STATE SPACE APPROACH TO GENERALIZED THERMOELASTIC PROBLEMS WITH A HEAT SOURCE

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ABSTRACT

In this work the state space formulation for generalized thermoelastic one dimensional problems with a heat source is presented. The technique is applied to a half-space problem with a plane distribution of heat sources on its boundary. Numerical results for the temperature, displacement and stress distributions are given and illustrated graphically.

NOMENCLATURE

λ,μ Lame's constants
α_t coefficient of linear thermal expansion
β $= [(\lambda+2\mu)/\mu]^{1/2}$
γ $= (3\lambda+2\mu)\alpha_t$
ρ density
V = velocity of propagation of longitudinal waves = $[(\lambda+2\mu)/\rho]^{1/2}$
σ' $= \sigma'_{xx}$ component of the stress tensor in x'-direction
u' component of displacement in x'-direction
c_E specific heat at constant strain
k thermal conductivity
δ $= \rho c_E/k$
t' time
T absolute temperature
T_o reference temperature chosen so that $|T - T_o| << 1$
b $= \gamma T_o/\mu$
g $= \gamma/\rho c_E$
τ_o relaxation time
q heat flux
Q intensity of the applied heat source per unit volume

INTRODUCTION

The theory of generalized thermoelasticity with one relaxation time was introduced by Lord and Shulman [1] for the special case of an isotropic body not subject to the effect of heat sources. This theory was extended [2] by Dhaliwal and Sherief to include both the effects of anisotropy and the presence of heat sources. In this theory a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associated with this theory is a hyperbolic one and hence automatically eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and the coupled theories of thermoelasticity. For many problems involving steep heat gradients and when short time effects are sought this theory is indispensable.

Due to the complexity of the partial differential equations of this theory the work done in this field is unfortunately limited in number. Among the theoretical contributions to the subject are the proofs of uniqueness theorems under different conditions by Ignaczak [3-4] and by sherief [5]. The state space formulation for problems not containing heat sources was done by Sherief and Anwar in [6] and the boundary element formulation was done by Anwar and Sherief in [7]. Some concrete problems have also been solved. The fundamental solutions for the spherically symmetric and the cylindrically symmetric spaces were obtained by Sherief [8] and by Sherief and Anwar [9], respectively. A two dimensional punch problem was considered by Sherief and Anwar in [10]. The same authors have also solved some problems involving cylindrical regions in [11] and [12].

In dealing with generalized or coupled thermoelastic problems the potential function approach is often used.

This is not always the most suitable approach. As was discussed in [13], this is mainly due to two reasons. The first is that it is preferable to formulate the problem in terms of the quantities with physical meaning since the boundary and initial conditions of the problem are related directly to these quantities. The second reason is that the solution for a physical problem is convergent while that of a potential function is ,unfortunately, not always so. The first to introduce the state space formulation in thermoelastic problems were Bahr and Hetnarski [13]. Their work dealt with coupled thermoelasticity in the absence of heat sources. This work was followed by the work of Sherief and Anwar [6] whose work dealt with generalized thermoelasticity when there are no heat sources. The present work is an attempt to generalize these results to include the effects of heat sources. The results obtained are used to solve a half-space problem with a plane distribution of heat sources on the boundary.

FORMULATION OF THE PROBLEM

We shall consider a homogeneous isotropic thermoelastic solid occupying the region $0 \leq x' \leq \infty$, whose state depends only on the space variable x' and the time variable t'. We shall also assume that the initial state of the medium is quiescent. The governing equations for generalized thermoelasticity with one relaxation time in the non-dimensional form consist of:

1. The equation of motion

$$\beta^2 \frac{\partial^2 u}{\partial x^2} - b\frac{\partial \theta}{\partial x} = \beta^2 \frac{\partial^2 u}{\partial t^2} \tag{1}$$

2. The generalized equation of heat conduction

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t} + \tau_o \frac{\partial^2 \theta}{\partial t^2} + g\left(\frac{\partial^2 u}{\partial x \partial t} + \tau_o \frac{\partial^3 u}{\partial x \partial^2 t}\right) - Q - \tau_o \frac{\partial Q}{\partial t} \tag{2}$$

3. The constitutive relations

$$\sigma = \beta^2 \frac{\partial u}{\partial x} - b\theta , \tag{3}$$

where in the above equations, we have used the following non-dimensional variables

$$x = V\,\delta\, x', \quad u = V\,\delta\, u', \quad t = V^2\,\delta\, t' ,$$

$$\tau_o = V^2\,\delta\,\tau_o', \sigma = \sigma'/\mu \quad \text{and } Q = \rho\, Q'/\,k\, T_o\, \delta^2\, V^2.$$

From now on we shall consider a heat source of the form

$$Q = Q_o\, \delta(x)\, H(t),$$

where $\delta(x)$ and $H(t)$ are the Dirac delta function and the Heaviside unit step function, respectively and Q_o is a constant.

Taking the Laplace transform with parameter s (denoted by a bar) of both sides of equations (1-3), we arrive at

$$\left(\frac{\partial^2}{\partial x^2} - s^2\right)\bar{u} = a\frac{\partial \bar{\theta}}{\partial x} , \tag{4}$$

$$\left(\frac{\partial^2}{\partial x^2} - s - \tau_o s^2\right)\bar{\theta} = g\, s\,(1 + \tau_o s)\frac{\partial \bar{u}}{\partial x} , \tag{5}$$

$$\bar{\sigma} = \beta^2 \frac{\partial \bar{u}}{\partial x} - b\,\bar{\theta} , \tag{6}$$

where $a = b/\beta^2$.

Choosing the state variables to be θ, u,θ' and u', where dashes denote differentiation with respect to x, equations (4-5) can be written as

$$\frac{\partial \bar{\theta}}{\partial x} = \bar{\theta}' , \tag{7}$$

$$\frac{\partial \bar{u}}{\partial x} = \bar{u}' , \tag{8}$$

$$\frac{\partial \bar{\theta}'}{\partial x} = (s + \tau_o s^2)\,\bar{\theta} + g(s + \tau_o s^2)\bar{u}' - Q_o\, \delta(x)\left(\frac{1 + \tau_o s}{s}\right), \tag{9}$$

$$\frac{\partial \bar{u}'}{\partial x} = s^2 \bar{u} + a \bar{\theta}'. \tag{10}$$

The above equations can be written in matrix form as

$$\frac{d \bar{v}(x,s)}{d x} = A(s) \bar{v}(x,s) + B(x,s) , \tag{11}$$

where

$$\bar{v}(x,s) = \begin{bmatrix} \bar{\theta}(x,s) \\ \bar{u}(x,s) \\ \bar{\theta}'(x,s) \\ \bar{u}'(x,s) \end{bmatrix},$$

$$A(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ s + \tau_o s^2 & 0 & 0 & g(s + \tau_o s^2) \\ 0 & s^2 & a & 0 \end{bmatrix}$$

and

$$B(x.s) = - Q_o\, \delta(x) \frac{(1 + \tau_o s)}{s} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

In order to solve the system (11) , we need first to find the form of the matrix exp(A(s) .x).

The characteristic equation of the matrix A has the form

$$k^4 - k^2 [(1+\epsilon)(s+\tau_o s^2) + s^2] + s^3 (1+\tau_o s) = 0 \tag{12}$$

where $\epsilon = g\, a$.

The roots of this equation, namely, k_1^2 and k_2^2 ,satisfy the relations

$$k_1^2 + k_2^2 = (1+\epsilon)(s+\tau_o s^2) + s^2 \tag{13-a}$$

$$k_1^2 k_2^2 = s^3 (1+\tau_o s) \tag{13-b}$$

The Taylor series expansion of the matrix exponential is

$$\exp[A(s) . x] = \sum_{n=0}^{\infty} \frac{1}{n!} [A(s) . x]^n. \tag{14}$$

Using the well known Cayley-Hamilton theorem, this infinite series can be truncated to

$$\exp [A(s).x] = a_0 (x,s) I + a_1 (x,s) A + a_2 (x,s) A^2 + a_3 (x,s) A^3. \tag{15}$$

To determine the coefficients a_0 - a_3 , we use Cayley-Hamilton theorem again to obtain

$$\exp (k_1.x) = a_0 + a_1 k_1 + a_2 k_1^2 + a_3 k_1^3 ,$$

$$\exp (- k_1.x) = a_0 - a_1 k_1 + a_2 k_1^2 - a_3 k_1^3 ,$$

$$\exp (k_2.x) = a_0 + a_1 k_2 + a_2 k_2^2 + a_3 k_2^3 ,$$

$$\exp (- k_2.x) = a_0 - a_1 k_2 + a_2 k_2^2 - a_3 k_2^3 .$$

The solution of the above system is given by

$$a_o = \frac{k_1^2 \cosh(k_2 x) - k_2^2 \cosh(k_1 x)}{k_1^2 - k_2^2},$$

$$a_1 = \frac{(k_1^2/k_2) \sinh(k_2 x) - (k_2^2/k_1) \sinh(k_1 x)}{k_1^2 - k_2^2},$$

$$a_2 = \frac{\cosh(k_1 x) - \cosh(k_2 x)}{k_1^2 - k_2^2}, \tag{16}$$

$$a_3 = \frac{k_2 \sinh(k_1 x) - k_1 \sinh(k_2 x)}{k_1 k_2 (k_1^2 - k_2^2)} .$$

Substituting the expressions (16) into (15) and computing A^2 and A^3 , we obtain after some lengthy algebraic manipulations, $\exp [A(s).x] = L(x,s) = [l_{ij}(x,s)], i,j = 1,2,3,4$

where the entries $l_{ij}(x,s)$ are given by

$$l_{11} = \frac{(k_1^2 - s - \tau_o s^2)\cosh k_2 x - (k_2^2 - s - \tau_o s^2)\cos}{k_1^2 - k_2^2}$$

$$l_{12} = \frac{g s^3 (1 + \tau_o s)[k_2 \sinh k_1 x - k_1 \sinh k_2 x]}{k_1 k_2 (k_1^2 - k_2^2)}.$$

$$l_{13} = \frac{k_2 (k_1^2 - s^2)\sinh k_1 x - k_1 (k_2^2 - s^2)\sinh k_2 x}{k_1 k_2 (k_1^2 - k_2^2)},$$

$$l_{14} = \frac{g s (1 + \tau_o s)(\cosh k_1 x - \cosh k_2 x)}{k_1^2 - k_2^2},$$

$$l_{21} = \frac{a s (1 + \tau_o s)(k_2 \sinh k_1 x - k_1 \sinh k_2 x)}{k_1 k_2 (k_1^2 - k_2^2)},$$

$$l_{22} = \frac{(k_1^2 - s^2)\cosh k_2 x - (k_2^2 - s^2)\cosh k_1 x}{k_1^2 - k_2^2},$$

$$l_{23} = \frac{a(\cosh k_1 x - \cosh k_2 x)}{k_1^2 - k_2^2},$$

$$l_{24} = \frac{k_2 (k_1^2 - s - \tau_o s^2)\sinh k_1 x - k_1 (k_2^2 - s - \tau_o s^2)\sinh k_2 x}{k_1 k_2 (k_1^2 - k_2^2)}, \quad (17)$$

$$l_{31} = \frac{s(1 + \tau_o s)[k_2 (k_1^2 - s^2)\sinh k_1 x - k_1 (k_2^2 - s^2}{k_1 k_2 (k_1^2 - k_2^2)}$$

$$l_{32} = \frac{g s^3 (1 + \tau_o s)(\cosh k_1 x - \cosh k_2 x)}{k_1^2 - k_2^2},$$

$$l_{33} = \frac{(k_1^2 - s^2)\cosh k_1 x - (k_2^2 - s^2)\cosh k_2 x}{k_1^2 - k_2^2},$$

$$l_{34} = \frac{g s (1 + \tau_o s)(k_1 \sinh k_1 x - k_2 \sinh k_2 x)}{k_1^2 - k_2^2},$$

$$l_{41} = \frac{a s (1 + \tau_o s)(\cosh k_1 x - \cosh k_2 x)}{k_1^2 - k_2^2},$$

$$l_{42} = \frac{s^2 [k_2 (k_1^2 - s - \tau_o s^2)\sinh k_1 x - k_1 (k_2^2 - s - \tau_o s}{k_1 k_2 (k_1^2 - k_2^2)}$$

$$l_{43} = \frac{a(k_1 \sinh k_1 x - k_2 \sinh k_2 x)}{k_1^2 - k_2^2},$$

$$l_{44} = \frac{(k_1^2 - s - \tau_o s^2)\cosh k_1 x - (k_2^2 - s - \tau_o s^2)\cosh k}{k_1^2 - k_2^2}$$

It should be noted here that we have used equations (13a) and (13b) repeatedly in order to write these entries in the simplest possible form. It should also be noted that this is a formal expression for the matrix exponential. In the physical problem $0 \le x \le \infty$, we should suppress the positive exponentials which are unbounded at infinity. Thus we should replace each sinh(kx) by $-\frac{1}{2}\exp(-kx)$ and each cosh(kx) by ½ exp(-kx).

We return now to system (11) whose formal solution can be written in the form

$$\bar{v}(x,s) = \exp(A(s).x)\left[\bar{v}(0,s) + \int_0^x \exp(-A(s).z)\, B(z,s)\, dz\right] \quad (18)$$

Using the integral property of the Dirac Delta function, namely

$$\int_0^x \delta(z) f(z)\, dz = \tfrac{1}{2} f(0) ,$$

equation (18) takes the form

$$\bar{v}(x,s) = L(x,s)\ [\bar{v}(0,s) + H(s)] , \quad (19)$$

where

$$H(s) = \frac{-Q_o(1+\tau_o s)}{2s}\begin{bmatrix} \dfrac{k_1 k_2 + s^2}{2 k_1 k_2 (k_1 + k_2)} \\ 0 \\ \tfrac{1}{2} \\ \dfrac{-a}{2(k_1+k_2)} \end{bmatrix}$$

Equation (18) gives the complete solution of the problem in the Laplace transform domain in terms of the boundary conditions of the problem represented by the vector $\bar{v}(0,s)$ and the applied heat source represented by H(s).

APPLICATION TO A HALF SPACE PROBLEM

We shall consider a half space subject to the following boundary conditions

(1) The surface x = 0 of the half-space is stress free, i.e.

$$\sigma(0,t) = 0. \quad \text{or } \bar{\sigma}(0,s) = 0 . \quad (20)$$

(2) The non-dimensional heat flow at the boundary is given by

$$q(0,t) = \tfrac{1}{2} H(t)\, Q_o , \text{ or } \bar{q}(0,s) = Q_o/2s . \quad (21)$$

Using the generalized Fourier's law of heat conduction, in the non-dimensional form, namely

$$q + \tau_o \dot{q} = -\frac{\partial \theta}{\partial x} ,$$

we get on taking Laplace transforms

$$\frac{\partial \bar{\theta}}{\partial x} = \frac{Q_o(1+\tau_o s)}{2s} . \quad (22)$$

This gives one of the four components of the initial state vector $\bar{v}(0,s)$. To find the remaining three components, we substitute the value x = 0 in both sides of equation (18) to get a system of equations whose solution is given by

$$\bar{\theta}(0,s) = \frac{Q_o a k_1 k_2 (k_1 + k_2)}{2s^2[s^2 + (1+\epsilon)k_1 k_2]} ,$$

$$\bar{u}(0,s) = \frac{-Q_o a k_1 k_2}{2s^2[s^2 + (1+\epsilon)k_1 k_2]} ,$$

$$\bar{\theta}'(0,s) = \frac{-Q_o(1+\tau_o s)}{2s} , \quad (23)$$

$$\bar{u}'(0,s) = \frac{Q_o a k_1 k_2 (k_1 + k_2)}{2s^2[s^2 + (1+\epsilon)k_1 k_2]} .$$

Substituting from equations (23) into equation (18), we obtain

$$\bar{\theta}(x,s) = \frac{Q_o k_1 k_2 [(k_1^2 - s^2) e^{-k_1 x} - (k_2^2 - s^2) e^{-k_2 x}]}{2s^2 (k_1 - k_2)[s^2 + (1+\epsilon)k_1 k_2]} , \quad (24)$$

$$\bar{u}(x,s) = \frac{Q_o a k_1 k_2 [k_1 e^{-k_1 x} - k_2 e^{-k_2 x}]}{2s^2(k_2 - k_1)[s^2 + (1+e)k_1 k_2]}. \qquad (25)$$

Substituting from equations (24) and (25) into equation (6), we get

$$\bar{\sigma}(x,s) = \beta^2 \frac{Q_o a k_1 k_2 [e^{-k_1 x} - e^{-k_2 x}]}{2(k_1 - k_2)[s^2 + (1+e)k_1 k_2]}. \qquad (26)$$

INVERSION OF THE LAPLACE TRANSFORM

In order to invert the Laplace transforms in equations (24)-(26) we shall use a numerical technique based on Fourier expansions of functions.

Let $\bar{g}(s)$ be the laplace transform of a given function g(t). The inversion formula of Laplace transforms states that

$$g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \bar{g}(s) ds,$$

where c is an arbitrary positive constant greater than all the real parts of the singularities of $\bar{g}(s)$. Taking $s = c + i y$, we get

$$g(t) = \frac{e^{ct}}{2\pi} \int_{-\infty}^{\infty} e^{ity} \bar{g}(c + i y) dy.$$

This integral can be approximated by

$$g(t) = \frac{e^{ct}}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikt\Delta y} \bar{g}(c + i k \Delta y) \Delta y.$$

Taking $\Delta y = \pi/t_1$, we obtain

$$g(t) = \frac{e^{ct}}{t_1} \left[\tfrac{1}{2} \bar{g}(c) + \mathrm{Re}\left(\sum_{k=1}^{\infty} e^{ik\pi t/t_1} \bar{g}(c + i k \pi/t_1) \right) \right]$$

For numerical purposes this is approximated by the function

$$g_N(t) = \frac{e^{ct}}{t_1} \left[\tfrac{1}{2} \bar{g}(c) + \mathrm{Re}\left(\sum_{k=1}^{N} e^{ik\pi t/t_1} \bar{g}(c + i k \pi/t_1) \right) \right], \qquad (27)$$

where N is a sufficiently large integer chosen such that

$$\frac{e^{ct}}{t_1} \mathrm{Re}\left[e^{iN\pi t/t_1} \bar{g}(c + i N \pi/t_1) \right] < \epsilon,$$

and ϵ is a preselected small positive number that corresponds to the degree of accuracy to be achieved. Formula (27) is the numerical inversion formula valid for $0 \le t \le 2 t_1$ [14]. In particular we choose $t = t_1$, getting

$$g_N(t) = \frac{e^{ct}}{t} \left[\tfrac{1}{2} \bar{g}(c) + \mathrm{Re}\left(\sum_{k=1}^{N} (-1)^k \bar{g}(c + i k \pi/t) \right) \right]. \qquad (28)$$

NUMERICAL RESULTS

The copper material was chosen for purposes of numerical evaluations. The constants of the problem were taken as

$$\epsilon = 0.0168, \; \beta^2 = 3.5 \text{ and } \tau_o = 0.02.$$

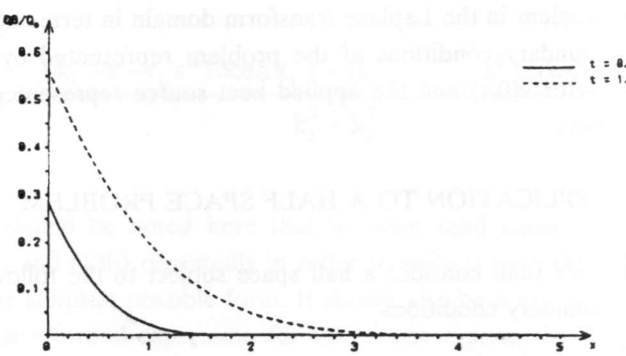

Figure 1. Temperature Distribution.

The computations were carried out for two values of time, namely for t = 0.25 and t = 1. Formula (28) was used to invert the Laplace transforms in equations (24) - (26) giving the functions $\theta(x,t)$, $u(x,t)$ and $\sigma(x,t)$, respectively. The results are illustrated graphically in Figures (1) and (3). These results show that the temperature increment θ decreases with increasing x for a given value of t while it increases with t for a fixed value of x. The graph of θ is shown in figure (1).

In Figure (2), the displacement distribution is drawn against x. It was found that u starts from a negative value at x = 0 and increases to reach a maximum positive value at a position given approximately by x = t. The value of u then decreases smoothly to reach zero.

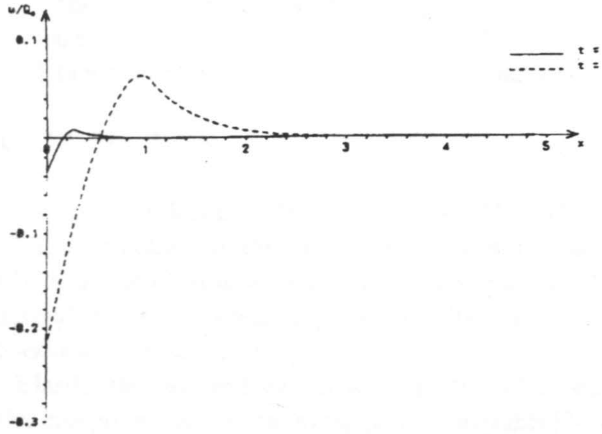

Figure 2. Displacement distribution.

The stress distribution is shown in figure (3). The stress starts with a zero value in accordance with the boundary conditions. Then decreases for values of x less than t approximately where it reaches a negative minimum value. The stress increases after this value to reach zero while remaining negative.

The important phenomenon observed in all computations is that the solution of any of the considered functions vanishes identically for $x > x^*(t)$, where $x^*(t)$ is a particular value of x depending only on the choice of t and is the same for all three functions. This value is approximately equal to 1.624 for t = 0.25 and equal to 3.745 for t = 1 and is the location of the wave front. This demonstrates clearly the difference between the coupled and the generalized theories of thermoelasticity. In the first and older theory the waves propagate with infinite speeds, so the value of any of the functions is not identically zero (though it may be small) for any large value of x. In the generalized theory the response to the thermal and mechanical effects does not reach infinity instantaneously but remains in a bounded region of space given by $0 \leq x \leq x^*(t)$.

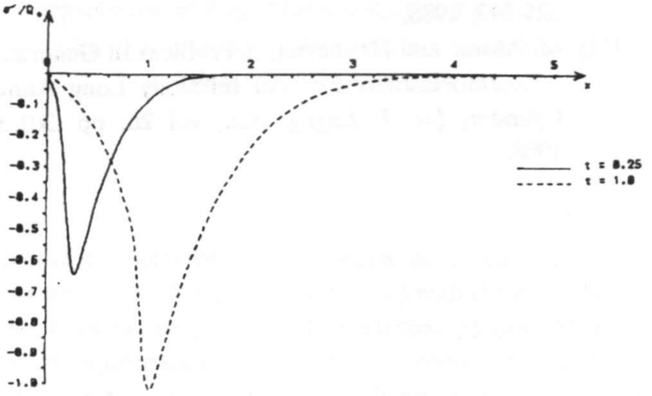

Figure 3. Stress distribution.

REFERENCES

[1] H. Lord And Y. Shulman, A Generalized Dynamical Theory Of Thermoelasticity, *J. Mech. Phys. Solid*, vol. 15, pp. 299-309, 1967.

[2] R. Dhaliwal and H. Sherief, Generalized Thermoelasticity For Anisotropic Media, *Quart. Appl. Math.*, vol. 33, pp. 1-8, 1980.

[3] J. Ignaczak, Uniqueness in Generalized Thermoelasticity, *J. Therm. Stress.*, vol. 2, pp. 171-175, 1979.

[4] J. Ignaczak, A Note On Uniqueness In Thermoelasticity With One Relaxation Time, *J. Therm. Stress.*, vol. 5, pp. 257-263, 1982.

[5] H.Sherief, On uniqueness And Stability in Generalized Thermoelasticity, *Quart. Appl. Math.*, vol. 45, pp. 773-778, 1987.

[6] M. Anwar and H. Sherief, State Space Approach To Generalized Thermoelasticity, *J. Therm. Stress.*, vol. 11, pp. 353-365, 1988.

[7] M. Anwar and H. Sherief, Boundary Integral Equation Formulation of Generalized Thermoelasticity In A Laplace Transform Domain, *Appl. Math. Model.*, vol. 12, pp. 161-166, 1988.

[8] H. Sherief, Fundamental Solution Of The Generalized Thermoelastic Problem For Short Times, *J. Therm. stress.*,vol. 9, pp. 151-164, 1986.

[9] H. Sherief and M. Anwar, Problem In Generalized

Thermoelasticity, *J. Therm. Stress.*, vol. 9, pp. 165-181, 1986.

[10] H. Sherief and M. Anwar, Two-Dimensional Problem Of A Moving Heated Punch In Generalized Thermoelasticity, *J. Therm. Stress.*, vol. 9, pp. 325-343, 1986.

[11] M. Anwar and H. Sherief, A Problem In Generalized Thermoelasticity For An Infinitely Long Annular Cylinder, *Int. J. Engng. Sci.*, vol 26, pp. 301-306, 1988.

[12] H. Sherief and M. Anwar, A Problem In Generalized Thermoelasticity For An Infinitely Long Annular Cylinder Composed Of Two Different Materials, *J. Therm. Stress.*, vol. 12, pp. 529-543, 1989.

[13] L. Bahar and R. Hetnarski, State Space Approach To Thermoelasticity, *J. Therm. Stress.*, vol. 1, pp. 135-145, 1978.

[14] G. Honig and U. Hirdes, A Method For The Numerical Inversion Of The Laplace Transform, *J. Comp. Appl. Math*, vol 10, pp. 113-132, 1984.