BOUNDARY ELEMENT METHOD APPLIED TO PLATE BENDING AND COMPARATIVE ANALYSIS WITH FINITE ELEMENT METHOD

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ABSTRACT

The boundary element method (BEM) as applied to flexure of thin plates is reviewed briefly, along with the derivation of the integral equations for the direct formulation. Besides, the main techniques used and difficulties encountered are discussed. The differences between the BEM and the displacement finite element method (FEM) are emphasized. The efficiency and accuracy of both methods are compared for several rectangular thin plates under transverse static loading. The merits of the BEM are highlighted. The BEM solution accuracy is better especially for higher derivatives of the function, a point of importance in inelastic analysis. Results are presented and conclusions drawn for the studied cases.

NOTATION

w	out of plane deflection, (L)
$\frac{\partial w}{\partial n}$	normal derivative at field point
M_i	bending moment per unit length in direction i, (FL/L)
E	Modulus of Elasticity, (F/L^2)
t,h	plate thickness, (L)
ν	Poisson's ratio
q (x,y)	uniformly distributed load on plate, (F/L^2)
ϵ_i	strain
ϵ	Global error measure
σ	stress, (F/L^2)
V,Q	shearing force, (F/L)
DOF	degrees of freedom
D	plate flexural rigidity, (FL^2/L), $\frac{Eh^3}{12(1-\nu^2)}$
∇^2	Laplace operator = $(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})$ in cylindrical coord.
$\nabla^4 = \nabla^2 \nabla^2$	biharmonic operator = $\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$
ϕ	displacement function
r	Eucledean distance from source to field point
n,t	normal and tangential directions to the boundary (at field point)
x,y,z	Cartesian coordinates
x_i, y_i	in plane point i coordinates
A,B	interior of the plate, plate x-sec.
∂B, C	the boundary
d s	boundary segment
N_s	no. of boundary elements
N_b	no. of boundary nodes
n_i	no. of internal elements
a,b	in plane plate dimensions in direction of coordinates, (L)
n_x, n_y	components of the outer unit normal to plate boundary
SS	simply supported
CL	clamped
p,q=i,j	denote source and field points, respectively (in equations)
p,q,k	denote internal points
P,Q=i,j	denote source and field points on the boundary, respectively
a_{ij}, d_{ij}, e_{ij}, f_{ij}, b_{ij}	matrix coefficients (i^{th} element, j^{th} element)

INTRODUCTION

The boundary element method (BEM, also called the boundary integral equation method, BIEM) is a powerful general-purpose numerical procedure for the solution of boundary value problems in two and three dimensional elastostics [1, 2, 3, 4, 5, 6]. It is an attractive alternative to the finite element method (FEM) for the numerical solution of transverse bending of elastic thin plates, the application of which is of vital importance in the design of many structures. The BEM is gaining popularity among other methods due to the possibility of reduced dimensionality of the problem, since only the discretization to the boundary is needed, resulting in a reduction of the system of equations and smaller input data required for computation.

PLATE BENDING

The governing differential equation of transverse deflections of thin elastic isotropic plates of uniform rigidity loaded by out-of-plane forces is the inhomogeneous biharmonic equation [7]:

$$\nabla^4 w = \nabla^2 \nabla^2 w = \frac{q(x,y)}{D} = g \qquad (x,y) \in \Omega \qquad (1)$$

assuming that the Poisson-Kirchhoff assumptions hold [7] and that the transverse displacement and rotations are dependent.

This equation can be described by a system of boundary integral equations [8,9,10,11]. It can be reduced exploiting Green's formula to the solution of an integral equation over the boundary which can be handled numerically by the usual techniques of replacing it by a corresponding set of simultaneous algebraic equations for functions on the boundary. In the BEM the discretization for the numerical solution is carried only on the boundary and not the whole domain, as done in the FEM. If values at interior points are required, they are calculated afterwards from the surface data. The resulting equations may be more complicated but the system of equations is smaller. The BEM has been applied to the problem of elastic [7,9,10,11,12,13,14] and inelastic [16,17] bending of thin plates. The associated elastic biharmonic equation has been solved indirectly [10,13] by using an integral representation of the solution in terms of certain known kernels and unknown source densities (functions) defined at the boundary. A different approach has been proposed [11] using solutions other than Green's function in an unbounded domain, by embedding the problem in another one for which the Green function for a concentrated load and moment are known. Direct BEM formulations relating the quantities

$w, \frac{\partial w}{\partial n}, \nabla^2 w$ and $\frac{\partial}{\partial n}\nabla^2 w$ to the physical quantities: deflection, normal slope, moment and shear, respectively, on the boundary have been used [9,14,15]. A more direct approach was employed [15,18] to solve directly for the required physical values on the boundary.

Usually the BEM formulation leads to a pair of coupled Fredholm type singular integral equations derived from the Rayliegh-Green identity for the biharmonic equation [16,19]. These two equations are usually the integral representation along the boundary of the deflection and its normal derivative. The second involves the derivatives of some singular kernels and their computation can be difficult [8]. An alternative second equation has been proposed [8] based upon properties of geometric similarities.

Certain difficulties are associated with plate problems. In case of polygonal boundaries, complications appear at the corners owing to their infinite curvature. Bending stresses become unbounded in the neighbourhood of a junction of two edges including an obtuse angle unless both edges are clamped or free. At free boundaries conditions containing higher derivatives of the displacement are involved which are not well adpated either to theoretical or numerical analysis [9,20]. Computation of moments and shears at the boundaries requires second and third differentiation of the deflection. In case of the indirect method this requires the differentiation of layer potentials at a source point on the boundary. These difficulties make an independent treatment of plate problems necessary and have limited the analysis in some early studies to clamped and/or simply supported edge conditions [7,13,16,17]. Nevertheless, plates with arbitrary shapes, loadings and edge conditions have been tackled [18, 20].

Various methods have been used to handle such situations: rounding the corners as a first approximation [10]; using double nodes at corners with zero length elements in between [16]; and using a multiple node concept with auxiliary relationships [1,5]. Other alternatives include: defining an integration contour which differs from the actual plate boundary to avoid numerical difficulties in case of free boundaries [20]; and requiring that the deflection and their derivatives up to the third order should be continuous and bounded [15]. Shu and Mukherjee [18] obtained three equations at each boundary node through the use of appropriate fundamental solutions for each node on the free edges, and used three unknowns ($w, \frac{\partial w}{\partial n}$ and $\frac{\partial w}{\partial t}$) requiring three independent boundary integral equations; the usual singular solution due to a point load plus two particular solutions corresponding to concentrated moments (tangential and normal).

Other difficulties include the inhomogeneous term in the governing differential equation. This term might give rise to a domain (area) integral which seems to spoil the inherent merits of the BEM as a boundary (line) discretization method. However, for some simple load cases using particular integrals can transform the domain integral to a boundary integral. Even if the domain integral is to be evaluated explicitly (still numerically) requiring the use of internal cells (elements), i.e., domain discretization, the topology of the internal elements is simpler than that of the FEM. Internal cells in the BEM do not add additional equations or unknowns to the overall system of equations and thus the number of equations depends only on the boundary discretization and not on the internal nodes (at least for elastic analysis). Besides, this internal discretization is done in BEM programs with the minimum user interference, thus the merits of the BEM as a boundary dicretization method are still preserved.

Furthermore, other difficulties include the "boundary layer effects". Since all approximations are confined to the boundary, as an interior point is too close to the boundary there would be rapid degeneration in the numerical accuracy of the stress and strain components as one samples quantities at an internal point very close to the boundary. Curvatures could be evaluated at an inside point analytically by careful analytical differentiation of the BE equations under the integral sign [16]. As a rule of thumb [5] degradation of accuracy may occur at distances less than one element length from the boundary.

DIRECT BEM FORMULATION [9,16,19]

Since the direct BEM formulation deals with quantities of physical significance, such formulation will only be mentioned here.

Fundamental Solution

The fundamental solution is that for the displacement at any point in a plate of infinite extent produced by a unit load acting at a point [1]. For an infinitely extended plate:

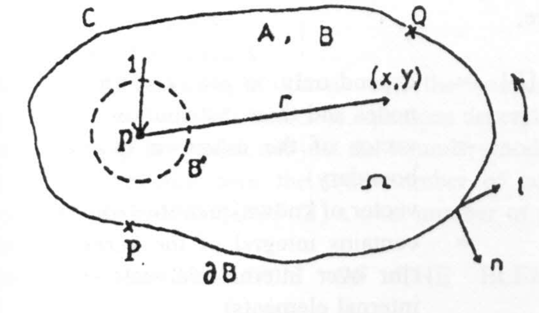


Figure 1. Fundamental solution.

$$w^* (x,y) = v = r^2 \ln r \tag{2}$$

where

$$D\nabla^4 w^*(x,y) = \delta\ (x - x_i)\ (y - y_i), \quad \text{Dirac Delta function}$$

$$r = \sqrt{(x-x_i)^2 + (y-y_i)^2}$$

using Green's second identity

$$\int_B (u\nabla^2 v - v\nabla^2 u)\, dA = \int_{\partial B} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds$$

let

$$u = \nabla^2 \phi$$

$$\nabla^2\nabla^2\phi = \phi^2 u = g = \frac{q}{D}(x,y)$$

$$\int_B (\nabla^2\phi\nabla^2 v - v g)\, dA = \int_{\partial B} [\nabla^2\phi\frac{\partial v}{\partial n} - v\frac{\partial}{\partial n}\nabla^2\phi]\, ds$$

interchange ϕ and v

$$\int_B (\phi\nabla^4 v - v g)\, dA = \int_{\partial B} [\phi\frac{\partial}{\partial n}(\nabla^2 v) - \frac{\partial\phi}{\partial n}\nabla^2 v - v\frac{\partial}{\partial n}\nabla^2\phi + \frac{\partial v}{\partial n}\nabla^2\phi]\, ds$$

$$\because\ v = r^2 \ln r$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

$$\nabla^2 r^2 \ln r = 4(1 + \ln r)$$

$$\nabla^4(r^2\ln r) = 4\nabla^2(\ln r) = 4(2\pi)\delta\ (p,q)$$

$$\nabla^4 v = \nabla^4(r^2 \ln r) = 8\pi\ \delta\ (p,q)$$

$$\therefore\ \alpha\ \pi\ \phi\ (p) = \int_B (r^2\ln r)\, g\, dA_q +$$

$$\int_{\partial B} [\frac{\partial}{\partial n}(\nabla^2 r^2 \ln r)\phi_Q - \nabla^2(r^2\ln r_p)\frac{\partial\phi_Q}{\partial n} + \frac{\partial}{\partial n}(r^2\ln r)\nabla^2\phi_Q - (r^2\ln r)\frac{\partial}{\partial n}\nabla^2\phi_Q]\, ds \tag{3}$$

in addition to another equation due to Poisson's equation. Finally for $\nabla^4 w = \dfrac{q(x,y)}{D} = g$

$$\alpha\pi\nabla^2 w(p) - \int_B (\ln r)\, g\, dA_q = \int_{\partial B} [\frac{\partial}{\partial n}(\ln r)\nabla^2 w_Q - \ln r\frac{\partial}{\partial n}\nabla^2 w_Q]\, ds \tag{4}$$

and

$$4\alpha\pi\, w(p) - \int_B (r^2\ln r)\, g\, dA_q = \int_{\partial B} [\frac{\partial}{\partial n}\{\nabla^2(r^2\ln r)\}\, w_Q - \nabla^2(r^2\ln r)\frac{\partial w_Q}{\partial n}$$

$$+ \frac{\partial}{\partial n}(r^2\ln r)\nabla^2 w_Q - r^2\ln r\frac{\partial}{\partial n}\nabla^2 w_Q]\, ds \tag{5}$$

n is the normal to $\partial B(C)$ the boundary at the field point

$\alpha = 1$, source point is on the boundary C where it is locally smooth,

$\alpha = 2$, source point inside the plate.

Discretized Equations

Assume the boundary $\partial B(C)$ of the plate to be divided into N_s straight boundary elements, N_b ($N_b = N_s$) boundary nodes and the interior A of the plate divided into n_i internal elements (say triangular cells). Therefore, there are ($2^* N_b$) unknown nodal boundary values since two boundary conditions out of the four quantities;

$w, \frac{\partial w}{\partial n}, \nabla^2 w$ and $\frac{\partial}{\partial n} \nabla^2 w$ are specified at each boundary. These ($2^* N_b$) unknown nodal boundary values can be calculated from two equations (equations (4) and (5)) written for each boundary node, i.e., $2^* N_b$ equations. Assuming that the previous quantities are <u>constants</u> [9] over each boundary element then:

$$4\pi w_i = \sum_{k=1}^{n_i} (r^2 \ln r)_{ik} (gA)_k + \sum_{j=1}^{N_b} \left(c_{ij} w_j + d_{ij} \frac{\partial w_j}{\partial n} + e_{ij} \nabla^2 w_j + f_{ij} \frac{\partial}{\partial n} \nabla^2 w_j\right) \quad (6)$$

and

$$\pi \nabla^2 w_i = \sum_{k=1}^{n_i} (\ln r)_{ik} (gA)_k + \sum_{j=1}^{N_b} \left(a_{ij} \nabla^2 w_j + b_{ij} \frac{\partial}{\partial n} \nabla^2 w_j\right) \quad (7)$$

$i = 1,2,\ldots N_b$

If the previous quantities are assumed to vary <u>linearly</u> [16] over each boundary element then:

$$\pi \nabla^2 w_i = \sum_{n_i} \int_{\Delta A_i} \ln r \, g \, dA_k + \sum_{N_s} \int_{\Delta B_i} \left\{\frac{\partial}{\partial n} \ln r \nabla^2 w_j - \ln r \frac{\partial}{\partial n} \nabla^2 w_j\right\} ds \quad (8)$$

and a similar discretized version of equation (6). For the discretization scheme and boundary conditions see Figures (2) and (3).

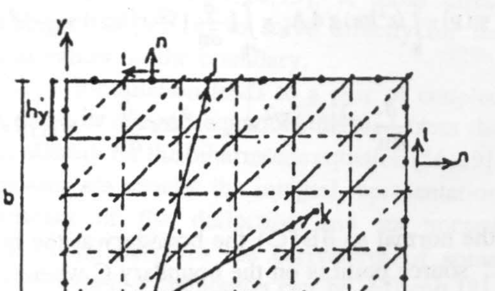
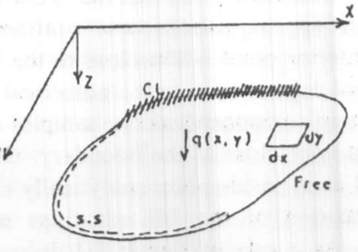


Figure 2. Plate discretization scheme.



Boundary Conditions [20]

Clamped edge: $w = 0, \frac{\partial w}{\partial n} = 0, \frac{\partial w}{\partial x} n_x + \frac{\partial w}{\partial y} n_y = 0$

Simply Supported: $w = 0, M_n = 0, \frac{\partial^2 w}{\partial x^2} n_x^2 + 2 \frac{\partial^2 w}{\partial x \partial y} n_x n_y + \frac{\partial^2 w}{\partial y^2} n_y^2 = 0$

n_x, n_y are components of the outer unit normal to the plate boundary.

For free edges, see references [9,20].

Figure 3. Plate boundary conditions.

The Final Form of The Discretized Equations

$$[A] \{x\} = [B] \{y\} + \{d\} \quad ($$

where,

[A], [B]	=	depend only on geometry (no. of bounda nodes and their distribution)
{x}	=	vector of the unknown quantities (on t boundary)
{y}	=	vector of known quantities (on the boundar
{d}	=	contains integral of the kernels r^2 lnr a rlnr over internal elements (say triangul internal elements)

These fully populated unsymmetrical set of simultaneo equations ($2^* N_b$ equations) are solved to obtain t unknowns on the boundary, i.e., 2 unknowns of any of $\frac{\partial w}{\partial n}, \nabla^2 w$ and $\frac{\partial}{\partial n} \nabla^2 w$ at each boundary nodal point. On this has been done the interior solution at any or every po could be easily found by writing the BE equation(s) for t required internal point(s).

BEM Computer Program

The BE numerical implementation of the previo formulation was done utilizing a BE plate program original developed by Morjaria and Mukherjee [16,17]. Line variation of $w, \frac{\partial w}{\partial n}, \nabla^2 w$ and $\frac{\partial}{\partial n} \nabla^2 w$ over each straig boundary element is assumed. Curvatures are obtained by analytical differentiation under the integral sign of the discretized appropriate BE equation. Special attention was paid to boundary curvatures to avoid the boundary layer effect.

THE DISPLACEMENT FEM FORMULATION

The state of deformation of the plate is assumed to be described entirely by one quantity; the lateral displacement of the middle plane. "C_0" continuous, non conforming rectangular elements as used by Melosh and Zienkiewicz [21] are used here. The individual stiffness matrix for each element is 12x12, i.e., 4 nodes of the rectangular element multiplied by 3 degrees of freedom (DOF) at each node, i.e., the transverse displacement of the plate and its derivatives in both cartesian coordinates.

Formulation of the Element Stiffness Matrix

The element stiffness matrix is easily formulated [21] as:

$$[K^e] = \int_V [B^T]\,[D]\,[B]\ \text{dvolume} \qquad (10)$$

where,

$[K^e]$ = element stiffness matrix
$[B]$ = strain matrix
$[D]$ = elasticity matrix

In the FEM the whole domain and the boundary are discretized thus the total number of equations depends on the total number of the nodes not just the boundary nodes as in the BEM. Typically here the total number of equations generated equals no. of DOF (3) x total number of nodes.

STRUCTURAL ANALYSIS OF THE ELEMENTS ASSEMBLAGE

The solution proceeds following the basic operations of the displacement matrix method. The individual element stiffness matrices contributing to each nodal point are superimposed to obtain the overall stiffness matrix of the whole plate. This is basically writing the equilibrium conditions at each node, and can be symbolically expressed as:

$$[K] = \sum_{i=1}^{n} [K_i^e] \qquad (11)$$

where,

$[K]$ = overalL global stiffness matrix of the whole plate structure
n = no. of nodes.

The equilibrium equation between the applied nodal forces [P], and the resulting nodal displacements, $\{\Delta\}$, can be expressed as:

$$[P] = [K]\,\{\Delta\} \qquad (12)$$

Finally the element stresses are determined using the stress matrix.

DIFFERENCES BETWEEN THE BEM AND THE FEM

- The BEM identically satisfies the governing differential equation and approximately satisfies the boundary conditions, thus the solution variables will vary continuously throughout the region and all approximation of geometry...etc., will occur at the outer boundaries. In the displacement FEM the compatability condition within elements and the displacement (geometric) boundary conditions are exactly satisfied while the equilibrium equations and boundary conditions on stresses are satisfied only approximately, and the stresses are discontinuous at nodal points.
- The FEM has constraints on the discretization process (interelement continuity) and its minimum requirements [21]. While in the BEM there are no constraints on interelement continuity or discretization process, and this might explain why constant elements work in the BEM.
- In the BEM more arithmatic calculations are involved in constructing each element or row of the overall matrix, but the overall system of equations are smaller than those of the FEM. In the BEM the final matrix while smaller is unsymmetrical and fully populated due to the coupling of all points and contributions from all boundary segments. In the FEM mainly the final matrix is larger but symmetrical and banded (sparse) as only the elements are related to their neighbours rather than to all other nodes, i.e., only coupling between adjacent neighbouring nodes.
- In the BEM the solution variables vary continuously over the domain (e.g., stresses are smooth) and can be obtained accurately at any particular interior point, after solving for the boundary unknowns, but does not require that the whole interior domain be solved. In the FEM stresses are (mainly) discontinuous at nodal points and schemes for averaging and smoothing them are required and the best values of stresses are obtained at integration points (3x3 scheme used here). In the FEM the whole problem should be solved before the stresses are obtained at a particular point.
- The BEM computational effort increases proportionately with the number of internal points at which the solution is required. Thus if the solution is required on the boundary and at only few internal points, the BEM for the same level of solution accuracy requires considerably

less computational effort than the FEM. If however the solution is required throughout the domain of the body, the FEM may be faster than the BEM.

NUMERICAL COMPARISON BETWEEN THE BEM AND FEM

The BEM and FEM solution accuracies for thin plate bending under transverse loading are compared as a function of mesh refinement, different boundary conditions and aspect ratios. The comparison is confined here to rectangular plates, most of which the exact or analytical solution is available, to make the comparison more meaningful. It is worth recalling that for the employed BEM the quantities w, $\frac{\partial w}{\partial n}$, $\nabla^2 w$ and $\frac{\partial}{\partial n}\nabla^2 w$ are assumed to vary linearly over boundary elements. For the employed FEM, w is cubic, slopes are quadratic and the strain within each element is linear.

Cases Studied

- Uniformly loaded simply supported and clamped square plates with several discretizations.
- Corner supported uniformly loaded square plates.
- Uniformly loaded square plates with a variety of boundary conditions.
- Uniformly loaded rectangular plates with several aspect ratios.

Unless otherwise mentioned the BEM used had 36 nodes on the boundary and 81 internal nodes. Symmetry conditions were not taken advantage of in both methods and the solution was carried out for the entire plate. For each case full comparison between both FEM and BEM regarding transverse deflection and bending moments at all points on the boundary and interior was made, however, only results at significant points and those of pronounced differences are reported hereafter.

DISCUSSION OF RESULTS

Numerical results and comparisons for both methods for the studied cases are shown in the given tables and figures.

- Table (1) for a uniformly loaded square plate reveals that for approximately the same level of discretization the BEM results are more accurate than those of the FEM. For example the BEM with 36 nodes and 81 internal points rendered better results for central deflection and bending moment at the centre and mid sides than the FEM with 8 x 8 mesh. For the same desired level of solution accuracy the BEM requires a coarser mesh than the FEM.
- For plates with different boundary conditions Table (2), the slight discrepancy of the BEM from the exact answer is less than that of the FEM, and the BEM yields obviously better results than the FEM especially for moments, in almost all the studied cases.
- In almost all cases the agreement between both methods is excellent and the results are within 5% of the exact answers, Tables (1 & 2), confirming the accuracy of the procedures used.
- For a clamped plate uniformly loaded with aspect ratios, a/b=2 and 4, higher discrepancies from the exact solution are obtained especially for the FEM case. This is expected as changing the FE aspect ratios beyond certain limits can affect the accuracy (same number of elements is used in each direction). While the BEM seems not to suffer and gives better results than the FEM, especially for bending moments. The situation would be reversed however for rectangular regions with high aspect ratios, since for the BEM in such cases substructuring of the region under consideration into several subregions would be inevitable to render good results, but would complicate the problem.
- When both the BEM and FEM results are compared to BEM results of reference [11], Figure (4), the boundary layer affected the results of the latter, at points located at distances less than the boundary segment length from the boundary. This confirms the need for special boundary curvature treatment and special extrapolation techniques to overcome such effect or the use of shorter elements to discretize the boundary.
- For a uniformly loaded simply supported square plate whose series solution is available, the "global error criteria" [22], for both methods are compared.

  Global error criterion for deflection,

$$\epsilon_w = \frac{\sum_{Nipt} [w_{prog} - w_{series}]}{N_{ipt}}$$

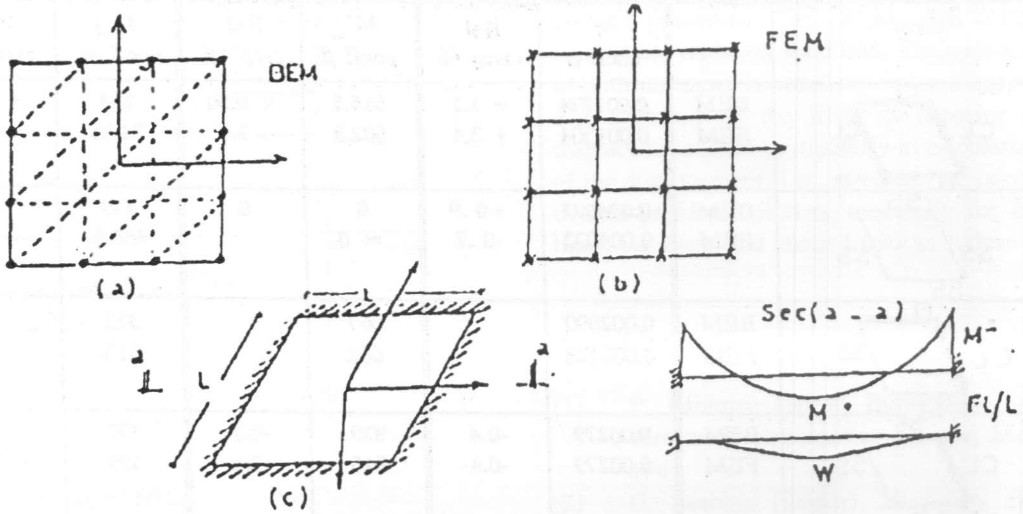
Global error criterion for bending moment,

$$\epsilon_M = \frac{\sum_{Nipt} [M_{program} - M_{series}]}{N_{ipt}}$$

where, N_{ipt} = no. of internal points in the BEM used. The BEM and FEM internal points were chosen to have the same locations so that the results can be compared at similar points. For the BEM (36 boundary nodes, 81 internal points) and the FEM (8x8 mesh)

Global error Criterion	ϵ_w	ϵ_M
BEM	0.9345×10^{-5}	0.01551×10^{-2}
FEM	0.9479×10^{-5}	0.11660×10^{-2}

Table 1. Uniformly distr. Load on a square clamped plate.



FEM (lumped lds.)

Table (1-a).

	Mesh	Total # of Nodes	w_{max} coeff α x10^2	w Rel. error %	M^-_{max} coeff β	M^- Rel. error %	M^+_{max} coeff γ	M^+ Rel.error %
1	2 x 2	9	0.1479	+ 17.0	355	-30.8	461.8	+ 99
2	4 x 4	25	0.1404	+ 11.4	476	-7.2	277.8	+ 20.2
3	6 x 6	49	0.13222	+ 5.7	495.5	-3.4	249.5	+ 8.0
4	8 x 8	81	0.1303	+ 3.4	502.8	-2.0	240.5	+ 4.1
5	12 x 12	169	0.1283	+ 1.8				
6	16 x 16	289	0.1275	+ 1.2				

BEM

Table (1-b).

Total no. of Bndry nodes	NIPT	coeff α	w Rel. error %	coeff β	M^- Rel. error %	M^+ coeff γ	M^+ Rel. error %
4 x 5 = 20	25	0.001291	+ 2.5	530.4	+ 3.4	233.8	+ 1.2
4 x 9 = 36	81	0.001274	+ 1.1	515.5	+ 0.5	230.1	-0.4

q = uniformly dist. load, NIPT = # of internal pnts. $w_{max} = \frac{\alpha q L^4}{D}, D = \frac{EI}{(1-\nu^2)}$, $M^-_x = \beta \frac{ql^2}{10^4}$, $M^+_x = \gamma \frac{ql^2}{10^4}$

Relative error % = $\frac{\text{solut} - \text{exact}}{\text{Exact}}$ % Exact solution: $\alpha = 0.001260$ [7], $\beta = 513$, $\gamma = 231$ [21]

Table 2 Uniformly loaded Plates.

a/b	Case		w coff α	Rel. error %	M^-_{x1} coff β	Rel. error %	M_{x2} coff. γ	Rel. error %
	CL CL CL CL	BEM FEM	0.001274 0.001304	+ 1.1 + 3.4	515.5 502.8	+ 0.50 - 2.0	234 240.5	+ 1.2 + 4.1
	SS SS SS SS	BEM FEM	0.004097 0.004033	+0 .9 -0 .7	0 ≃ 0	0	479 480.5	0 + 0.4
1	CL CL SS SS	BEM FEM	0.002090 0.002128		667 662		313 313	
	SS CL SS SS	BEM FEM	0.00279 0.00279	-0.4 -0.4	839 816	-0.1 -2.5	392 399	+ 0.5 + 2.3
	CL CL CL SS	BEM FEM	0.001581 0.00160	+ 0.7 + 1.9	599 589	-0.3 -2.0	279 292	
2	CL CL CL CL	BEM FEM	0.002565 0.002565	+ 1.0 + 1.0	575 490	+0.7 - 14.2	159.4 159.3	+ 0.9 + 0.9
	SS SS SS SS	BEM FEM	0.01006 0.01010	-0.9 -0.5	0 ≃ 0	0	465.5 465	+0.3 +0.2
4	CL CL CL CL	BEM FEM	0.00266 0.00260	+ 5.5 + 3.2	580 333	+ 1.6 - 41.7	125 127.5	0 + 2.0
	SS SS SS SS	BEM FEM	0.01290 0.01278	-0.2 -1.1	0 ≃0	0	381 381.3	-0.8 -0.8

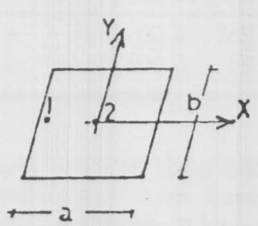
Exact solut Ref. [7], w, M_{x2} at pnt 2, M^-_{x1} at pnt 1.

FEM mesh 8x8, (lumped loads), whole plate no symmetry utilized.

BEM 9 nodes on each side, no. of internal points = 81

$$M^-_{x1} = -\beta\, q\, l^2/10^4,\; M_{x2} = \gamma q l^2/10^4,\; w = \alpha q \frac{l^4}{D},\; D = \frac{Et^3}{12(1-\nu^2)}$$

$$\text{relative error } \% = \frac{\text{solut} - \text{exact}}{\text{exact}}\ \%,\ l = b$$



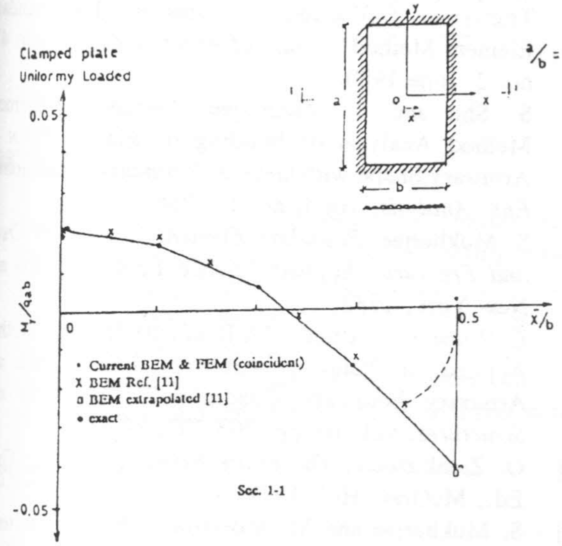


Figure 4. Boundary layer effect.

This reveals that the BEM although slightly better for calculating deflections is obviously superior to the FEM for calculating bending moments (stresses, derivatives of w). This may be an important issue especially for inelastic problems, as the failure criteria may be based on stress calculations, the accuracy of which at all plate points can influence the overall solution. Small discrepancies in the solution at points of interest may be within the acceptable accuracy, nevertheless, the analyst may be uncertain as to the total effect of such discrepancies on the global overall problem.

- In terms of assessing the efficiency of both methods a trade off between solution accuracy, computational aspects, input/output, and human effort should be considered for a particular problem. This impairs any generalized statement. In fact for large, infinite domains the BEM would be better. While the FEM is sometimes better for finite size bodies with fine details, plates with non-homogeneous material properties, multilayered plates, shells (combination of flexure and membrane actions), thick plates and narrow thin strips. Indeed the two methods might well be combined to advantage in suitable situations.

CONCLUSIONS

The BEM is an accurate and efficient numerical technique for the analysis of thin plate bending. It leads to reduced dimensionality of the problem, since only the discretization of the boundary is needed, resulting in a reduction of the system of equations and smaller input data required for computation. Even when domain integrals are to be evaluated using internal cells (FE technique) they do not add to the total number of equations and the merits of the BEM are still preserved. The BEM often complements, and in some cases replaces, the FEM. The appropriate combination of both strategies is beneficial in handling suitable situations. The accuracy of the BEM is superior than that of the displacement FEM particularly in calculating the derivatives of the displacement (i.e., stresses), an important issue to the overall problem solution, especially for inelastic problems based on stress and strain limits as failure criteria.

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