FINITE ELEMENT SIMULATION OF TURBULENT FLOW OVER A FENCE

A.A.A. Moustafa and M.A. El Attar
Department of Engineering Mathematics, Faculty of Engineering,
Alexandria University, Alexandria, Egypt.

ABSTRACT

A finite element model, for turbulent flow over a fence, based on a penalty function approximation and a Petrov-Galerkin formulation which retains high accuracy for fully irregular meshes, is presented. The model is applied to the simulation of turbulent flow of an incompressible fluid over a fence located in a closed channel. The results are compared with reported experimental data.

INTRODUCTION

The separation of turbulent flow due to abrupt change in the geometry of a solid boundary is quite a common problem and is a basic one for many engineering devices such as cooling fins, valves, buildings and shelter fences. The flow field produced by a fence is highly complex and consists of several regions with different characteristics. The recirculation zone behind the fence, the free stream flow, the turbulent free shear layer between these two regions and the boundary layer at the wall. The numerical solution of the governing equations presents a difficult challenge and has been addressed by many investigators in the past, [1-3].

The main objective of this paper is to describe a two-dimensional Petrov-Galerkin finite element model based on bilinear rectangular elements for the solution of flow problems in arbitrary geometries. In principle, the model can be directly extended to three spatial dimensions. To show the capabilities of the model, it is applied to the simulation of turbulent flow of an incompressible fluid over a fence located in a closed channel. The results are compared with reported experimental data.

THE MATHEMATICAL MODEL

We consider steady two-dimensional flow, with turbulence represented using an isotropic turbulent viscosity hypothesis. The governing equations are the Navier-Stokes equations of conservation of linear momentum; the mass conservation equation for incompressible flow and the transport equations for turbulence kinetic energy and turbulence dissipation rate. These are, in Cartesian coordinates,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\nu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu\frac{\partial v}{\partial y}\right) \tag{3}$$

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial y}\right) + G - \varepsilon \tag{4}$$

$$u\frac{\partial \varepsilon}{\partial x} + v\frac{\partial \varepsilon}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right) + \frac{\varepsilon}{k}(C_1 G - C_2 \varepsilon) \tag{5}$$

where u and v are velocities in the x- and y-directions, respectively, p is the pressure, ν is the kinematic viscosity, ρ is the density, k is the turbulence kinetic energy, ε is the rate of dissipation of turbulence energy, and C_1, C_2, σ_k and σ_ε are constants. The kinematic viscosity ν is given by

$$\nu = \frac{\mu + \mu_t}{\rho} \tag{6}$$

where μ is the molecular viscosity and μ_t is the turbulent viscosity. Using the k-ε turbulence model,

[2], we have:

$$\mu_t = \frac{C_\mu k^2}{\varepsilon} \qquad (7)$$

where C_μ is a constant. The function G in (4) and (5) is defined as:

$$G = \mu_t \left(2\left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right) \qquad (8)$$

Equations (1)–(5) are assumed to be valid for all points in a domain R in the x-y plane with boundary ∂R. For simplicity of presentation, we assume that only Dirichlet-type boundary conditions are imposed on ∂R. Other types of boundary conditions can be directly incorporated by the model.

PENALTY FUNCTION APPROXIMATION

The penalty function approximation is especially attractive in the solution of confined recirculating flows and is described in [4,5]. Following the work of Fukumori and Wake [6], we write

$$p = p_s - \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \qquad (9)$$

where p_s is the static component of pressure and λ is the penalty parameter, assumed to be large. Equation (9) incorporates the basic fact that, under conditions of no motion, the pressure must be the static pressure, a condition which is not satisfied by penalty formulations found in the literature that assume $p_s = 0$. u and v in equation (9) represent some approximation of the exact solutions hence $(\partial u/\partial x + \partial v/\partial y)$ equals some very small residue. A full discussion can be found in [6]. As a result of the penalty approximation, the pressure and the mass conservation equation are eliminated from the system of equations (1)–(5) and the governing equations become:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\lambda}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left(\frac{\mu}{\rho} + \frac{C_\mu k^2}{\rho \varepsilon} \right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$
$$+ \frac{2C_\mu k}{\rho \varepsilon}\left(\frac{\partial k}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial k}{\partial y}\frac{\partial u}{\partial y} \right) - \frac{C_\mu k^2}{\rho \varepsilon^2}\left(\frac{\partial \varepsilon}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \varepsilon}{\partial y}\frac{\partial u}{\partial y} \right) \qquad (10)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{\lambda}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left(\frac{\mu}{\rho} + \frac{C_\mu k^2}{\rho \varepsilon} \right)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$
$$+ \frac{2C_\mu k}{\rho \varepsilon}\left(\frac{\partial k}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial k}{\partial y}\frac{\partial v}{\partial y} \right) - \frac{C_\mu k^2}{\rho \varepsilon^2}\left(\frac{\partial \varepsilon}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial \varepsilon}{\partial y}\frac{\partial v}{\partial y} \right) \qquad (11)$$

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = \frac{C_\mu k^2}{\sigma_k \varepsilon}\left(\frac{\partial^2 k}{\partial x^2} + \frac{\partial^2 k}{\partial y^2} \right) + \frac{2C_\mu k}{\sigma_k \varepsilon}\left[\left(\frac{\partial k}{\partial x} \right)^2 + \left(\frac{\partial k}{\partial y} \right)^2 \right]$$
$$- \frac{C_\mu k^2}{\sigma_k \varepsilon^2}\left(\frac{\partial \varepsilon}{\partial x}\frac{\partial k}{\partial x} + \frac{\partial \varepsilon}{\partial y}\frac{\partial k}{\partial y} \right) + \frac{C_\mu k^2}{\varepsilon}\left\{ 2\left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right\} - \varepsilon \qquad (12)$$

$$u\frac{\partial \varepsilon}{\partial x} + v\frac{\partial \varepsilon}{\partial y} = \frac{C_\mu k^2}{\sigma_\varepsilon \varepsilon}\left(\frac{\partial^2 \varepsilon}{\partial x^2} + \frac{\partial^2 \varepsilon}{\partial y^2} \right) - \frac{C_\mu k^2}{\sigma_\varepsilon \varepsilon^2}\left[\left(\frac{\partial \varepsilon}{\partial x} \right)^2 + \left(\frac{\partial \varepsilon}{\partial y} \right)^2 \right]$$
$$+ \frac{2C_\mu k}{\sigma_\varepsilon \varepsilon}\left(\frac{\partial \varepsilon}{\partial x}\frac{\partial k}{\partial x} + \frac{\partial \varepsilon}{\partial y}\frac{\partial k}{\partial y} \right) + C_1 C_\mu k\left\{ 2\left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right\} - \frac{C_2 \varepsilon^2}{k} \qquad (13)$$

Once the velocity fields are known, the pressure variable is calculated a posteriori if desired by solving the equation;

$$-\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right) = \rho\left[\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right) \right] \qquad (14)$$

Over the region R, subject to homogeneous Neumann conditions along the boundary ∂R; i.e.

$$-\frac{\partial p}{\partial x} n_x - \frac{\partial p}{\partial y} n_y = 0 \quad \text{for } (x,y) \in \partial R \qquad (15)$$

where $(n_x, n_y)^t$ is the unit vector normal to the boundary and pointing in the outward direction. Because only Neumann boundary conditions are given, the solution of (14) subject to (15) is determined only up to an additive constant. In order to obtain a unique pressure field, one more condition must be imposed. This is usually done by setting the pressure at one point in the domain equal to a reference pressure, i.e.

$$p(x_o, y_o) = p_o \tag{16}$$

where (x_o, y_o) is a point in R and p_o is a prescribed value.

PETROV–GALERKIN FORMULATION

We use a Petrov–Galerkin formulation based on bilinear quadrilateral elements that has been developed from the work of Kelly et al [7] and Heinrich [8,9]. Assume that the domain $\bar{R} = R \cup \partial R$ has been partitioned into a number of quadrilateral elements $\bar{e}_m$ whose interiors are sets denoted by e_m and the boundaries by ∂e_m so that

$\bar{e}_m = e_m \cup \partial e_m$ and $e_m \cap \partial e_m = \emptyset$, the empty set. The partition also satisfies

$e_i \cap e_j = \emptyset$ if $i \neq j$ and $\bigcup_m \bar{e}_m = \bar{R}$. We will need the following definitions:

(i) We denote by $H^1(R)$ the space of functions in R that are square integrable over R, together with their first partial derivatives.

(ii) We denote by $H_o^1(R)$ the subspace of functions of $H^1(R)$ that vanish on the boundary δR

(iii) We define the test functions:

$$T = U + P_1,\ V = U + P_2,\ W = U + P_3 \tag{17}$$

Where U is in $H_o^1(R)$ and P_1, P_2 and P_3 are piecewise continuous functions, possibly discontinuous across the boundaries δe_k of the above-defined partition such that P_1, P_2 and P_3 go to zero as the sizes of the elements in the partition become small.

The weak formulation for (10)–(13) can now be stated as: find a velocity vector field $(u, v)^t$, a scalar turbulence kinetic energy k and a scalar rate of dissipation of turbulence energy ε, where u,v,k and ε are in $H_o^1(R)$ such that:

$$\begin{aligned}
&\int_R \left\{ U\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\lambda}{\rho}\frac{\partial U}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\mu}{\rho} + \frac{C_\mu k^2}{\rho\varepsilon}\right)\left(\frac{\partial U}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial u}{\partial y}\right)\right. \\
&\left. - \frac{2C_\mu kU}{\rho\varepsilon}\left(\frac{\partial k}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial k}{\partial y}\frac{\partial u}{\partial y}\right) + \frac{C_\mu k^2 U}{\rho\varepsilon^2}\left(\frac{\partial \varepsilon}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \varepsilon}{\partial y}\frac{\partial u}{\partial y}\right)\right\} dR \\
&+ \sum_m \int_{e_m} P_1 \left\{ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{\lambda}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\mu}{\rho} + \frac{C_\mu k^2}{\rho\varepsilon}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right. \\
&\left. - \frac{2C_\mu k}{\rho\varepsilon}\left(\frac{\partial k}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial k}{\partial y}\frac{\partial u}{\partial y}\right) + \frac{C_\mu k^2}{\rho\varepsilon^2}\left(\frac{\partial \varepsilon}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \varepsilon}{\partial y}\frac{\partial u}{\partial y}\right)\right\} dR = 0
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\int_R \left\{ U\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\lambda}{\rho}\frac{\partial U}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\mu}{\rho} + \frac{C_\mu k^2}{\rho\varepsilon}\right)\left(\frac{\partial U}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial v}{\partial y}\right)\right. \\
&\left. - \frac{2C_\mu kU}{\rho\varepsilon}\left(\frac{\partial k}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial k}{\partial y}\frac{\partial v}{\partial y}\right) + \frac{C_\mu k^2 U}{\rho\varepsilon^2}\left(\frac{\partial \varepsilon}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial \varepsilon}{\partial y}\frac{\partial v}{\partial y}\right)\right\} dR \\
&+ \sum_m \int_{e_m} P_1 \left\{ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \frac{\lambda}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\mu}{\rho} + \frac{C_\mu k^2}{\rho\varepsilon}\right)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right. \\
&\left. - \frac{2C_\mu k}{\rho\varepsilon}\left(\frac{\partial k}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial k}{\partial y}\frac{\partial v}{\partial y}\right) + \frac{C_\mu k^2}{\rho\varepsilon^2}\left(\frac{\partial \varepsilon}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial \varepsilon}{\partial y}\frac{\partial v}{\partial y}\right)\right\} dR = 0
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\int_R \left\{ U\left(u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y}\right) + \frac{C_\mu k^2}{\sigma_k\varepsilon}\left(\frac{\partial U}{\partial x}\frac{\partial k}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial k}{\partial y}\right)\right. \\
&- \frac{2C_\mu kU}{\sigma_k\varepsilon}\left[\left(\frac{\partial k}{\partial x}\right)^2 + \left(\frac{\partial k}{\partial y}\right)^2\right] + \frac{C_\mu k^2 U}{\sigma_k\varepsilon^2}\left(\frac{\partial \varepsilon}{\partial x}\frac{\partial k}{\partial x} + \frac{\partial \varepsilon}{\partial y}\frac{\partial k}{\partial y}\right) \\
&\left. - \frac{C_\mu k^2 U}{\varepsilon}\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right] + \varepsilon U\right\} dR \\
&+ \sum_m \int_{e_m} P_2 \left\{ u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} + \frac{C_\mu k^2}{\sigma_k\varepsilon}\left(\frac{\partial^2 k}{\partial x^2} + \frac{\partial^2 k}{\partial y^2}\right) + \frac{2C_\mu k}{\sigma_k\varepsilon}\left[\left(\frac{\partial k}{\partial x}\right)^2 + \left(\frac{\partial k}{\partial y}\right)^2\right]\right. \\
&+ \frac{C_\mu k^2}{\sigma_k\varepsilon^2}\left(\frac{\partial \varepsilon}{\partial x}\frac{\partial k}{\partial x} + \frac{\partial \varepsilon}{\partial y}\frac{\partial k}{\partial y}\right) \\
&\left. - \frac{C_\mu k^2}{\varepsilon}\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right\} + \varepsilon\right\} dR = 0
\end{aligned} \tag{20}$$

$$
\begin{aligned}
\int_R & \left\{ U\left(u\frac{\partial \varepsilon}{\partial x} + v\frac{\partial \varepsilon}{\partial y}\right) + \frac{C_\mu k^2}{\sigma_\varepsilon \varepsilon}\left(\frac{\partial U}{\partial x}\frac{\partial \varepsilon}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial \varepsilon}{\partial y}\right)\right. \\
& + \frac{C_\mu k^2 U}{\sigma_\varepsilon \varepsilon^2}\left[\left(\frac{\partial \varepsilon}{\partial x}\right)^2 + \left(\frac{\partial \varepsilon}{\partial y}\right)^2\right] - \frac{2C_\mu k U}{\sigma_\varepsilon \varepsilon}\left(\frac{\partial \varepsilon}{\partial x}\frac{\partial k}{\partial x} + \frac{\partial \varepsilon}{\partial y}\frac{\partial k}{\partial y}\right) \\
& \left. - C_1 C_\mu k U\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right] + \frac{C_2 \varepsilon^2 U}{k}\right\} dR \\
& + \sum_m \int_{e_m} P_3 \left\{ u\frac{\partial \varepsilon}{\partial x} + v\frac{\partial \varepsilon}{\partial y} - \frac{C_\mu k^2}{\sigma_\varepsilon \varepsilon}\left(\frac{\partial^2 \varepsilon}{\partial x^2} + \frac{\partial^2 \varepsilon}{\partial y^2}\right)\right. \\
& + \frac{C_\mu k^2}{\sigma_\varepsilon \varepsilon^2}\left[\left(\frac{\partial \varepsilon}{\partial x}\right)^2 + \left(\frac{\partial \varepsilon}{\partial y}\right)^2\right] - \frac{2C_\mu k}{\sigma_\varepsilon \varepsilon}\left(\frac{\partial \varepsilon}{\partial x}\frac{\partial k}{\partial x} + \frac{\partial \varepsilon}{\partial y}\frac{\partial k}{\partial y}\right) \\
& \left. - C_1 C_\mu k \left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right\} + \frac{C_2 \varepsilon^2}{k}\right\} dR = 0 \qquad (21)
\end{aligned}
$$

for all test functions T,V and W of the form (17), where m ranges over the number of elements in the partition. A more detailed discussion of this type of weak formulation can be found in Hughes el al [10].

The weak formulation for the pressure equation (14) is the standard formulation for second-order elliptic operators: Find a pressure field p in $H^1(R)$ such that p satisfies (15) and

$$
\int_R \left\{\frac{\partial P}{\partial x}\frac{\partial p}{\partial x} + \frac{\partial P}{\partial y}\frac{\partial p}{\partial y}\right\} dR = \rho \int_R \left\{\frac{\partial P}{\partial x}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial P}{\partial y}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\right\} dR \qquad (22)
$$

is satisfied for all P in $H^1(R)$ such that $P(x_o, y_o) = 0$.

It should be noticed that the computation of the velocity field using the penalty function formulation does not require the explicit calculation of the pressure. Hence, (22) is solved only if the pressure field is needed. Also, it does not feedback into the solution of the Navier-Stokes equations, so any inaccuracies in the pressure solution will not affect the development of the flow. Again, to obtain the pressure from (22) requires the inversion of the Laplacian operator, which can be done very efficiently, compared with the cost of retaining the pressure in the momentum equations in a standard mixed formulation.

The discrete Petrov-Galerkin approximation is obtained by approximating the functions u,v,k and ε in expressions (18)-(21) over the subspace $H^1_o(R)$ such that:

$$
\left.
\begin{aligned}
u &= \sum_j u_j \phi_j(x,y) \\
V &= \sum_j v_j \phi_j(x,y) \\
k &= \sum_j k_j \phi_j(x,y) \\
\varepsilon &= \sum_j \varepsilon_j \phi_j(x,y)
\end{aligned}
\right\} \qquad (23)
$$

where j ranges over the number of nodes and u_j, v_j, k_j and ε_j denote the values of the dependent variables at the nodes. The functions $\phi_j(x,y)$ are the well-known bilinear isoparametric shape functions [11]. The component U of the test functions (17) is defined node-wise equal to the shape functions, i.e.

$$
U_j(x,y) = \phi_j(x,y) \qquad \text{for} \quad \text{all } j \qquad (24)
$$

The perturbation functions P_1, P_2 and P_3 are given by:

$$
P_i^j(x,y) = r_i\left(u\frac{\partial \phi_j}{\partial x} + v\frac{\partial \phi_j}{\partial y}\right), \quad i = 1,2,3 \qquad (25)
$$

where

$$
r_i = \frac{\alpha_i h}{\varepsilon\,|W|} \qquad (26)
$$

$|W|$ is the magnitude of the local velocity W, and h is an average element length whose definition is

$$
h = \Delta x\,|\cos\theta| + \Delta y\,|\sin\theta| \qquad (27)
$$

where Δx and Δy are the dimensions of the rectangle and θ is the angle of the velocity vector W to the positive x-axis, for more details see [4]. The parameters α_i are given by:

$$
\alpha_i = \coth\gamma_i - \frac{1}{\gamma_i} \qquad (28)
$$

Here, γ_i is the element cell Reynolds number defined as:

$$
\gamma_i = \frac{|W|\,h}{2\,|\nu|} \qquad (29)
$$

while

$$\gamma_2 = \frac{\sigma_k |v|}{|\mu_t|}\gamma_1 \qquad (30)$$

and

$$\gamma_3 = \frac{\sigma_\varepsilon |v|}{|\mu_t|}\gamma_1 \qquad (31)$$

It should be noticed here that the Petrov–Galerkin treatment depends on physical parameters for each operator, and therefore three independent parameters, α_1,α_2 and α_3 are calculated at each element.

SOLUTION OF THE NON-LINEAR SYSTEM

The procedure described in the previous section leads to a system of non-linear equations that we write as:

$$Aw + F_1(w,k,\varepsilon) = 0 \qquad (32)$$

$$Bk + F_2(w,k,\varepsilon) = 0 \qquad (33)$$

$$C\varepsilon + F_3(w,k,\varepsilon) = 0 \qquad (34)$$

where the vectors $w=(u_1,v_1,u_2,v_2,........)^t$, $k=(k_1, k_2,......)^t$ and $\varepsilon=(\varepsilon_1, \varepsilon_2,...)^t$ contain the nodal values of velocities, turbulence kinetic energy and rate of dissipation of turbulence energy respectively. The matrices A,B and C contain the coefficients from the penalty and momentum as well as $k-\varepsilon$ model. The vectors F_1,F_2 and F_3 are the non-linear terms.

The system of equations (32-34) is solved sequentially using a basic Newmark algorithm such as presented by Hughes et al.[10]. The algorithm is described as follows:

Step1. Predict initial guesses for u,v,k and ε

Step2. Form the residual vector ΔF_3

Step3. Solve for incremental corrections to the predicted ε

Step4. Correct the dissipation rate ε

Step5. Form the residual vector ΔF_2

Step6. Solve for incremental corrections to the predicted k

Step7. Correct the turbulence kinetic energy k

Step8. Form the residual vector ΔF_1

Step9. Solve for incremental corrections to the predicted velocities w

Step10. Correct the velocities w

Step11. Check for accuracy, if the desired accuracy has not been achieved, repeat the process starting from step2.

NUMERICAL EXAMPLE

The proposed model is applied to the simulation of turbulent flow of an incompressible fluid over a fence located in a closed channel Figure(1). The boundary conditions are:

(i) For the inlet boundary, u=0; v=0; p=0; $\varepsilon = \varepsilon_o$ and $k=k_o$

(ii) For the out boundary, $\frac{\partial u}{\partial x}=0$; v=0; p=0; and $\varepsilon = k = 0$

(iii) For the solid boundary, u=0; v=0 while

$$k = W_T^2(C_v C_D)^2 \quad \text{and} \quad \varepsilon = \frac{W_T^3}{(\kappa y)}$$

where C_μ=0.5478; C_D=0.1643; κ=0.41; $W_T = (\frac{\tau_s}{\rho})^{0.5}$ and τ_s is the wall shear stress. The constant parameters in the turbulent model are recommended in [12] as: C_1=1.44, C_2=1.92, σ_k=1.0 and σ_ε=1.314.

The flow pattern has been studied with Reynolds number γ_1=600, 2000 and the fence non-dimensional height S/H=0.25. The stream lines pattern for γ_1=600 and γ_1=2000 are shown in figure (2) and figure (3) respectively while figure (4) shows the velocity profiles at several x/s locations. It is clear, from these patterns, that the flow approaching the fence is deflected by the build-up of the pressure as the fluid impinges on the front face. The adverse pressure gradient ahead of the fence acts on the slower moving fluid in the thin floor boundary layer. Deceleration of the flow at the front face and its deflection around the top of the fence causes a zone of high positive pressure particularly near the centre of the front face where the flow approaches stagnation. The existence of a vertical gradient of location of the maximum surface pressure nearer to the top of the fence because of the higher dynamic pressure associated with faster moving fluid near the top of the fence. For the downstream region, the fluid forms a large recirculation zone characterized with low velocity and pressure. In Figure(5), a comparison between the calculated velocity profiles and the reported experimental data [13] is illustrated. An excellent agreement is observed.

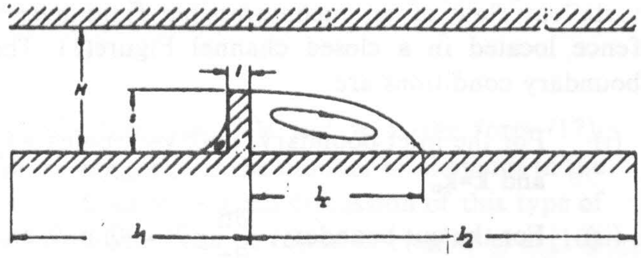

Figure 1. Geometric representation of the fence.

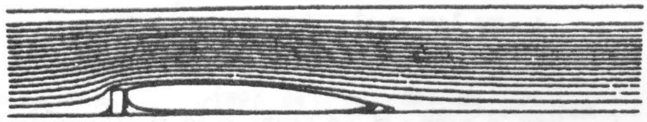

Figure 2. Streamlines of the flow field for Reynolds number=600 and S/H= 0.25.

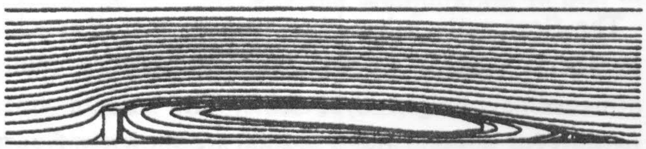

Figure 3. Streamlines of the flow field for Reynolds number=2000 and S/H= 0.25.

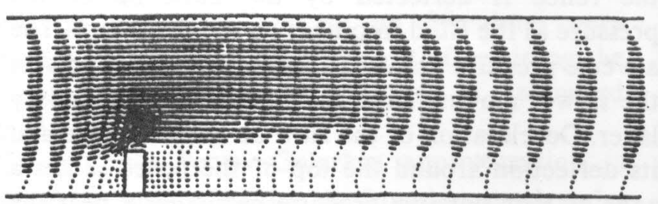

Figure 4. Velocity vector field for Reynolds number=600 and S/H=0.25.

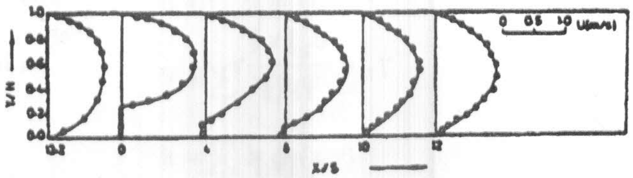

Figure 5. Axial velocity component: Comparison between predictions and experimental data (o experimental; — prediction), for Reynolds number=600 and S/H=0.25.

CONCLUSION

We have presented a finite element method for turbulent flow that has been shown to be robust and efficient in practical calculations. The method is based on a penalty function approximation and a Petrov-Galerkin formulation. As an illustration, it has been applied in the simulation of turbulent flow of an incompressible fluid over a fence located in a closed channel. The results were in good agreement with reported experimental data.

REFERENCES

[1] S.M. Fraser and M.H. Siddig, "Turbulent Flow Over a Plane Normal Wall", *J. Mech. Eng.*, Vol. 22, No. 4, pp.417-431, 1980.

[2] A. Liakopoulous, "Computation of High Speed Turbulent Boundary Layer Flow Using The k-ε Turbulence Model", *Int. J. for Numerical Methods in Fluids*, Vol. 15, pp. 81-97, 1985.

[3] C.A.J. Fletcher, "On an Alternating Direction Implicit Finite Element Method for Flow Problems", *Comp. Meth. Appl. Mech. Eng.*, Vol. 30, pp. 307-312, 1982.

[4] T.J.R. Hughes, W.K. Liu and A. Brooks, "Finite element analysis of incompressible viscous flows by the penalty function formulation", *J. Comput. Phys.*, Vol. 30, pp. 1-60, 1979.

[5] J.C. Heinrich and R.S. Marshall, "Viscous Incompressible Flow by a Penalty Function Finite Element Method", *Comput. and Fluids*, Vol. 9, pp. 73-83, 1981.

[6] E. Fukumori and A. Wake, "A Note on the Penalty Function Method Applied to the Navier-Stokes Equations", *Preprint*, Department

of Civil Engineering, University of Buffalo, NY, 1986.

[7] D.W. Kelly, S. Nakazawa, O.C. Zienkiewicz and J.C. Heinrich, "A Note on Upwinding and Anisotropic Balancing Dissipation in Finite Element Approximation to Convective Diffusion Problems, *Int. J. Num. Meths. Eng.* Vol. 15, pp. 1705-1711, 1980.

[8] J.C. Heinrich, "A Finite Element Model for Double-Diffusive Convection", *Int. J. Num. Meths. Eng.*, Vol. 20, pp. 447-464, 1984.

[9] J.C. Heinrich and C.C. Yu, "Finite Element Simulation of Buoyancy-Driven Flows with Emphasis on Natural Convection in a Horizontal Circular Cylinder", *Comp. Meth. Appl. Mech. Eng.*, Vol. 69, pp. 1-27, 1988.

[10] T.J.R. Hughes, M. Mallet and A. Mikuzami, "A New Finite Element Formulation for Computational Fluid Dynamics", *Comp. Meths. Appl. Mech. Eng.*, Vol. 54, pp. 341-355, 1986.

[11] R. Wait and A.R. Mitchel, *Finite Element Analysis and Applications*, Wiley, Chichester, 1985.

[12] W. Rodi, *Turbulent Models and their Applications in Hydraulics - A State of the Art Review*, Book Publication, Delft, Netherlands, 1980.

[13] M.G. Carvalho, F. Durst and J.C.F. Pereira, "Predictions and Measurements of Laminar Flow over Two Dimensional Obstacles", *Appl. Math. Modeling*, Vol. 11, 1987.