. It was found that the error bound for this method is of the order $O(h^8)$.

In a previous paper [1] a cubic on cubic splines method was studied and the error was of the order $O(h^6)$.

The comparison between the two studies shows a noticeable improvement in the present paper which is confirmed by the numerical results obtained.

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