STATE SPACE APPROACH TO MAGNETOHYDRODYNAMIC UNSTEADY FREE CONVECTION FLOW

Magdy A. Ezzat
Department of Mathematics Faculty of Science
University of Alexandria, Alexandria, Egypt

ABSTRACT

The equations of magnetohydrodynamic unsteady free convection boundary layer flow past an infinite vertical plate for one-dimensional problems are cast into matrix form using the state space and Laplace-transform techniques. The results obtained can be used to generate solutions in Laplace-transform domain to a broad class of problems in magnetohydrodynamic free convection flow. The technique is applied to a heated vertical plate problem and to a problem pertaining a plate under uniform heating. The inversion of the Laplace transforms is carried out using a numerical approach. Numerical results for the temperature, velocity and skin friction distributions are given and illustrated graphically for both problems.

INTRODUCTION

The unsteady laminar free convection flow has been studied by many authors. Nanda and Sharma [1] studied free convection laminar boundary layer flows from a vertical flat plate when its temperature oscillates in time about a constant non-zero mean.

Soundalgekar, Patel and Pop [2] have solved magnetohydrodynamic unsteady free convection one-dimensional flow past an infinite oscillating vertical plate.

Soundalgekar [3] and Megahed [4] have solved few problems on unsteady free convection flow in the presence of a magnetic field. Mishra and Mohapatra [5] have solved magnetohydrodynamic unsteady free convection flow past a vertical porous plate with variable suction.

In most of these attempts the unknown quantities (velocity components and temperature) were assumed to be oscillating functions. This assumption facilitates the solution of the problem rendering it to an ordinary differential equation without time.

In the present paper the state space approach is developed for magnetohydrodynamic unsteady free convection past an infinite vertical plate. This approach enables one to use the methodology of modern control theory in dealing with problems in fluid dynamics. This method was developed Bahar and Hetnarski [6] to deal with coupled thermoelasticity problems. The solutions obtained in a closed form in the Laplace domain. The inversion of the Laplace transform is carried out using a numerical technique [7]. The resulting formulation is applied to two different problems to show how this approach is applied to concrete problems. The first deals with a heated vertical plate problem, while the second deals with a problem pertaining a plate under uniform heating. Numerical results are presented.

FORMULATION OF THE PROBLEM

Let us consider the unsteady one-dimensional flow of a fluid of density ρ', viscosity μ, and electrical conductivity σ occupying the region $y' \geq 0$, where y'_ axis is taken perpendicular to the infinite, vertical plate. A magnetic field of uniform strength is applied transversely to the direction of the flow (to the plate). We assume that the magnetic Reynolds number of the flow is small enough so that the induced magnetic field can be neglected. The influence of the density variations with temperature is considered only in the body force term. In the energy equation, terms representing viscous and Jole dissipation are neglected as they are assumed to be very small in free convection flows. We note that since the plate is infinite in extent conditions depend upon y and the time t only, and that the velocity vector has components {u(y,t),0,0}. With this assumptions the boundary layer equations that govern the unsteady one-dimensional, free convection, flow through a viscous incompressible fluid bounded by an infinite, vertical, plate in the presence of a magnetic field [2] are

$$\rho' \frac{\partial u'}{\partial t'} = \rho' g \beta^* (T' - T'_\infty) + \mu \frac{\partial^2 u'}{\partial y'^2} - \sigma B_o^2 u' , \quad (1)$$

$$\rho' C' \frac{\partial T'}{\partial t'} = \lambda' \frac{\partial T'^2}{\partial y'^2} \quad . \quad (2)$$

In the above equations t' is the time variable, B_o is the applied magnetic field strength, g is the acceleration due to gravity, β^* is the coefficient of volume expansion, λ' is the thermal conductivity, C' is the specific heat at constant pressure, T' is the temperature in the boundary layer, and T'_∞ is the temperature far away from the plate.

Let us introduce the following non-dimensional variables

$$y = \frac{y' u_o}{\nu^*}, t = \frac{u_o^2 t'}{\nu^*}, u = \frac{u'}{u_o}, \theta = \frac{T' - T'_\infty}{T_o' - T'_\infty},$$

$$P = \frac{\mu C'}{\lambda'} \text{ (The Prandtl number) },$$

$$G = \frac{\nu^* g \beta^* (T'_o - T'_\infty)}{u_o^3} \text{ (The Grashoff number)}, \quad (3)$$

$$M = \frac{\sigma B_0^2 \nu^*}{\rho' u_o^2} \text{ (The Magnetic number)},$$

where T'_o is the temperature at the plate.

In view of transformation (3), (1) and (2) yield:

$$\left(\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t} - M \right) u = - G \theta \quad , \quad (4)$$

$$\left(\frac{\partial^2}{\partial y^2} - P \frac{\partial}{\partial t} \right) \theta = 0 \quad . \quad (5)$$

To simplify the algebra, only problems with zero initial conditions are considered.

Taking the Laplace transform with parameter s (denoted by a bar) of both sides of equations (4) and (5), we arrive at

$$\left(\frac{\partial^2}{\partial y^2} - s - M \right) \bar{u} = - G \bar{\theta} \quad ,$$

$$\left(\frac{\partial^2}{\partial y^2} - P s \right) \bar{\theta} = 0 \quad .$$

We shall choose as state variables the temperature increment θ, the velocity component u and their gradients. Equations (6) and (7) can be written as

$$\frac{\partial \bar{\theta}}{\partial y} = \bar{\theta}' \quad , \quad (8)$$

$$\frac{\partial \bar{u}}{\partial y} = \bar{u}' \quad , \quad (9)$$

$$\frac{\partial \bar{\theta}'}{\partial y} = P s \bar{\theta} \quad , \quad (10)$$

$$\frac{\partial \bar{u}'}{\partial y} = (s + M) \bar{u} - G \bar{\theta} \quad . \quad (11)$$

The above equations can be written in matrix form as

$$\frac{d \bar{v}(y,s)}{d y} = A(s) \bar{v}(y,s) \quad , \quad (12)$$

where

$$\bar{v}(y,s) = \begin{bmatrix} \bar{\theta}(y,s) \\ \bar{u}(y,s) \\ \bar{\theta}'(y,s) \\ \bar{u}'(y,s) \end{bmatrix} ,$$

and

$$A(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ Ps & 0 & 0 & 0 \\ -G & s + M & 0 & 0 \end{bmatrix}$$

Equation (12) can be integrated by means of the matrix exponential to yield

$$\bar{v}(y,s) = \exp[A(s).y]\ \bar{v}(0,s) \quad , \quad (13)$$

following the development given in [8].

To determine the matrix exponential explicitly, it is observed that its Taylor-series expansion terminates with the term containing the cube of the matrix A on account of the Cayley-Hamilton theorem. This is due to the fact that the characteristic equation corresponding to the matrix A in equation (12) is given by

$$k^4 - k^2 (M + s + P s) + P s (M+s) = 0, \quad (14)$$

where k is a characteristic root. The Cayley-Hamilton theorem states that the matrix A satisfies its own characteristic equation in the matrix sense. Therefore, it follows that

$$A^4 - A^2 (M + s + P s) + P s (M+s) I = 0, (15)$$

Equation (15) shows that A^4 and all higher powers of A can be expressed in terms of A^3, A^2, A and I, the unit matrix of order 4. The matrix exponential can now be written in the form

$$\exp[A y] = a_o(y,s) I + a_1(y,s) A(s) + a_2(y,s) A^2(s) + a_3(y,s) A^3(s) \quad (16)$$

The scalar coefficients of equation (16) are now evaluated by replacing the matrix A by its characteristic roots $\pm k_1$ and $\pm k_2$, which are the roots of the biquadratic equation (14), satisfy the relations

$$k_1^2 + k_2^2 = Ps + M + s, \quad (17a)$$

$$k_1^2\, k_2^2 = Ps(M+s). \quad (17b)$$

This leads to the system of equations

$$\exp(k_1.y) = a_0 + a_1 k_1 + a_2 k_1^2 + a_3 k_1^3 ,$$

$$\exp(-k_1.y) = a_0 - a_1 k_1 + a_2 k_1^2 - a_3 k_1^3 ,$$

$$\exp(k_2.y) = a_0 + a_1 k_2 + a_2 k_2^2 + a_3 k_2^3 ,$$

$$\exp(-k_2.y) = a_0 - a_1 k_2 + a_2 k_2^2 - a_3 k_2^3 .$$

The solution of the above system is given by

$$a_o = \frac{k_1^2 \cosh(k_2 y) - k_2^2 \cosh(k_1 y)}{k_1^2 - k_2^2},$$

$$a_1 = \frac{(k_1^2/k_2)\sinh(k_2 y) - (k_2^2/k_1)\sinh(k_1 y)}{k_1^2 - k_2^2},$$

$$a_2 = \frac{\cosh(k_1 y) - \cosh(k_2 y)}{k_1^2 - k_2^2}, \quad (18)$$

$$a_3 = \frac{k_2 \sinh(k_1 y) - k_1 \sinh(k_2 y)}{k_1 k_2 (k_1^2 - k_2^2)}.$$

Substituting the expressions (18) into equation (16) and computing A^2, and A^3, we obtain after some lengthy algebraic manipulations.

$$\exp[A(s).y] = L(y,s) = [L_{ij}(y,s)], i,j = 1,2,3,4. \quad (19)$$

where the elements $L_{ij}(y,s)$ are given by

$$L_{11} = L_{13} = \cosh k_1 y \quad ,$$

$$L_{12} = L_{14} = L_{32} = L_{34} = 0 \ , \quad (20)$$

$$L_{13} = \frac{\sinh k_1 y}{k_1} \quad ,$$

$$L_{21} = L_{43} = \frac{G\,(\cosh k_2 y - \cosh k_1 y)}{k_1^2 - k_2^2} \quad ,$$

$$L_{41} = \frac{G\,(k_2 \cosh k_2 y - k_1 \cosh k_1 y)}{k_1^2 - k_2^2} \quad ,$$

$$l_{22} = l_{44} = \cosh k_2 y \quad ,$$

$$L_{23} = \frac{G\,(k_1 \sinh k_2 y - k_2 \sinh k_1 y)}{k_1 k_2 (k_1^2 - k_2^2)} \quad \text{(20)continue,}$$

$$L_{24} = \frac{\sinh k_2 y}{k_2} \quad ,$$

$$L_{31} = k_1 \sinh k_1 y \quad ,$$

$$L_{42} = k_2 \sinh k_2 y \quad ,$$

It should be noted here that we have used equations (17a) and (17b) repeatedly in order to write these entries in the simplest possible form. It should also be noted that this is a formal expression for the matrix exponential. In the physical problem $\infty \geq y \geq 0$, we should suppress the positive exponentials which are unbounded at infinity. Thus we should replace each sinh(ky) by - ½ exp(-ky) and each cosh(ky) by ½ exp(-ky).

We return now to system (12) whose formal solution can be written in the form

$$\bar{v}(y,s) = L(y,s)\, \bar{v}(0,s) \quad . \tag{21}$$

APPLICATIONS

(I) A HEATED PLATE PROBLEM

We shall consider the free convection flow of an electrically conducting, non magnetic fluid occupying a semi-infinite region $y \geq 0$ of the space bounded by an infinite vertical plate $(y=0)$, with the condition

$$u(0,t) = 0 \quad . \tag{22}$$

We assume that temperature of the form

$$\theta(0,t) = \theta_o H(t) \quad , \tag{23}$$

is applied to the plate at time t=o, where θ_o is a constant and H(t) is Heaviside unit step function. All initial conditions are assumed to be zero.

We now apply the state space approach described above to the same problem. The two components of the transformed initial state vector $\bar{v}(0,s)$ are known, namely,

$$\bar{\theta}(0,s) = \frac{\theta_o}{s} \quad , \tag{24}$$

$$\bar{u}(0,s) = 0 \quad . \tag{25}$$

In order to obtain the remaining two components $\bar{u}'(0,s)$ and $\bar{\theta}'(0,s)$, we substitute y=0 into equations (21) and (20). We obtain a system of two simultaneous linear equations whose solution is

$$\bar{u}'(0,s) = \frac{\theta_o G}{s(k_1 + k_2)} \quad ,$$

$$\bar{\theta}'(0,s) = -\frac{\theta_o (Ps + k_1 k_2)}{s(k_1 + K_2)} \quad .$$

Equations (17) were used again to simplify the forms (26) and (27). Finally, we substitute from (24)-(27) into (21) to arrive at

$$\bar{\theta}(y,s) = \frac{\theta_o \exp(-k_1 y)}{s} \quad , \tag{28}$$

$$\bar{u}(y,s) = \frac{-\theta_o G\,[\exp(-k_1 y) - \exp(-k_2 y)]}{s(k_1^2 - k_2^2)} \quad , \tag{29}$$

(II) PLATE UNDER UNIFORM HEATING

We consider magnetohydrodynamic free convection flow of an electrically conducting, non magnetic fluid past a heated infinite, vertical plate, at whose surface the heat flux is uniform q, with the conditions

$$\theta'(0,t) = -q H(t) \quad , \tag{30}$$

$$u(0,t) = 0 \quad . \tag{31}$$

The two components of the transformed initial state vector $\bar{v}(0,s)$ are known, namely,

$$\bar{\theta}'(0,s) = -\frac{q}{s} \quad , \tag{32}$$

$$\bar{u}(0,s) = 0 \quad . \tag{33}$$

To obtain the remaining two components $\bar{u}'(0,s)$ and $\bar{\theta}(0,s)$, we substitute $y=0$ on both sides of equations (21) getting a system of linear equations whose solution gives

$$\bar{u}'(0,s) = \frac{qG}{s(Ps + k_1 k_2)} \quad , \tag{34}$$

$$\bar{\theta}(0,s) = \frac{q(k_1 + k_2)}{s(Ps + k_1 k_2)} \quad . \tag{35}$$

Substituting from equations (32)-(35) into the right handside of equation (21) and performing the matrix multiplications, we finally obtain the solution of the problem in the Laplace transform domain as

$$\bar{\theta}(y,s) = \frac{q}{s k_1} \exp(-k_1 y) \quad , \tag{36}$$

$$\bar{u}(y,s) = -\frac{qG}{s(Ps + k_1 k_2)(k_1 - k_2)} \left[\exp(-k_1 y) - \exp(-k_2 y)\right] . \tag{37}$$

INVERSION OF THE LAPLACE TRANSFORM

In order to invert the Laplace transforms in the above equations we shall use a numerical technique based on Fourier expansions of functions.

Let g(s) be the laplace transform of a given function g(t). The inversion formula of Laplace transforms states that

$$g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st}\, \bar{g}(s)\, ds \quad ,$$

where c is an arbitrary positive constant greater than all the real parts of the singularities of g(s). Taking $s = c + i\,y$, we get

$$g(t) = \frac{e^{ct}}{2\pi} \int_{-\infty}^{\infty} e^{ity}\, \bar{g}(c + i\,y)\, dy \quad .$$

this integral can be approximated by

$$g(t) = \frac{e^{ct}}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikt\Delta y}\, \bar{g}(c + ik\Delta y)\Delta y \; .$$

Taking $\Delta y = \pi/t_1$, we obtain

$$g(t) = \frac{e^{ct}}{t_1}\left[\tfrac{1}{2}\bar{g}(c) + \mathrm{Re}\left[\sum_{k=1}^{\infty} e^{ik\pi t/t_1}\, \bar{g}(c + ik\pi/t_1) \right]\right]$$

For numerical purposes this is approximated by the function

$$g_N(t) = \frac{e^{ct}}{t_1}\left[\tfrac{1}{2}\bar{g}(c) + \mathrm{Re}\left(\sum_{k=1}^{N} e^{ik\pi t/t_1}\, \bar{g}(c + ik\pi/t_1) \right)\right], \tag{38}$$

where N is a sufficiently large integer chosen such that

$$\frac{e^{ct}}{t_1} \mathrm{Re}\left[e^{iN\pi t/t_1}\, \bar{g}(c + iN\pi/t_1) \right] < \epsilon \quad ,$$

and ϵ is a preselected small positive number that corresponds to the degree of accuracy to be achieved. Formula (38) is the numerical inversion formula valid for $2t_1 \geq t \geq 0$ [7]. In particular we choose $t = t_1$, getting

$$g_N(t) = \frac{e^{ct}}{t}\left[\tfrac{1}{2}\bar{g}(c) + \mathrm{Re}\left(\sum_{k=1}^{N} (-1)^k\, \bar{g}(c + ik\pi/t) \right)\right] . \tag{39}$$

CONCLUSIONS

In this paper the state space approach is adopted for the solution of one-dimensional problems of magnetohydrodynamic unsteady free convection flow past an infinite vertical plate. The technique is applied to a heated vertical plate problem and to a problem pertaining a plate under uniform heating. The inversion of the Laplace transforms is carried out using a numerical approach. The effects of the magnetic field and the Prandtl number on flow characteristics have been studied and are illustrated graphically. Figures (1-4) represent the first problem while Figures (5-7) represent the second one.

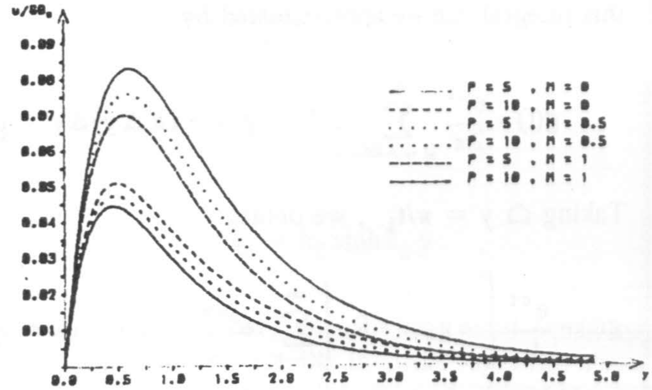

Figure 1. Velocity distribution.

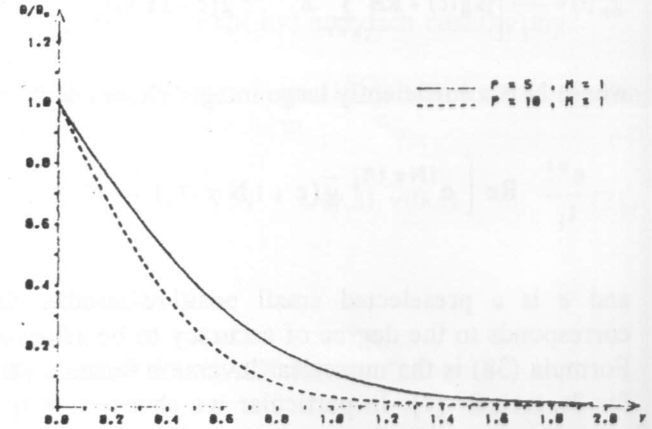

Figure 2. Temperature distribution.

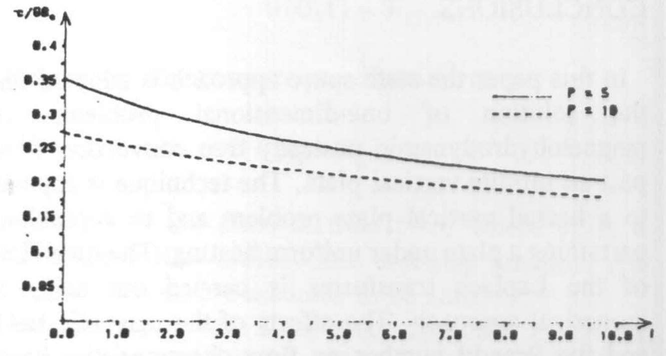

Figure 3. Skin friction against M.

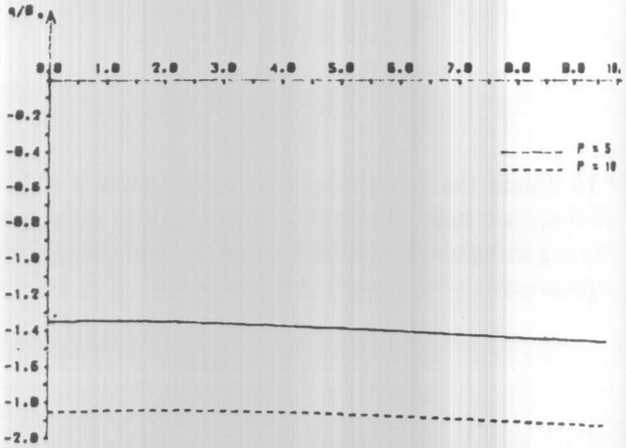

Figure 4. Heat flux against M.

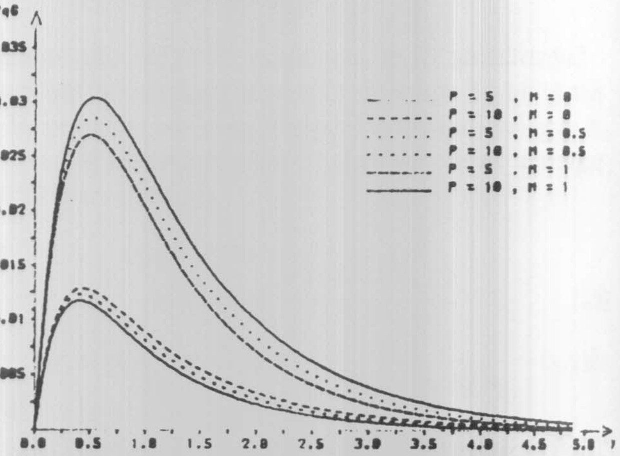

Figure 5. Velocity distribution.

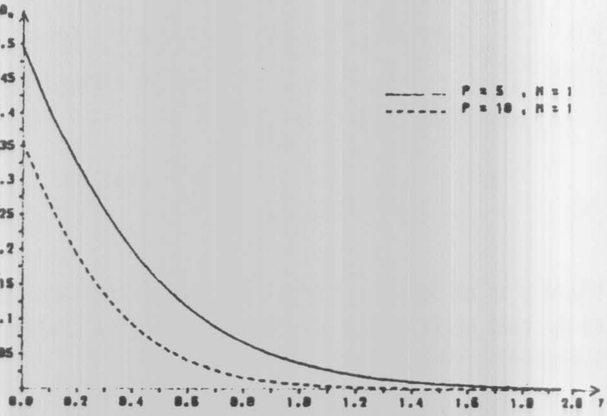

Figure 6. Temperature distribution.

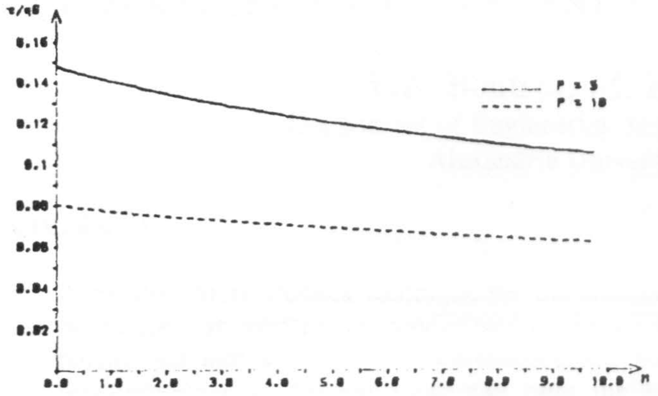

Figure 7. Skin friction against M.

Figures (1) and (5) show that the velocity at any point in the fluid decreases as M, the magnetic field strength, increases. A similar effect is observed as the Prandtl number, P, increases. We observe from Figures (2) and (6) that the temperature at any point decreases as P increases. It is to be noted from Figures (3) and (7) that the skin friction τ decreases as M increases and also as P. It is also seen from figure (4) that the heat flux decreases with an increase in P and the effect of increasing M is to decrease the heat flux slightly.

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