A CONFORMAL MAPPING-MARCHING NUMERICAL METHOD FOR LAMINAR FLOW IN THE ENTANCE LENGTH-APPLICATION TO ELLIPTIC AND ANNULAR ECCENTRIC CIRCULAR DUCTS

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ABSTRACT

A conformal mapping-marching numerical method is proposed to solve the laminar hydrodynamic entrance length problem in a straight duct. The cross section of the duct with curved contour is transformed,using the conformal mapping,into a rectangular shape. The mass and momentum conservation equations in the original plane are re-expressed in terms of the transverse velocity components along the new coordinate system in the transformed plane. The resulting equations casted in a finite difference form were solved using the Patankar-Spalding marching technique. The method was applied to calculate the axial pressure drop in the entrance of elliptic and annular eccentric circular ducts. The novelty of the method is the use of the finite difference numerical technique in the case of curved boundary ducts without the need to very fine large size grids.

NOMENCLATURE

a,b	Ellipse semi-major and semi-minor axes.
D_H	Hydraulic diameter.
e	Eccentricity of annular eccentric circular duct.
f	Transformation function .
h	Transformation parameter.
k	Geometric scaling factor.
L_e	Dimensionless entrance length for pressure development.
l_e	Pressure developing entrance length.
P	Dimensionless pressure.
p	Pressure.
p_O	Duct inlet pressure.
Re	Reynolds number.
$f.\ Re_{f\ D}$	Fully developed Friction factor* Reynolds number.
R_O,R_i	Outer and inner radii of annular duct.
u,v	Transverse velocity components in x-y plane.
W	Dimensionless axial velocity component in z direction.
w	Axial velocity component in z direction.
w_O	Average velocity of flow.
x,y	Transverse coordinates in x-y plane.
Z	Dimensionless axial coordinate.
z	Axial coordinate.
Ξ,H	Dimensionless transverse coordinates in $\xi - \eta$ plane.
ξ,η	Transverse coordinates in $\xi - \eta$ plane.
Ψ,Φ	Dimensionless transverse velocity components in $\xi - \eta$ plane.
ψ,ϕ	Transverse velocity components in $\xi - \eta$ plane.
Θ	Angle of rotation.
μ	Viscosity.
ρ	Density.
ϵ	Aspect ratio.

INTRODUCTION

The mathematical handling of the hydrodynamic flow in the entrance length of a duct is complicated by the presence of transverse velocities due to the flow development. The complexity further increases if the transverse field is of the two dimensional type. The mathematical structure of such problems is founded either on the boundary layer idealization or the full Navier-Stokes equations. Boussinesq [1] divided the flow field into two regions. The first is near the duct entrance,where the boundary layer is growing on the duct wall and the fluid in the core is accelerating,while the second is adjecent to the fully developed part. He described the flow field by perturbations imposed on the Blasius solution in the first and on the fully developed solution in the second. The two solutions are then joined smoothly or matched at an axial location between them. The above approach was used by [2,3] for the parallel gap, by [4] for the circular tube,and [5,6] improved the solution of [2] by considering more terms in the expansion. Later,[7] proved that the series solution of [2] applies only downstream from the entrance section and proposed new series solution. The improvement of the work of [2] was further extended in [8] by employing a second order boundary layer analysis. The Von Karman-Pohlhausen integral method,with a parabolic velocity profile in the boundary layer and the Bernoulli's equation in the inviscid core was employed to predict the flow inside the circular tube and the parallel gap [9]. Cubic and quartic profiles were tried in [10,11] to refine the analysis of [9], and the logarithmic profile was checked in the circular tube [12]. The effects of the viscous forces in the entire flow field was taken into account by replacing the Bernoulli's equation by the mechanical energy equation [13]. The parabolic profile was used to determine the characteristics of the flow field in the circular tube [14] and in the parallel gap [15], and the power law profiles in the parallel gap [16]. The integral method was also used to tackle the full set of the Navier-Stokes equations in the circular tube and parallel gap [17]. The linearization of the nonlinear inertia terms in the momentum boundary layer equation,having only one independent transverse coordinate,was introduced by Langhaar [18] , Targ [19,20], and Sparrow et al. [21]. The simplification enables the analytic solution of the momentum boundary layer equation. The linearization model [19,20] provides slower flow development near the duct entrance. One advantage of the linearization technique is the absence of discontinuities in the velocity and pressure fields. The linearization [18] was employed by [22,23] for the parallel gap and the equilateral triangle. The annular duct was solved by [24,25,26] using the linearization [18],by [27,28] using the linearization [19,20],and by [29,30] using the linearization [21]. The extension of the above linearization schemes to include situations with two independent transverse coordinates can be found in [31,32,33]. The linearization method is restricted to simple flow situations without geometric asymmetries and strong property variations. Linearization of the complete Navier-Stokes equations can be found in the parallel gap case by [34].

The utilization of the numerical techniques eliminates the need for velocity profiles or linearization schemes to integrate the governing equations. According to the flow situation,the duct geometry,and the computational accuracy,the governing equations can be mathematically formulated both transversely and axially parabolic,transversely elliptic-axially parabolic,or both transversely and axially elliptic. The axially parabolic formulations were casted in a finite difference form and solved by the marching technique for the parallel gap [35],circular tube [36,37,38],rectangular duct [39,40],square duct [41,42] ,concentric annular circular duct [43,44,45,46,47] ,and eccentric annular circular duct [48]. The elliptic- elliptic formulation was solved by an iterative scheme in the case of the circular tube [49,50,51,52],the parallel gap [53,54,55, 56,57],and the concentric annular circular duct [58].

Ducts with curved boundaries or geometric asymmetries as the elliptic and the annular eccentric circular ducts,require fine calculation grids to obtain accurate results with the finite difference method. If a transformation can be sought which renders the boundaries straight,then the number of grid points can be reduced. In order to demonestrate this idea,the boundaries of the elliptic and annular eccentric circular ducts were made straight and the

THEORETICAL ANALYSIS

The equations governing the transport of mass and momentum of a constant property fluid flowing without dissipation through the entrance of a uniform cross sectional area duct are:-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = O \quad (1)$$

$$\rho [u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}] = -\frac{\partial p}{\partial x} + \mu [\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}] \quad (2)$$

$$\rho [u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}] = -\frac{\partial p}{\partial y} + \mu [\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}] \quad (3)$$

$$\rho [u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}] = -\frac{\partial p}{\partial z} + \mu [\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}] \quad (4)$$

The boundary conditions of the above set of equations are:-

$$u = v = w = O \quad \text{,on the duct wall} \quad (5)$$

$$u = v = O, w = w_O, p = p_O \; z = O \quad (6)$$

$$u = v = O \quad , \quad \frac{\partial w}{\partial z} = O \quad z > l_e \quad (7)$$

The above set of equations has to be solved numerically. In the case of a curved duct, it is convenient to make the duct sides straight and perpendicular by a special transformation. If the transformation function $(x + iy) = f(\eta + i\xi)$ is analytic over the duct cross section, then the transformation is conformal. A total component of the velocity vector V in the transverse plane x-y and inclined an angle α to the x-axis, when represented on the ξ-η plane will make an angle $(\alpha + \Theta)$ to the ξ-axis. The angle of rotation Θ equals the argument of the gradient of the transformation function f, i. e. $\Theta = \arg. f'$. The transverse velocity components u and v of V along the axes x and y are related to the components ψ and ϕ of V along the axes ξ and η by:

$$u = \psi \cos\Theta + \phi \sin\Theta$$
$$V = -\psi \sin\theta + \phi \sin\theta$$

The above two equations together with the transformation fuction f, can be used to obtain the following transformed version of the flow governing equations:

$$\frac{\partial \psi}{\partial \xi} + \frac{\partial \phi}{\partial \eta} + k\frac{\partial w}{\partial z} - \psi\frac{\partial \Theta}{\partial \eta} - \phi\frac{\partial \Theta}{\partial \xi} = O \quad (8)$$

$$\rho [\psi\frac{\partial \psi}{\partial \xi} + \phi\frac{\partial \psi}{\partial \eta} + kw\frac{\partial \psi}{\partial z} + \phi^2\frac{\partial \Theta}{\partial \eta} + \psi\phi\frac{\partial \Theta}{\partial \xi}] =$$
$$-\frac{\partial p}{\partial \xi} + \frac{\mu}{k}[\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} + 2\frac{\partial \Theta}{\partial \xi}\frac{\partial \phi}{\partial \xi} + 2\frac{\partial \Theta}{\partial \eta}\frac{\partial \phi}{\partial \eta}$$
$$-\psi[(\frac{\partial \Theta}{\partial \xi})^2 + (\frac{\partial \Theta}{\partial \eta})^2]] \quad (9)$$

$$\rho [\psi\frac{\partial \phi}{\partial \xi} + \phi\frac{\partial \phi}{\partial \eta} + kw\frac{\partial \phi}{\partial z} - \psi^2\frac{\partial \Theta}{\partial \xi} - \psi\phi\frac{\partial \Theta}{\partial \eta}] =$$
$$-\frac{\partial p}{\partial \eta} + \frac{\mu}{k}[\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} - 2\frac{\partial \Theta}{\partial \xi}\frac{\partial \psi}{\partial \xi} - 2\frac{\partial \Theta}{\partial \eta}\frac{\partial \psi}{\partial \eta}$$
$$-\phi[(\frac{\partial \Theta}{\partial \xi})^2 + (\frac{\partial \Theta}{\partial \eta})^2]] \quad (10)$$

$$\rho [\psi\frac{\partial w}{\partial \xi} + \phi\frac{\partial w}{\partial \eta} + kw\frac{\partial w}{\partial z}] = -k\frac{\partial p}{\partial z} + \frac{\mu}{k}[\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2}] \quad (11)$$

where k is a geometric scaling factor equals $\sqrt{(\frac{\partial x}{\partial \xi})^2 + (\frac{\partial x}{\partial \eta})^2}$. The group of equations(8) through (11) can be casted in a dimensionless form by introducing the following dimensionless variables:

$$\Psi = \frac{\psi}{w_O}, \Phi = \frac{\phi}{w_O}, W = \frac{w}{w_O}, P = \frac{(p - p_O)}{\rho w_O^2}$$
$$, \Xi = \frac{\xi}{D_H}, H = \frac{\eta}{D_H}, Z = \frac{z}{D_H k}$$

The new set is:

$$\frac{\partial \Psi}{\partial \Xi} + \frac{\partial \Phi}{\partial H} + \frac{\partial W}{\partial Z} - \Psi \frac{\partial \Theta}{\partial H} - \Phi \frac{\partial \Theta}{\partial \Xi} = O \qquad (12)$$

$$\Psi \frac{\partial \Psi}{\partial \Xi} + \Phi \frac{\partial \Psi}{\partial H} + W \frac{\partial \Psi}{\partial Z} + \Phi^2 \frac{\partial \Theta}{\partial H} + \Psi \Phi \frac{\partial \Theta}{\partial \Xi} =$$

$$-\frac{\partial P}{\partial \Xi} + \frac{1}{Re}\left[\frac{\partial^2 \Psi}{\partial \Xi^2} + \frac{\partial^2 \Psi}{\partial H^2} + 2\frac{\partial \Theta}{\partial \Xi}\frac{\partial \Phi}{\partial \Xi} + 2\frac{\partial \Theta}{\partial H}\frac{\partial \Phi}{\partial H} \qquad (13)\right.$$

$$\left. -\Psi\left[\left(\frac{\partial \Theta}{\partial \Xi}\right)^2 + \left(\frac{\partial \Theta}{\partial H}\right)^2\right]\right]$$

$$\Psi \frac{\partial \Phi}{\partial \Xi} + \Phi \frac{\partial \Phi}{\partial H} + W \frac{\partial \Phi}{\partial Z} - \Psi^2 \frac{\partial \Theta}{\partial \Xi} - \Psi \Phi \frac{\partial \Theta}{\partial H} =$$

$$-\frac{\partial P}{\partial H} + \frac{1}{Re}\left[\frac{\partial^2 \Phi}{\partial \Xi^2} + \frac{\partial^2 \Phi}{\partial H^2} - 2\frac{\partial \Theta}{\partial \Xi}\frac{\partial \Psi}{\partial \Xi} - 2\frac{\partial \Theta}{\partial H}\frac{\partial \Psi}{\partial H}\right.$$

$$\left. -\Phi\left[\left(\frac{\partial \Theta}{\partial \Xi}\right)^2 + \left(\frac{\partial \Theta}{\partial H}\right)^2\right]\right] \qquad (14)$$

$$\Psi \frac{\partial W}{\partial \Xi} + \Phi \frac{\partial W}{\partial H} + W \frac{\partial W}{\partial Z} = -\frac{\partial P}{\partial Z} + \frac{1}{Re}\left[\frac{\partial^2 W}{\partial \Xi^2} + \frac{\partial^2 W}{\partial H^2}\right] \qquad (15)$$

The Reynolds number $Re = \frac{(\rho \; w_O \; D_H \; k)}{\mu}$, includes the geometric factor k, which is a function of the transverse location. The dimensionless transformed version of the boundary conditions (5) through (7) is:

$$\Psi = \Phi = W = O \text{ on the solid boundaries} \qquad (16)$$

$$\Psi = \Phi = O, W = 1, P = O \quad Z = O \qquad (17)$$

$$\Psi = \Phi = O, \frac{\partial W}{\partial Z} = O, Z > L_e = \frac{l_e}{(D_H k)} \qquad (18)$$

The above method will be checked for flow situations inside the elliptic and annular eccentric circular ducts. The transformations appropriate for these ducts and the corresponding geometric parameters are summarized below.

The Elliptic Duct:

The transformation $x + iy = h \sin(\eta + i\xi)$ will change the ellipse curved contour $(x^2/a^2 + y^2/b^2) = 1$ in the x−y plane into a straight line ($\xi = \xi_O$) in $\xi - \eta$ plane, Figure (1). Where "h" is the semi focal distance $(a^2 - b^2)^{1/2}$ and ξ_O equals $\tanh^{-1}(b/a)$. The transverse velocity vector rotation angle Θ and the geometric scaling factor k in the case of the ellipse are respectively :

$$\Theta = \arctan(\coth\xi \; \cot\eta)$$

$$\text{and} \quad k = \frac{h}{D_H}\sqrt{\sinh^2\xi + \cos^2\eta}$$

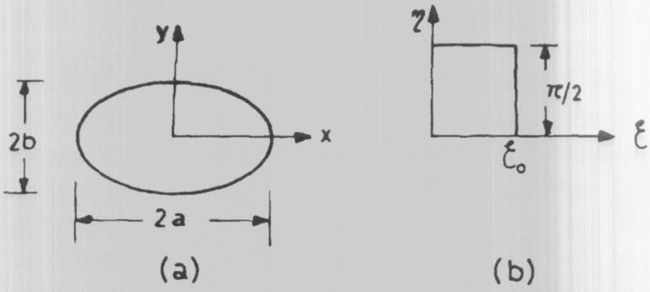

Figure 1. The transformation of the elliptical duct.

The Annular Eccentric Circular Duct:

The transformation:

$$(x + iy) = h\cot\frac{(\eta + i\xi)}{2}$$

will change the duct subtened between the two eccentric circles in Figure (2-a), into the rectangle in Figure (2-b), where:

$$h = \frac{1}{2e}\left[(R_O^2 + R_i^2 - e^2)^2 - 4R_O^2 R_i^2\right]^{1/2}$$

The transformed boundaries of the duct are represented in the new plane by the following straight lines:

$$\xi_o = \sinh^{-1}(-\frac{h}{R_o}), \xi_i = \sinh^{-1}(-\frac{h}{R_i})$$

The eccentricity of the duct is related to the above variables by:

$$e = -h [\coth \xi_o - \coth \xi_i]$$

The Θ and k values for this geometry are:

$$\Theta = \arctan [\frac{\cosh \xi \cos \eta - 1}{\sinh \xi \sin \eta}]$$

and

$$k = \frac{h/D_H}{[\cosh \xi - \cos \eta]}$$

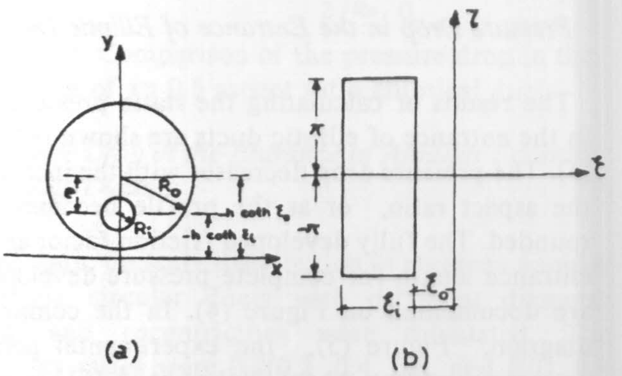

Figure 2. The transformation of the annular eccentric circular duct.

RESULTS AND DISCUSSIONS

The group of equations (12) through (15) subjected to the boundary conditions, equations(16), (17) and (18), were casted in a finite difference form and solved using the marching technique described in [40]. The derivatives $\partial\Theta/\partial\Xi$ and $\partial\Theta/\partial H$ as well as the Reynolds number, which contains the factor k, depend on the transverse location only, and can be calculated once and for all, at all grid points, from the geometric relations of a given duct shape. The extra terms, appearing in the group (12) through (15) , other than those in a standard form, as for example, equations (1) through (4), will be handled as source terms in the solving program. The gradients of the velocity components Ψ and Φ in the source terms will be calculated from the slope of a quadratic polynomial fitted through the respective grid point and the two neighbouring points. At a boundary point, the gradients are to be obtained from the slope of a cubic polynomial. In the transverse plane, the size of the computational grid was 80x80. The grid points were uniformly distributed. The accuracy of the numerical computation was increased by benefitting from the geometric symmetry of the ellipse and the annular eccentric ducts. The calculation domain is one quarter of duct cross section in the first and one half in the latter. The axial step $\Delta Z/D_H Re$ was uniform and equals 10^{-5}. The axial derivatives were first computed with respect to the axial increment $\Delta Z/D_H$, which is the same at all transverse points, and are then scaled by the factor k before being used in the finite difference balance equations. So long as the curved duct geometries will be changed in this method to rectangular shapes, it seemed essential to compare the pressure drop predictions obtained by the marching technique with the available data in the literature. The static pressure drop in three rectangular ducts having aspect ratios 0.2 , 0.5 , and 1.0 were calculated and plotted in Figure (3). The experimentally measured pressures by Beaver et al. [59] are very close to those calculated in the present work.

Table 1. Comparison of the fully developed friction factor in rectangular ducts.

Rectangular duct aspect ratio	f. Re beavers et al. [59]	f. Re present work	Percentage difference
1. 00	56. 908	56. 895	-0. 0228
0. 50	62. 192	62. 182	-0. 0160
0. 25	76. 282	76. 273	-0. 0110

The comparison in Table (1) reveals that the experimental fully developed friction factor times the Reynolds number are higher than the numerical ones by only 0.02%. The Langhaar's linearization technique used by Han [60] predicts faster flow development in the three ducts. The pressure distribution computed by the linearization method devised by Fleming and Sparrow [32] is in excellent agreement with the experiments [59] and the difference can not be distinguished on the graph, while those of Wiginton [31] lies between Han [60] and the experiments [59]. The pressure drop of Curr et al. [40], which was obtained by the finite difference method assuming a uniform velocity at inlet, is also in excellent agreement with the experiments [59]. The foregoing argument gives support to the marching technique as an appropriate method for calculating the flow in the entrance of rectangular ducts. It also revealed that some of the approximate methods always overestimate the pressure drop in the entrance of these ducts.

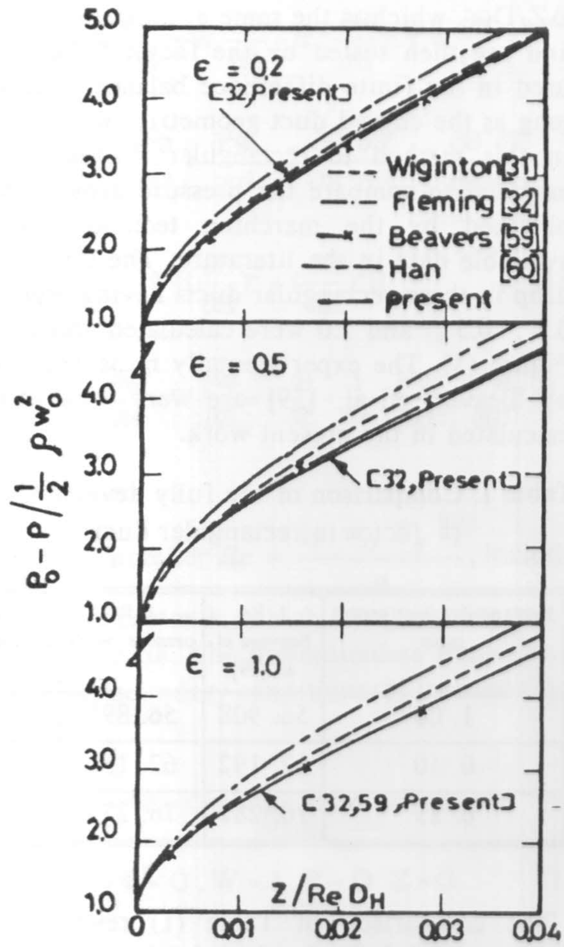

Figure 3. The static pressure. Distribution in rectangular ducts having aspect ratios 0.2, 0.5 and 1.0.

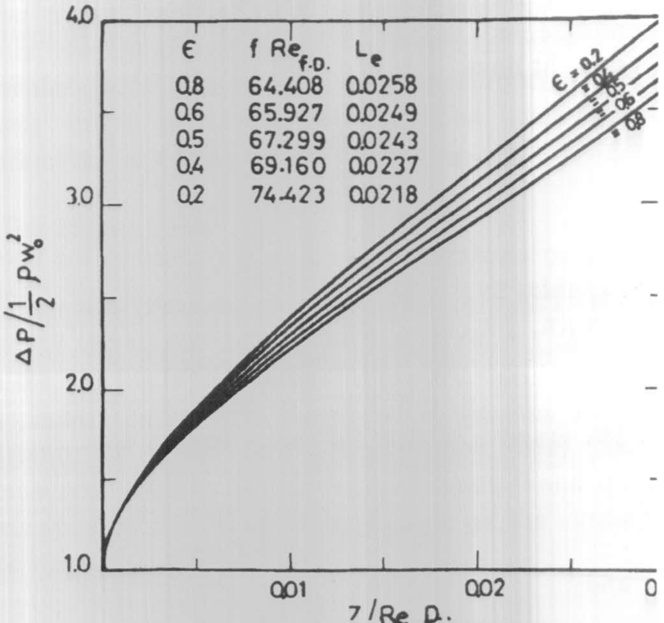

Figure 4. Pressure drop in the entrance of elliptical ducts.

Pressure Drop in the Entrance of Elliptic Ducts:

The results of calculating the static pressure drop in the entrance of elliptic ducts are shown in Figure (4). The pressure drop decreases with the increase of the aspect ratio, or as the profile becomes more rounded. The fully developed friction factor and the entrance length for complete pressure development are documented on Figure (4). In the comparison diagram, Figure (5), the experimental pressure drop measured by Abdel-Wahed et al. [61] in an 0.5 aspect ratio duct is higher than those calculated by Bhatti [62] and by the present investigators. The experimental set-up [61] incorporates a large thin flat plate attached to the test section at its entrance. Therefore, the fluid inlet velocity in the elliptic duct is expected to be non-uniform. A vena-contracta will establish near the duct inlet. The pressure decrement due to such contraction in a circular tube amounts approximately to one half dynamic head. In an elliptic duct, the loss is expected to be more than one half. The constant difference between the experimental pressure [61] and that computed in the present investigation in the fully developed regime is about 0.7 times the dynamic head. Consequently, one can claim that the present calculation method, based on a rigorous numerical analysis, predicts accurately the development of pressure in the entrance of elliptical ducts. On the contrary, the approximate integral approach due to Bhatti [62] founded on an assumed velocity profile in the wall boundary layer overpredicts the pressure in the developing part.

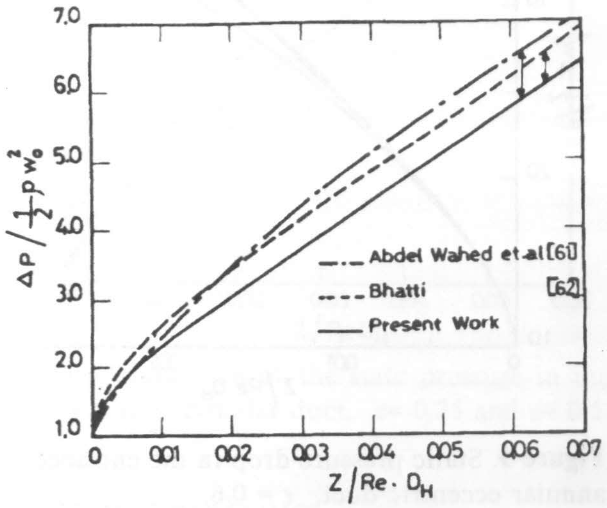

Figure 5. Comparison of the pressure drop in the entrance of an 0.5 aspect ratio elliptical duct.

Pressure Drop in the Entrance of Annular Eccentric Circular Ducts:

The static pressure distribution in eighteen annular eccentric circular ducts with different diameter ratios and eccentricities were calculated. The diameter ratios were 0.2(0.2 ,0.4, 0.6, and 0.7), 0.4 (0.1, 0.2, 0.4, and 0.5), 0.5 (0.1, 0.2, 0.25, and 0.4), 0.6 (0.1, 0.2, and 0.3), and 0.8 (0.05, 0.1, and 0.15). The numbers in brackets are the eccentricities corresponding to these diameter ratios. The pressures were plotted in Figures (6) through (10). On the same figures, the fully developed friction factors and the entrance lengths necessary for the pressure development are written. For a given diameter ratio, the fully developed friction factor, on one hand, decreases with the increase of the eccentricity and on the other, the entrance length increases with it. The friction factor and entrance length show similar behaviour with the diameter ratio if the eccentricity is kept constant. The pressure drop in the duct having diameter ratio and eccentricity 0.5 and 0.25 respectively, is compared with that obtained by Feldman [48]. The comparison is shown graphically in Figure (11). The pressure prediction of Feldman lies above that found in this research. Feldman in his analysis postulates that the transverse flow in the entrance length emerges radially from the two cylinders forming the annular duct. The present method does not contain such hypotheses. Unfortunately, our pressure drop in the annular eccentric circular duct could not be compared with experimental results due to the lack of this information in the literature. Despite this lack, one can relay, at least at the present time, on the results of this investigation to estimate the laminar pressure drop in the entrance of annular eccentric circular ducts. The reported friction factor [63] in an annular concentric circular duct, of a specific diameter ratio, is larger than that in an annular eccentric duct having the same ratio. The friction factor decreases as the inner cylinder moves off the center of the outer one. The hydrodynamic entrance length for the velocity development is slightly larger than that required for the pressure development. The velocity entrance length of Feldman in the eccentric duct is about ten times that obtained by Roy [28] , Sparrow [29] , Liu [45], and Coney [47] in the concentric duct. The pressure developing entrance length in the eccentric ducts of this investigation are of the same order of magnitude as the velocity entrance lengths computed by [28, 29, 45, 47].

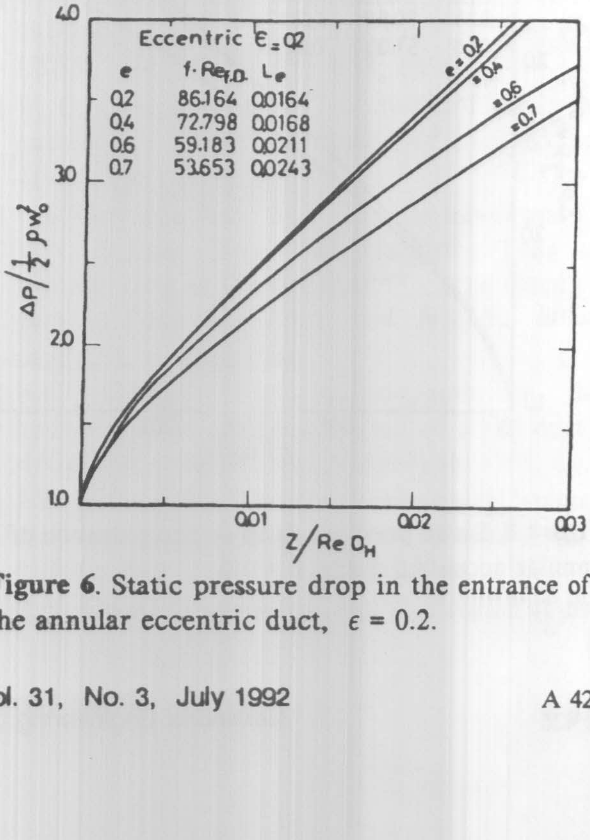

Figure 6. Static pressure drop in the entrance of the annular eccentric duct, $\epsilon = 0.2$.

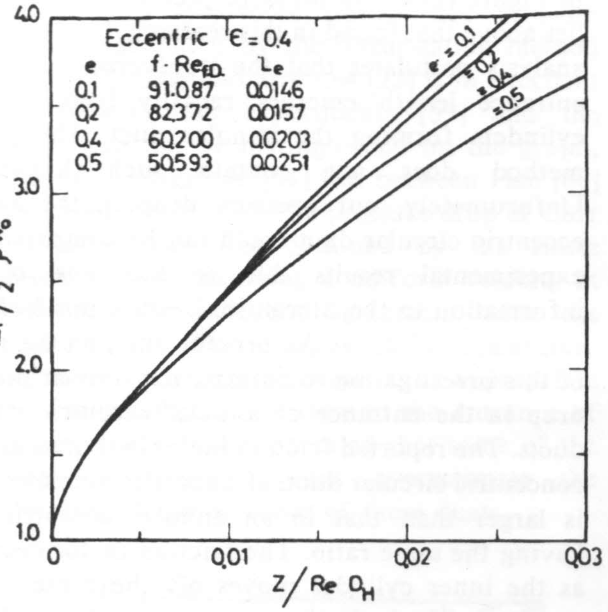

Figure 7. Static pressure drop in the entrance of the annular eccentric duct, ϵ = 0.4.

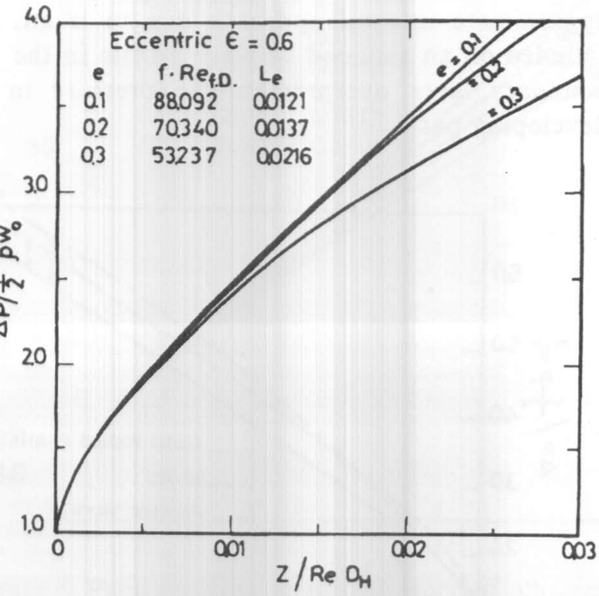

Figure 9. Static pressure drop in the entrance of the annular eccentric duct, ϵ = 0.6.

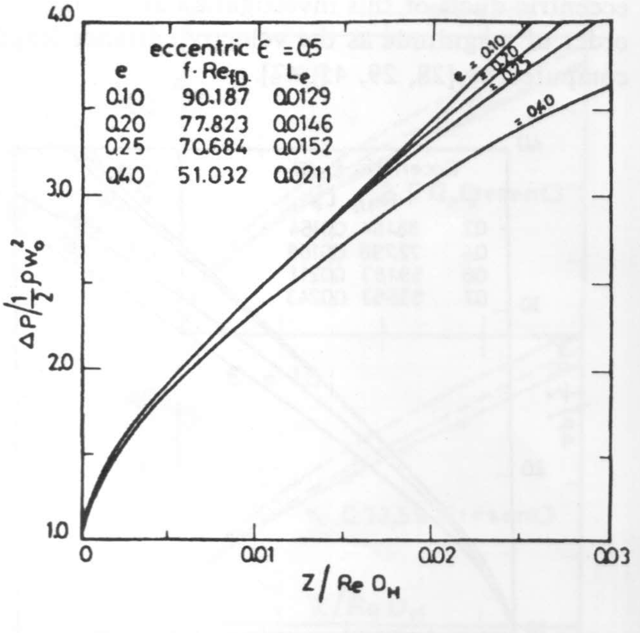

Figure 8. Static pressure drop in the entrance of the annular eccentric duct, ϵ = 0.5.

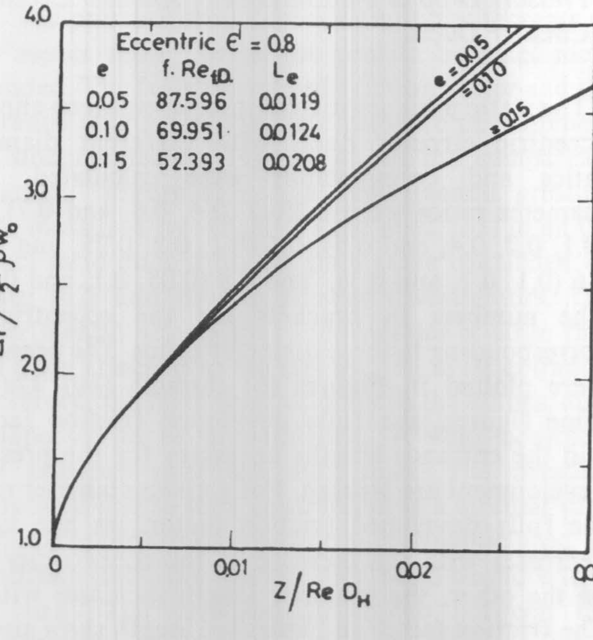

Figure 10. Static pressure drop in the entrance of the annular eccentric duct, ϵ = 0.8.

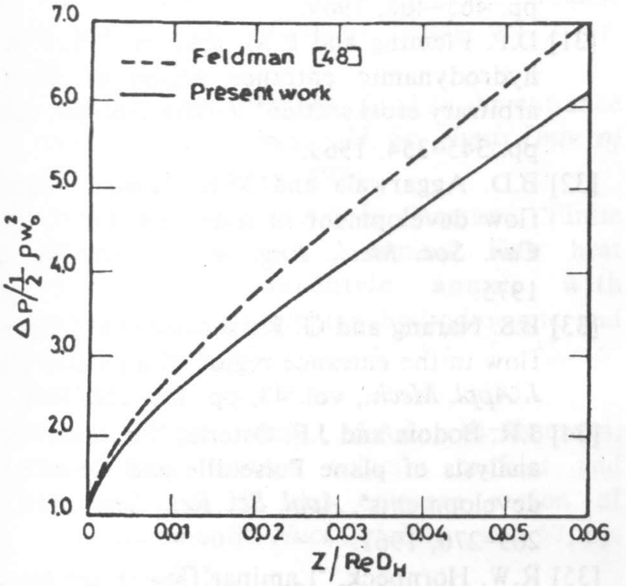

Figure 11. Comparison of the statc pressure in the annular eccentric circular duct, e= 0.25 and ε= 0.5.

CONCLUDING REMARKS

1- The problem of laminar flow in the entrance of a straight duct with curved boundaries can be conveniently solved by the present method.
2- The novelty of the method is the use of the finite difference numerical technique in the case of curved boundary ducts without the need to very fine large size grids.
3- The pressure drop in the entrance of an elliptical duct increases with the decrease of the duct aspect ratio.
4- The pressure drop in the entrance of an annular eccentric circular duct decreases with the eccentricity and increases with the diameter ratio.

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