LARGE DISPLACEMENT ANALYSIS OF STRUCTURES

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ABSTRACT

A general finite element technique, based on the Lagrangian approach, is presented for the non-linear analysis of spatial framed structures subjected to static loads. The non-linearity is due to finite deformation (large rotation is included) and elasto-plasticity. A new finite element model is developed in which the axial displacement component is represented by a hermite polynomial and a more exact strain model is also presented which includes the effect of large rotation. The governing matrices are integrated using Gauss and Trapezoidal rule integration schemes. A comparative study has been made between numerical results of the proposed method and other published solutions which indicates that the proposed technique is very effective in all problems solved.

1- INTRODUCTION

The use of the incremental variational principles together with the finite element method and a sophisticated computer machine make it possible to formulate general finite element techniques for the non-linear analysis of framed structures and to include both types of nonlinearities (geometric and material nonlinearities).

In those techniques; fundamental incremental equilibrium equations of the problem are formulated using either the Eulerian approach in which all quantities are referred to the deformed configuration [Kassimali (1), Kam (2) and others], or the Lagrangian approach in which all quantities are referred to the initial configuration [El-Zanaty and Murry (3), Cichon (4) and Keck (5)].

Available finite element techniques, for the analysis of nonlinear systems which are based on Lagrangian approach are restricted to the cases of small angles of rotation. The present work is a trial to overcome this deficiency. Therefore, a new finite element model is developed, in which the axial displacement component is represented by cubic polynomial (Hermite polynomial) as the transverse displacement components, and a more exact strain displacement relations are presented which include the effects of large rotation.

A computer program has been written which includes the above effects. Several numerical examples have been solved and the results are compared with other published solutions.

2- ASSUMPTIONS

The following assumptions in the derivation of the strain displacement relations are made:

1. The member length is much greater than the cross section dimensions.
2. Plane sections, which are normal to the member x-axis before deformation, remain plane and normal to the deformed member x-axis after deformation (i.e. shear deformation is excluded).
3. Cross sections are doubly symmetric and constant throughout its length (prismatic members).
4. Angles of twist are assumed small and therefore, warping effects can be neglected.

3- FINITE ELEMENT MODEL AND STRAIN DISPLACEMENT RELATIONS

Since the axial displacement component, U_0 is represented by cubic polynomial as the transverse displacement components (V_0, W_0) and the displacement derivative U_0' is also used as a nodal degree of freedom as in V_0' and W_0', see Figure (1), the displacement function components (U_0, V_0, W_0 and θ_{t0}) can be written in terms of the coordinate variable, X, and the nodal degrees of freedom (q_1--> q_{14}), as follows:

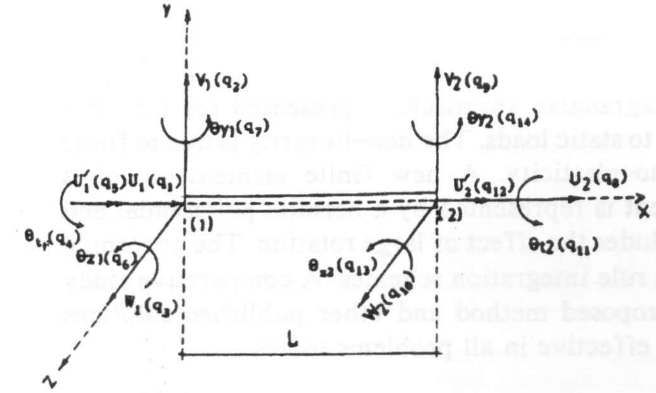

Figure 1-a.

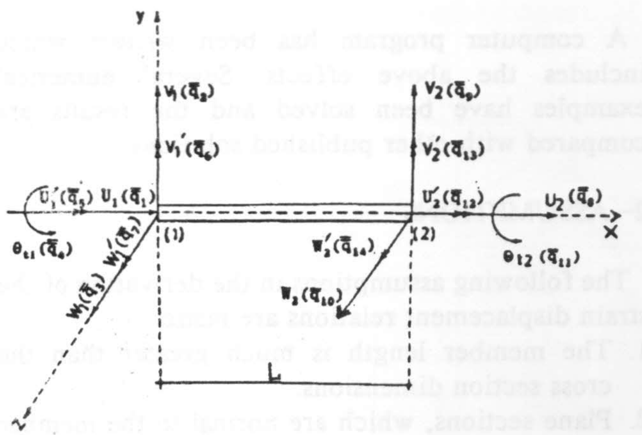

Figure 1-b.

$$[U_o, V_o, W_o, \theta_{t0}]^T = [N] \{q\} \quad (1)$$

or,

$$U_0 = \sum_{i=1}^{14} N_{1,i}\, q_i, \quad V_0 = \sum_{i=1}^{14} N_{2,i}\, q_i,$$

$$W_0 = \sum_{i=1}^{14} N_{3,i}\, q_i, \quad \theta_{t0} = \sum_{i=1}^{14} N_{4,i}\, q_i \quad (2)$$

where [N] is the shape function matrix, which has the following non-zero components:

$$N_{1,1} = N_{2,2} = N_{3,3} = 1 - 3S^2 + 2S^3$$

$$N_{1,5} = N_{2,6} = N_{3,7} = L * (S - 2S^2 + S^3)$$

$$N_{1,8} = N_{2,9} = N_{3,10} = (3S^2 - 2S^3) \quad (3)$$

$$N_{1,12} = N_{2,13} = N_{3,14} = L * (-S^2 + S^3)$$

$$N_{4,4} = (1 - S),\ N_{4,11} = S$$

$$\text{and, } S = X/L$$

The relations between the angles of rotations (θ_y, θ_z) and the derivatives of the displacement components (U_0, V_0, W_0) can be obtained if, as shown in Figure (2), two arbitrary infinitely near points A and B are considered on the x-axis of a frame element before deformation and due to deformation they move to A^* and B^*.

From the geometry of Figure (2):

$$W_0' = -(1+U_0')\tan\theta_y$$

$$V_0' = (1+U_0')\tan\theta_z \quad (4)$$

From eqns. (4), the relations between the displacement vectors {q} and $\{\bar{q}\}$, defined in Figure (1), can be written as:

$$q_1 = \bar{q}_1 \quad q_8 = \bar{q}_8$$

$$q_2 = \bar{q}_2 \quad q_9 = \bar{q}_9$$

$$q_3 = \bar{q}_3 \quad q_{10} = \bar{q}_{10}$$

$$q_4 = \bar{q}_4 \quad q_{11} = \bar{q}_{11} \quad (5)$$

$$q_5 = \bar{q}_5 \quad q_{12} = \bar{q}_{12}$$

$$q_6 = (1+\bar{q}_5)\tan\bar{q}_6 \quad q_{13} = (1+\bar{q}_{12})\tan\bar{q}_{13}$$

$$q_7 = -(1+\bar{q}_5)\tan\bar{q}_7 \quad q_{14} = -(1+\bar{q}_{12})\tan\bar{q}_{14}$$

Differentiating eqns. (5), gives the relations between the incremental displacement vectors $\{\Delta q\}$ and $\{\Delta \bar{\bar{q}}\}$, which can be written as:

$$\{\Delta q\} = [T_1]\{\Delta \bar{\bar{q}}\} \qquad (6)$$

where the transformation matrix $[T_1]$ has the following non-zero components:

$$T_{1,1} = T_{2,2} = T_{3,3} = T_{4,4} = T_{5,5} = 1$$

$$T_{8,8} = T_{9,9} = T_{10,10} = T_{11,11} = T_{12,12} = 1$$

$$T_{6,5} = \tan \bar{\bar{q}}_6 \, , \; T_{6,6} = (1 + \bar{\bar{q}}_5)(1 + \tan^2 \bar{\bar{q}}_6) \qquad (7)$$

$$T_{7,5} = -\tan \bar{\bar{q}}_7, \; T_{7,7} = -(1+\bar{\bar{q}}_5)(1+ \tan^2 \bar{\bar{q}}_7)$$

$$T_{13,12} = \tan\bar{\bar{q}}_{13}, T_{13,13} = (1+\bar{\bar{q}}_{12})(1+\tan^2 \bar{\bar{q}}_{13})$$

$$T_{14,12} = -\tan \bar{\bar{q}}_{14} \, , \; T_{14,14} = -(1 + \bar{\bar{q}}_{12})(1 + \tan^2 \bar{\bar{q}}_{14})$$

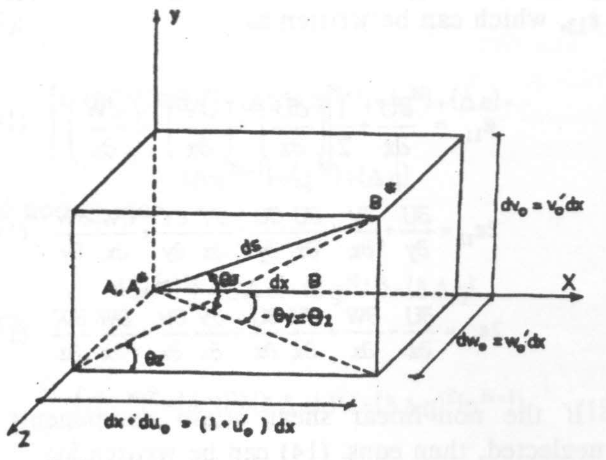

Figure 2. Spatial centroidal deformation.

Considering a generic material point, a, lies on an arbitrary cross section, A, its centroid is, O, before deformation, and due to deformation A, a and O move to A^*, a^* and O^*, respectively, as shown in Figure (3).

The axes x_0, y_0 and z_0 are aligned with the axes x, y and z respectively, while the axes x_3, y_3 and z_3 are used to define the orientation of the deformed cross section A^*, where y_3 and z_3 are the principal axes of A^* and x_3 is the outward normal to them.

The displacement components of the material point, 'a', (U, V, W) can be expressed in terms of the displacement components of the centroid, 'O', (U_0, V_0, W_0), using three angles of rotation $(\theta_1, \theta_2, \theta_3)$, known as the Euler angles, as follows:

$$U = U_0 - S_3C_2[C_1y - S_1z] + S_2[S_1y + C_1z]$$

$$V = V_0 - y + C_3[C_1y - S_1z] \qquad (8)$$

$$W = W_0 - z + S_2S_3[C_1y - S_1z] + C_2[S_1y + C_1z]$$

where:

$$S_1 = \sin \theta_1, \; S_2 = \sin \theta_2, \; S_3 = \sin \theta_3$$

$$C_1 = \cos \theta_1, \; C_2 = \cos \theta_2, \; C_3 = \cos \theta_3$$

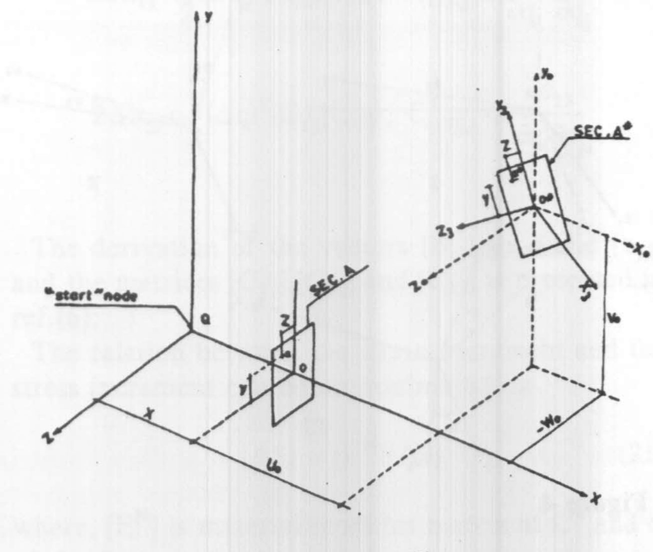

Figure 3.

The Euler angles $(\theta_1, \theta_2, \theta_3)$ consist of a sequence of three plane rotations about instantaneous axes as shown in Figure (4), see ref. (6).

Eqns. (8) can be simplified by ignoring the terms of sine products, i.e. putting

$$S_1S_2 = 0, \; S_1S_3 = 0, \; S_2S_3 = 0$$

then:

$$U = U_0 - S_3C_2C_1y + S_2C_1z$$

$$V = V_0 - y + C_3C_1y - C_3S_1z \qquad (9)$$

$$W = W_0 - z + C_2 S_1 y + C_2 C_1 z$$

The Euler angles (θ_2, θ_3) can be expressed in terms of (U_0, V_0, W_0) as follows, see Figure (2)

$$\sin\theta_2 = -W_0'/h_2, \quad h_2 = \sqrt{(1+U_0')^2 + W_0'^2}$$

$$\cos\theta_2 = h_1 / h_2, \quad h_1 = (1+U_0') \qquad (10)$$

$$\sin\theta_3 = V_0' / h_3, \quad h_3 = \sqrt{(1+U_0')^2 + V_0'^2 + W_0'^2}$$

$$\cos\theta_3 = h_2 / h_3$$

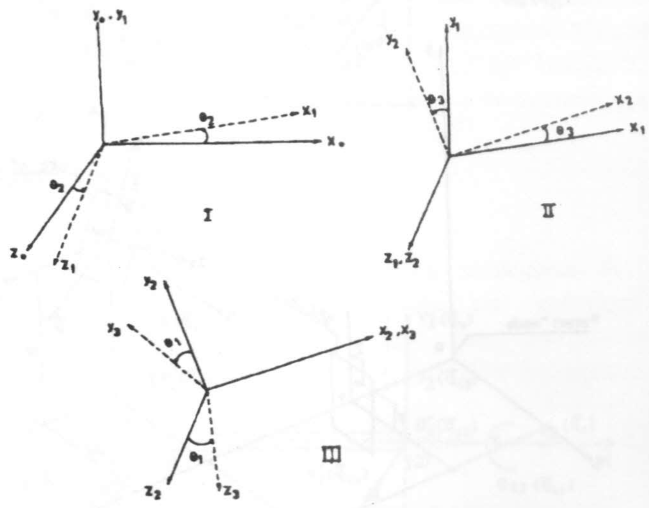

Figure 4.

Note that the angle θ_1 (not shown in Figure (2)) is the angle of twist, θ_t, defined before and is assumed to be small, then:

$$\sin\theta_1 \approx \theta_t, \quad \cos\theta_1 = 1 \qquad (11)$$

substituting eqns. (10) and (11) into eqns. (9) yields:

$$U = U_0 + [-(h_1/h_2h_3)V_0']\, y + [-W_0'/h_2]z \quad (12\text{-}1)$$

$$V = V_0 + [(h_2/h_3) - 1]y + [-(h_2/h_3)\,\theta_t]z \quad (12\text{-}2)$$

$$W = W_0 + [(h_1/h_2)\,\theta_t]y + [(h_1/h_2) - 1]z \quad (12\text{-}3)$$

From eqns. (12), the displacement derivatives U′, V′ and W′ can be written in an approximate form as follows:

$$U' = U_0' - P_1V_0''y - P_2W_0'z$$

$$V' = V_0' + P_3U_0''y - P_4\theta_t'z \qquad (13)$$

$$W' = W_0' + P_5\theta_t'y + P_6U_0''z$$

where,

$$P_1 = \frac{h_1}{h_2h_3},\ P_2 = \frac{1}{h_2},\ P_3 = \frac{h_1}{h_3}\left[\frac{1}{h_2} - \frac{h_2}{h_3^2}\right]$$

$$P_4 = \frac{h_2}{h_3},\ P_5 = \frac{h_1}{h_2},\ P_6 = \frac{1}{h_2}\left[1 - \frac{h_1^2}{h_2^2}\right]$$

In the spatial frame element, the only non-zero Green strain components are the longitudinal component, ε_{11}, and the shear components ε_{12} and ε_{13}, which can be written as:

$$\epsilon_{11} = \frac{\partial U}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2\right] \qquad (14\text{-}1)$$

$$2\epsilon_{12} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} + \frac{\partial U}{\partial x}\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\frac{\partial V}{\partial y} + \frac{\partial W}{\partial x}\frac{\partial W}{\partial y} \qquad (14\text{-}2)$$

$$2\epsilon_{13} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} + \frac{\partial U}{\partial x}\frac{\partial U}{\partial z} + \frac{\partial V}{\partial x}\frac{\partial V}{\partial z} + \frac{\partial W}{\partial x}\frac{\partial W}{\partial z} \qquad (14\text{-}3)$$

If the non-linear shear strain components are neglected, then eqns. (14) can be written as:

$$\epsilon_{11} = \frac{\partial U}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2\right],$$

$$2\epsilon_{12} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x},\ 2\epsilon_{13} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} \qquad (15)$$

Substituting eqns. (12) and (13) into eqns. (15) and neglecting terms which contain the squares and the products of the coordinates y and z, yields:

$$\epsilon_{11} = U_0' + \tfrac{1}{2}(U_0'^2 + V_0'^2 + W_0'^2) + y[-P_1(1+U_0')V_0'' + P_3V_0'U_0'' + P_5W_0'\theta_t'] + z[-P_2(1+U_0')W_0'' - P_4V_0'\theta_t' + P_6W_0'U_0'']$$

$$2\epsilon_{12} = V_0' - P_1V_0' + yP_3U_0'' - zP_4\theta_t' \simeq -zP_4\theta_t' \quad (16)$$

$$2\epsilon_{13} = W_0' - P_2W_0' + yP_5\theta_t' + zP_6U_0'' \simeq yP_5\theta_t'$$

Eqns. (16) represent the strain displacement relations considered in the proposed technique.

4-1 Incremental equilibrium equations

The principle of virtual displacements can be written in a vector form at configuration C^{N+1} as follows:

$$\int_V \{S^{N+1}\}^T\{\delta e^{N+1}\}dV=\{\delta q^{N+1}\}^T\{p^{N+1}\} \quad (17)$$

where, $\{S^{N+1}\}$ is the 2nd Piola-Kirchhoff stress vector at configuration C^{N+1}, $\{P^{N+1}\}$ is the external load vector at C^{N+1} and V is the initial volume.

If the following substitutions can be made into eqn. (17):

$$\{S^{N+1}\}=\{S^N\}+\{\Delta S\},\ \{e^{N+1}\}=\{e^N\}+\{\Delta e\},$$

$$\{\Delta q^{N+1}\}=\{q^N\}+\{\Delta q\}$$

and noting that

$$\{\delta e^{N+1}\}=\{\delta\Delta e\},\ \{\delta q^{N+1}\}=\{\delta\Delta q\}$$

then,

$$\int_V (\{S^N\}^T+\{\Delta S\}^T)\{\delta\Delta e\}dV=\{\delta\Delta q\}^T\{p^{N+1}\} \quad (18)$$

The strain increment can be expressed as the summation of linear and quadratic components, as follows:

$$\{\Delta\epsilon\} = \{\Delta e\} + \{\Delta h\}$$

or,

$$\begin{bmatrix}\Delta\epsilon_{11}\\ 2\Delta\epsilon_{12}\\ 2\Delta\epsilon_{13}\end{bmatrix}=\begin{bmatrix}\Delta e_{11}\\ 2\Delta e_{12}\\ 2\Delta e_{13}\end{bmatrix}+\begin{bmatrix}\Delta h_{11}\\ 2\Delta h_{12}\\ 2\Delta h_{13}\end{bmatrix} \quad (19)$$

where

$$\Delta e_{11}=\{b_{11}\}^T\{\Delta q\},\ b_{(11)i}=\frac{\partial e_{11}}{\partial q_i}$$

$$2\Delta e_{12}=\{b_{12}\}^T\{\Delta q\},\ b_{(12)i}=2\frac{\partial e_{12}}{\partial q_i}$$

$$2\Delta e_{13}=\{b_{13}\}^T\{\Delta q\},\ b_{(13)i}=2\frac{\partial e_{13}}{\partial q_i} \quad (20)$$

$$\Delta h_{11}=\frac{1}{2}\{\Delta q\}^T[C_{11}]\{\Delta q\},\ C_{(11)i,j}=\frac{\partial e_{11}}{\partial q_i\partial q_j}$$

$$2\Delta h_{12}=\frac{1}{2}\{\Delta q\}^T[C_{12}]\{\Delta q\},\ C_{(12)i,j}=2\frac{\partial e_{12}}{\partial q_i\ \partial q_j}$$

$$2\Delta h_{13}=\frac{1}{2}\{\Delta q\}^T[C_{13}]\{\Delta q\},\ C_{(13)i,j}=2\frac{\partial e_{13}}{\partial q_i\partial q_j}$$

The derivation of the vectors $\{b_{11}\}$,$\{b_{12}\}$ and $\{b_{13}\}$ and the matrices $[C_{11}]$,$[C_{12}]$ and $[C_{13}]$ is presented in ref.(6).

The relation between the strain increment and the stress increment can be approximated as:

$$\{\Delta S\} = [E^N]\{\Delta e\} \quad (21)$$

where, $[E^N]$ is material modulus matrix at C^N and is of the form:

$$[E^N]=\begin{bmatrix}E^N & 0 & 0\\ 0 & G^N & 0\\ 0 & 0 & G^N\end{bmatrix}$$

where,

E^N = tangent longitudinal modulus at C^N

G^N = tangent shear modulus at C^N

Substituting eqns. (19), (20) and (21) into eqn. (18), and ignoring terms of third and higher orders in incremental variables (so that a linear equation will result), leads to:

$$\int_V [E^N \{b_{11}^N\}^T\{\Delta q\}\delta(\{b_{11}^N\}^T\{\Delta q\}) + G^N\{b_{12}^N\}^T\{\Delta q\}\delta(\{b_{12}^N\}^T\{\Delta q\})$$
$$G^N\{b_{13}^N\}^T\{\Delta q\}\delta(\{b_{13}^N\}^T\{\Delta q\})]dV + \int_V [S_{11}^N\delta(\frac{1}{2}\{\Delta q\}^T[C_{11}^N]\{\Delta q\})$$
$$+S_{12}^N\delta(\frac{1}{2}\{\Delta q\}^T[C_{12}^N]\{\Delta q\}) + S_{13}^N\delta(\frac{1}{2}\{\Delta q\}^T[C_{13}^N]\{\Delta q\})]dV \quad (22)$$
$$=\{\delta\Delta q\}^T\{p^{N+1}\}$$
$$-\int_V [S_{11}^N\delta(\{b_{11}^N\}^T\{\Delta q\}) + S_{12}^N\delta(\{b_{12}^N\}^T\{\Delta q\}) + S_{13}^N\delta(\{b_{13}^N\}^T\{\Delta q\})]dV$$

The incremental equilibrium equations can be obtained by taking the first variation of equation (22) with respect to the incremental nodal displacements, as follows:

$$\int_V [E^N\{b_{11}^N\}\{b_{11}^N\}^T + G^N\{b_{12}^N\}\{b_{12}^N\}^T + G^N\{b_{13}^N\}\{b_{13}^N\}^T]\{\Delta q\}dV$$
$$+\int_V [S_{11}^N[C_{11}^N] + S_{12}^N[C_{12}^N] + S_{13}^N[C_{13}^N]]\{\Delta q\}dV \quad (23)$$
$$=\{p^{N+1}\} - \int_V [S_{11}^N\{b_{11}^N\} + S_{12}^N\{b_{12}^N\} + S_{13}^N\{b_{13}^N\}]dV$$

which can be written in the form:

$$[k_t^N]\{\Delta q\} = \{P^{N+1}\} - \{f^N\} \quad (24)$$

where,

$[k^N_t]$ = tangent stiffness matrix at C^N
$\{f^N\}$ = internal force vector at C^N

The computation of the tangent stiffness matrix and the internal force vector requires integration over the volume of the element. In this work, the integration is performed using numerical integration. The integration over the cross-section of the element is performed using the trapezoidal rule, while the integration over the element length is performed using Gauss Numerical integration scheme. [ref. (6)].

The element incremental equilibrium equations in global coordinates at C^N, can be written as:

$$\left[K_t^N\right]\{\Delta Q\} = \{P^{N+1}\} - \{F^N\} \quad (25)$$

The transformation of the local equilibrium equations is carried out in two steps. In the first step a transformation is made from the system of degrees of freedom defined in Figure (1-a) to the other system of Figure (1-b), as follows:

$$\{\bar{\bar{\Delta q}}\} = [T_1]^T\{\Delta q\}$$
$$\{\bar{\bar{f}}\} = [T_1]^T\{f\} \quad (26)$$
$$[\bar{\bar{k}}_t] = [T_1]^T[k_t][T_1]$$

In the second step, a transformation is made using the conventional linear transformation matrix [T], which is commonly used in the linear analysis, as follows:

$$\{F\} = [T]^T\{\bar{\bar{f}}\}$$
$$\{Q\} = [T]^T\{\bar{\bar{q}}\} \quad (27)$$
$$[K_t] = [T]^T[\bar{\bar{k}}_t][T]$$

The structural incremental equilibrium equations can be established by using the standard assembly procedure which is used in the linear analysis, these equations can be written as:

$$[\overline{K}_t^N]\{\Delta\overline{Q}\} = \{\overline{P}^{N+1}\} - \{\overline{F}^N\} \quad (28)$$

where

$[K_t^N]$ = the structure tangent stiffness matrix at C^N
$\{\Delta Q\}$ = the structure displacement increment vector between C^N and C^{N+1}.
$\{P^{N+1}\}$ = the structure external load vector at C^{N+1}
$\{F^N\}$ = the structure internal force vector at C^N

Once eqns. (28) are assembled, the Newton-Raphson method can be used to obtain the load-displacement characteristics of the frames.

5- NUMERICAL EXAMPLES

Three numerical examples have been solved to examine the accuracy of the proposed solution technique. The first example is a simply supported beam subjected to a concentrated transverse mid-span load, P, and an axial load, Q = 16 * 10^3 (lbs), as shown in Figure (5). The problem has an elastic-plastic material with E = 28,985 KSI (200 KN/mm^2) and σ_y = 40 KSI (0.267 KN/mm^2). The beam is modelled using 4-submembers as shown in Figure(5). The resulting load-displacement curves are compared with solutions given by ref.(7) and ref.(8) these curves are presented in Figure (6).

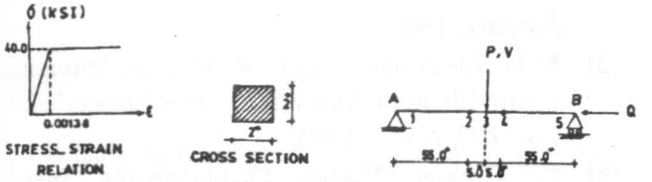

Figure 5.

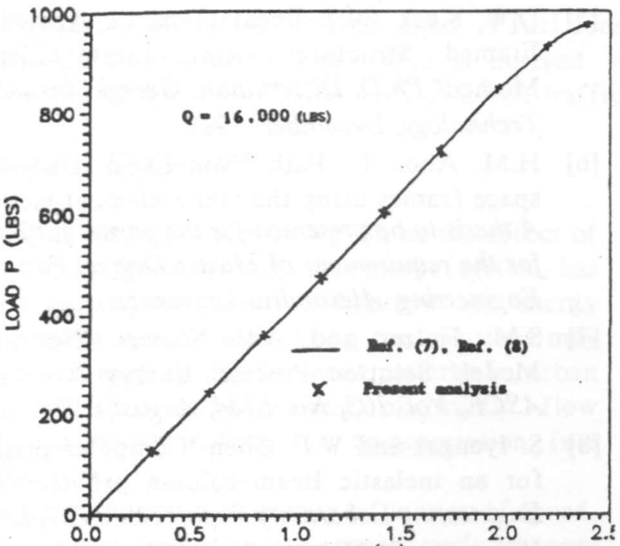

Figure 6.

The second example is a cantilever beam subject to a transverse concentrated load at the free end, see Figure (7). The beam has a linear elastic material with E = 30,000 KSI (206.9 KN/mm^2). The convergence of the proposed solution technique for large displacement only, is tested by modelling the cantilever with 1, 2 and 4 elements. The results are presented in Figure (8) along with the solution given in [ref. (5)]. From Figure (8) it is clear that the use of 4 elements yields a solution similar to the elastica solution.

The third example is a beam curved in the horizontal plane and subjected to a concentrated load, P, at its free end as illustrated in Figure (9).

The beam has a linear elastic material with E = 10,000 KSI (68.97 KN/mm^2). The problem was solved by the proposed technique with 4 and 8-submembers. The resulting load-displacement curves for large displacements analysis, are compared with the corresponding solutions given by ref. (9). These curves are presented in Figure (10).

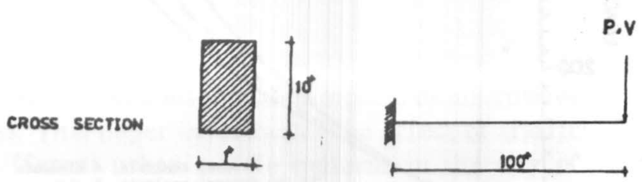

Figure 7.

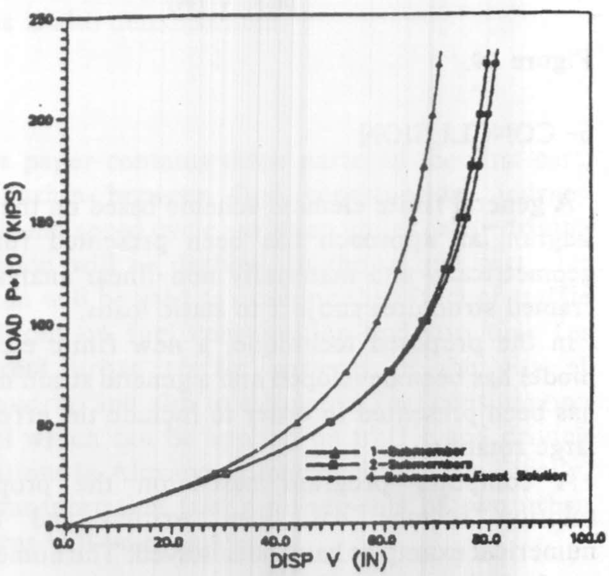

Figure 8.

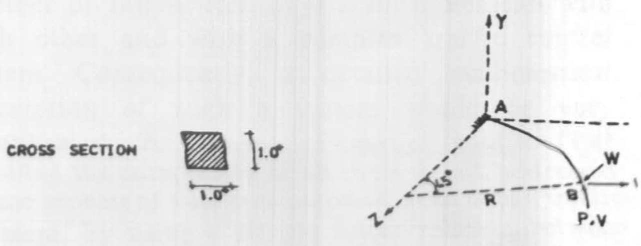

Figure 9.

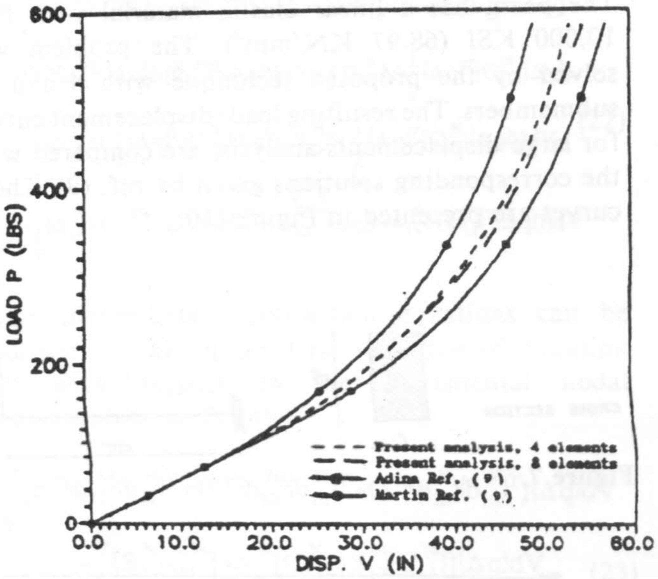

Figure 10.

6- CONCLUSION

A general finite element scheme based on the total Lagrangian approach has been presented for the geometrically and materially non-linear analysis of framed structures subject to static loads.

In the proposed technique, a new finite element model has been developed and a general strain model has been presented in order to include the effect of large rotation.

A computer program based on the proposed solution scheme has been written and three numerical examples have been solved. The numerical results indicate the effectiveness and accuracy of the proposed technique in solving small and large rotation framed structures.

Appendix I. References

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