ON ADJUSTMENT OF ANGLES IN TRIANGULATION

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ABSTRACT

This paper is mainly concerned with local adjustment of angles observed at one point. The determination of the Most Probable Values (MPV) of angles is the main purpose of this investigation. Modified formulae are used to allow the conversion between different methods of adjustment. Such adjustment may be performed easily and simply by using matrix algebra.

INTRODUCTION

The several directions radiating from an occupied station may be observed in several ways. For example, consider the station A Figure (1) from which there are three directions to observe points B, C and F. Normally, for the best results each angle should be observed independently, namely the angles FAG, GAB and BAF would be separately observed. Such routine implies fresh pointings each time, and not accepting the pointing to one station say to G for both angles FAG and GAB. In a similar way the angles FAB and GAF could be observed independently. This procedure gives values of each single angle which are

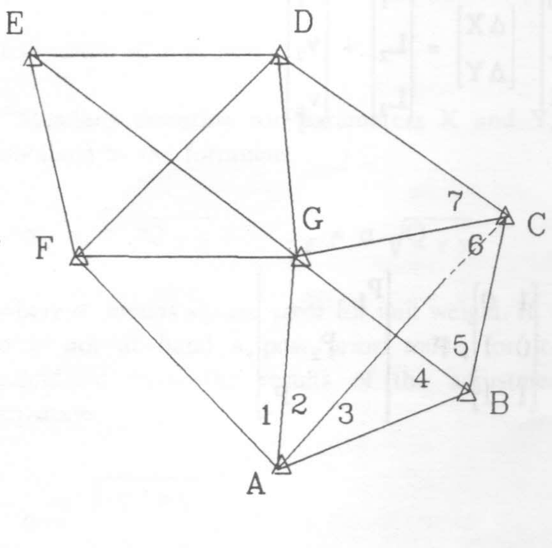


Figure 1.

truly independent and the results could be considered to be consistent, reliable and time-loose. However, such technique may be time-wasty [2,3].

On other hand readings can be taken round the various points without resetting. In this case the accuracy of the resulting angles would be less but the time will be saved. Besides more complicated methods such as Gauss-Schreiber in which all combination observations are available, observed between direction at a station and in addition each direction of the complete survey should observe the same number of times so that a consistent standard of precision is obtained throughout the entire survey. This method gives a considerable rise to a complicated programme prior to fieldwork which is less adaptable in bad weather, than the less precise methods.

Generally speaking any of these methods will offer an acceptable result on condition that Most Careful observations should be made each of which is as genuinely independent as it can be. It is generally false to insist on rigid standards of agreement between successive measures of the same angle if by so doing the observations cease to be independent.

Practically, it is found that the final mean values of the observed angles are inconsistent within themselves, and with external data, such as old surveys, to which they have to be related. An adjustment is therefore required to bring them about consistency. Consistency of data to be used in computations is desirable because it permits to check the whole work conclusively. It will be noted that the adjustment process does not correct the observation though it may improve them. The adjusted angles are still in error.

There are two main methods of adjustment:

(i) By first adjusting the observed angles, etc. and thereafter computing final coordinates.

(ii) By computing provisional coordinates with unadjusted observed angles, and finally adjusting the coordinates.

In both methods, the adjustment may be performed by either semi-rigorous methods, or by rigorous methods (least squares).

As this paper is noted to investigate the angle adjustment, each case will be herein treated separately.

Case I: Angles β_1, β_2 and β are variable

Consider in Figure (2) β is the inner angle at point A between B and C, and is divided internally into two parts angle β_1 and β_2 where β, β_1 and β_2 are measured separately the standard deviations σ_β, $\sigma_{\beta 1}$ and $\sigma_{\beta 2}$ for each angle represents the precision of measuring each one. From this figure, it can be noticed that, there are two values for the angle at point A, one is β and the other is the summation of β_1 and β_2. Taking all types observation errors into consideration, these two values will differ from each other.

Moreover β will be assumed an angle in first order net while β_1 and β_2 are in second order net.

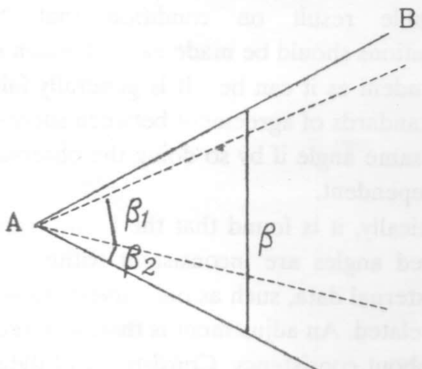
(I.1) Parametric method of adjustment

The observation equations can be written as :

$\beta_1 = X + V_1$, $\beta_2 = Y + V_2$

and $\beta = (X + Y) + V_\beta$

where X and Y are unknown parameters.

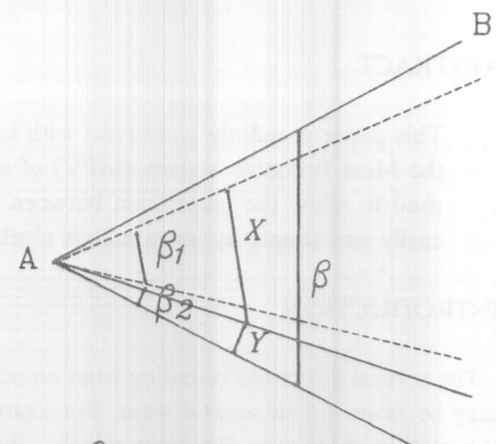


β_1 , β_2 and β are variable

Figure 2.

In Figure (3), where x and y are the unknown parameters, observation equations as can be written in the form [2]:

$$A x = L + v \quad .... \quad p \qquad (I.1)$$



β_1 , β_2 and β are variable

Figure 3.

Let X_o and Y_o be approximate values for the two unknown parameters. Combining the terms in the residuals together, and those in the parameter corrections together, and expressing the results in matrix notation we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_\beta \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & & \\ & P_2 & \\ & & P_3 \end{bmatrix},$$

and

$$X^T = [\Delta X \quad \Delta Y],\ L^T = [L_1 \quad L_2 \quad L_\beta],$$

$$v^T = [v_1 \quad v_2 \quad v_\beta]$$

The normal equations will be

$$(A^TPA)\,X = A^TPL$$

that is

$$\begin{bmatrix} P_1 + P_\beta & P_\beta \\ P_\beta & P_2 + P_\beta \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix} = \begin{bmatrix} P_1L_1 + P_\beta L_\beta \\ P_2L_2 + P_\beta L_\beta \end{bmatrix}$$

Multiplying the matrices out gives

$$\begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix} = \begin{bmatrix} P_1 + P_\beta & P_\beta \\ P_\beta & P_2 + P_\beta \end{bmatrix}^{-1} \begin{bmatrix} P_1L_1 + P_\beta L_\beta \\ P_2L_2 + P_\beta L_\beta \end{bmatrix}$$

Finally, the parameters are obtained by the applying following equations, i.e. (MPV)

$$\left.\begin{aligned} X &= X_o + \Delta X \\ Y &= Y_o + \Delta Y \end{aligned}\right\} \qquad (I.2)$$

Estimation of Precision

Standard deviation for parameters X and Y, can be obtained by the formulae:

$$\sigma_X = \sigma\sqrt{Q_{XX}}, \qquad \sigma_Y = \sigma\sqrt{Q_{YY}} \qquad (I.3)$$

where σ means square error for unit weight. If, however, σ is not at hand a post priori value for it can be calculated from the results of the adjustment using equation:

$$\sigma = \sqrt{\frac{V^TPV}{n - u}} \qquad (I.4)$$

where V is the vector of observational residuals, P is the a priori weight matrix of the observations, n is the number of condition equations, u is the number of parameters, and the cofactor matrix of the parameter estimates, we obtain [1,2,3]

the cofactor matrix Q_x is given by:

$$Q_X = N^{-1}\begin{bmatrix} P_1 + P_\beta & P_\beta \\ P_\beta & P_2 + P_\beta \end{bmatrix}^{-1}$$

$$= \frac{1}{P_1P_2 + P_1P_\beta + P_2P_\beta}\begin{bmatrix} P_2 + P_\beta & -P_\beta \\ -P_\beta & P_1 + P_\beta \end{bmatrix}$$

and

$$Q_{XX} = \frac{1}{P_x} = \frac{P_2P_\beta}{P_1(P_2 + P_\beta) + P_2P_\beta}$$

$$Q_{YY} = \frac{1}{P_Y} = \frac{P_1P_\beta}{P_1(P_2 + P_\beta) + P_2P_\beta}$$

$$(N = A^TPA)$$

So, the priori weights for parameters can be given as:

$$P_X = \frac{1}{P_1} + \frac{P_2P_\beta}{P_1 + P_\beta}$$

$$(I.5)$$

$$P_Y = \frac{1}{P_2} + \frac{P_1P_\beta}{P_1 + P_\beta}$$

(I.2) Correlative method of adjustment

Consider each estimated observation, β_i', as a corrected observation value which obtained from the measurement β_i by adding a correction v_i to it, i.e.

$$\beta_1' = \beta_1 + v_1 \ , \ \beta_2' = \beta_2 + v_2 \ \text{and} \ \beta' = \beta + v_\beta \quad (I.6)$$

According to condition $\beta' = \beta_1' + \beta_2'$, we obtain

$$(\beta_1 + v_1) + (\beta_2 + v_2) = \beta + v_\beta \quad (I.7)$$

i.e. $v_1 + v_2 - v_\beta + L = 0$

Equation (I.6) can be rewritten in matrix form as follows:

$$A\,v = L \quad (I.8)$$

where A coefficient matrix for the observations vector unknowns (residuals) is

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = P^{-1}Ak = \begin{bmatrix} 1/P_1 \\ 1/P_2 \\ 1/P_\beta \end{bmatrix} k \quad (I.9)$$

and

$$k = -\frac{L}{\left[\frac{1}{P}\right]};$$

where

$$\left[\frac{1}{P}\right] = \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_\beta}$$

Estimation of Precision

The cofactor matrix $Q_{\beta'}$ calculated by the following formula [1]

$$Q_{\beta'} = P^{-1} - P^{-1}AN^{-1}A^TP^{-1} \quad (I.10)$$

$$\text{i.e. } Q_{\beta'} = \frac{1}{\left[\frac{1}{P}\right]} \begin{bmatrix} \frac{1}{P_1}\left(\left[\frac{1}{P}\right] - \frac{1}{P_1}\right) & -\frac{1}{P_1P_2} & -\frac{1}{P_1P_\beta} \\ -\frac{1}{P_2P_1} & \frac{1}{P_2}\left(\left[\frac{1}{P}\right] - \frac{1}{P_2}\right) & -\frac{1}{P_2P_\beta} \\ -\frac{1}{P_\beta P_1} & -\frac{1}{P_\beta P_2} & \frac{1}{P_\beta}\left(\left[\frac{1}{P}\right] - \frac{1}{P_\beta}\right) \end{bmatrix}$$

Standard deviation for unknown values β_1', β_2' and β' are calculated by the following formulae respectively

$$\sigma_{\beta_1'} = \sigma \sqrt{\frac{1}{P_1}\left(\left[\frac{1}{P}\right] - \frac{1}{P_1}\right)\left[\frac{1}{P}\right]^{-1}} \quad (I.11)$$

$$\sigma_{\beta_2'} = \sigma \sqrt{\frac{1}{P_2}\left(\left[\frac{1}{P}\right] - \frac{1}{P_2}\right)\left[\frac{1}{P}\right]^{-1}} \quad (I.12)$$

$$\sigma_{\beta'} = \sigma \sqrt{\frac{1}{P_\beta}\left(\left[\frac{1}{P}\right] - \frac{1}{P_\beta}\right)\left[\frac{1}{P}\right]^{-1}} \quad (I.13)$$

where

$$\sigma = \sqrt{\frac{V^TPV}{r}} \quad (I.14)$$

and r is the redundancy, or the number of statistical degrees of freedom.

(I.3) Adjustment by Simple Method

Consider two values for each unknown X and Y, in Figure (3), with different weight; namely,

$X = \beta_1$ as weight P_1

or

$= \beta - \beta_2$ as weight $P_2' = \dfrac{P_2P_\beta}{P_2 + P_\beta}$

and

$Y = \beta_2$ as weight P_2

or

$= \beta - \beta_1$ as weight $P_1' = \dfrac{P_1 P_\beta}{P_1 + P_\beta}$,

where the values X and Y can be computed from the following equations

$$X = \frac{P_1 \beta_1 + P_2'(\beta - \beta_2)}{P_1 + P_2'} \quad \text{(I.15)}$$

$$Y = \frac{P_1' \beta_2 + P_1'(\beta - \beta_1)}{P_1' + P_2} \quad \text{(I.16)}$$

The weights for values X and Y, can be written in the form:

$$P_X = P_1 + P_2' = P_1 + \frac{P_2 P_\beta}{P_2 + P_\beta}$$

(I.17)

$$P_Y = P_2 + P_1' = P_2 + \frac{P_1 P_\beta}{P_1 + P_\beta}$$

Estimation of precision:

Standard deviation for unknown X and Y, can be easily obtained by the following formulae, respectively

$$\sigma_X = \sigma / \sqrt{P_X} \quad \text{(I.18)}$$

$$\sigma_Y = \sigma / \sqrt{P_Y} \quad \text{(I.19)}$$

where

$$\sigma = \sqrt{\frac{V^T P V}{n - u}}$$

$$= \sqrt{V^T P V} \quad \text{(I.20)}$$

The values of residuals for the unknowns X, Y and β can be calculated directly from the relations

$v_1 = X - \beta_1$, $v_2 = Y - \beta_2$,

and lastly

$$v_3 = (X + Y) - \beta \quad \text{(I.21)}$$

Case II: β_1 and β_2 are variables, and β is constant

In this case angles β_1, β_2 are of a lower order network are adjusted to meet the conditions of a previously adjusted higher order network including angle β. Unlike case I, β is considered invariable. The corrected values for each angle β_1 and β_2 must equal to the value of β, i.e., angle β is errorless and the adjustment will done only for the value of β, therefore $\overline{X}$ and $\overline{Y}$ in Figure (4) are not consistent with X and Y from other method.

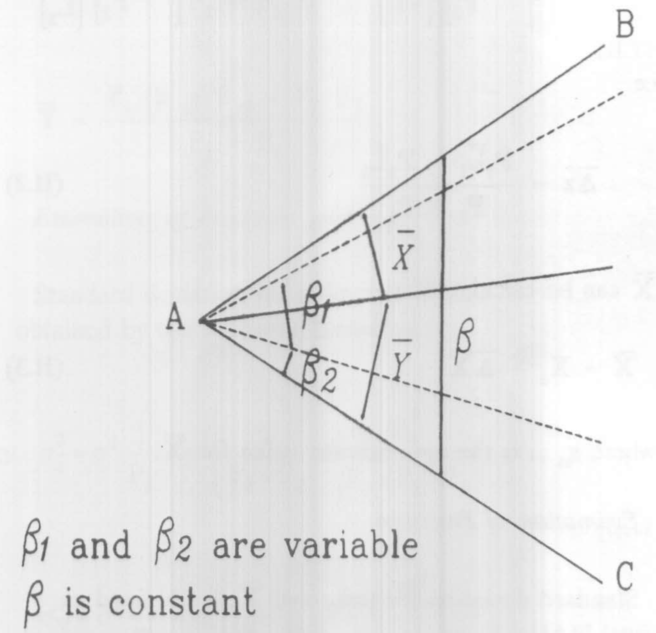


Figure 4.

(II.1) Adjustment by Parametric Method

In Figure (4), $\overline{X}$ and $\overline{Y}$ are the unknown parameters, and observation equations can be written as follows:

$$\overline{\Delta x} = \overline{L}_1 - \overline{v}_1 \; , \; .... \; P_1 \, ,$$

(II.1)

$$\overline{\Delta Y} = \overline{L}_2 - \overline{v}_2 \; , \; .... \; P_2$$

where

$$\overline{\Delta x} = \overline{L}_1 - \overline{v}_1 \quad , \quad .... \quad P_1 ,$$

$$\overline{\Delta Y} = \overline{L}_2 - \overline{v}_2 \quad , \quad .... \quad P_2 \tag{II.1}$$

where

$$\overline{L}_1 = \beta_1 - x_o$$

$$\overline{L}_2 = \beta_2 - (\beta - x_o)\beta = \text{const.}$$

Correspondingly the normal equations can be written in matrix form as follows:

$$(1 \quad -1)\begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\overline{\Delta x} = (1 \quad -1)\begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix}\begin{bmatrix} \overline{L}_1 \\ \overline{L}_2 \end{bmatrix},$$

i.e.

$$\overline{\Delta x} = \frac{P_1\overline{L}_1 - P_2\overline{L}_2}{P_1 + P_2} \tag{II.2}$$

$\overline{X}$ can be calculated as:

$$\overline{X} = X_o + \overline{\Delta X} \tag{II.3}$$

where x_o ... is the approximate value for $\overline{X}$.

Estimation of Precision

Standard deviation for unknown $\overline{X}$, is obtained as usual [4,5]

$$\sigma_{\overline{x}} = \sigma\sqrt{Q_{\overline{xx}}} \tag{II.4}$$

where σ ... Standard deviation for a unit weight, $Q_{\overline{XX}}$ ... is the cofactor matrix.
and [I]

$$\sigma = \sqrt{\frac{\overline{v}^T P \overline{v}}{n - u + B^T D^T P D B Q}} \tag{II.5}$$

where

$$D = E - AN^{-1}A^TP, N = A^TPA, A^T = (1 -1, B^T = (0 -1),$$

$$A^TPA = P_1 + P_2 ,$$

$$P = \begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix}, \quad Q_\xi = \frac{1}{P_\beta}, \quad (A^TPA)^{-1} = \frac{1}{P_1 + P_2}$$

$$D = \frac{1}{P_1 + P_2}\begin{bmatrix} P_2 & P_2 \\ P_1 & P_1 \end{bmatrix}$$

Substituting in equation (II.5), we obtain

$$\sigma = \sqrt{\frac{\overline{v}^T P \overline{v}}{1 + \frac{P_1 P_2}{(P_1 + P_2)P_\beta}}} \tag{II.6}$$

Applying the general law of cofactor propagation, we get

$$Q_{\overline{xx}} = N^{-1} + N^{-1} A^T P B Q B^T P A N^{-1}$$

$$= \frac{1}{P_1 + P_2} + \frac{1}{P_1 + P_2}(1 \quad -1)\begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{P_\beta}\times$$

$$\times (0 \quad -1)\begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix}\begin{bmatrix} +1 \\ -1 \end{bmatrix}\frac{1}{P_1 + P_2},$$

$$\text{i.e., } Q_{\overline{x}\overline{x}} = \frac{1}{P_{\overline{x}}} = \frac{1}{P_1 + P_2} + \frac{P_2^2}{P_1 + P_2} \cdot \frac{1}{P_\beta} \tag{II.7}$$

(II.2) Correlative Method Adjustment

According to the condition ($\beta_1' + \beta_2' = \beta$), we obtain

$$(\beta_1 + \overline{v_1}) + (\beta_2 + \overline{v_2}) = \beta$$

i.e.,

$$\overline{v_1} + \overline{v_2} + \overline{L} = 0$$

where $L = (\beta_1 + \beta_2) - \beta$

correlate $\overline{K} = - \dfrac{\overline{L}}{\dfrac{1}{P_1} + \dfrac{1}{P_2}}$,

and corrections $\overline{v_1}$, $\overline{v_2}$ are :

$$\overline{v_1} = \frac{1}{P_1}\overline{K}\ ,\ \overline{v_2} = \frac{1}{P_2}\overline{K}$$

And finally, values β_1' and β_2' are :

$$\beta_1' = \beta_1 + \overline{v_1}\ ,\ \beta_2' = \beta_2 + \overline{v_2}$$

(check $\beta_1' + \beta_2' = \beta$)

Estimation of Precision

Standard deviation for a unit weight is generally :

$$\overline{\sigma} = \sqrt{\frac{\overline{v}^T P \overline{v}}{r}} \qquad \text{(II.8)}$$

and

$$\sigma = \sqrt{\frac{\overline{v}^T P \overline{v}}{1 + \dfrac{P_1 P_2}{(P_1 + P_2) P_\beta}}} \qquad \text{(II.9)}$$

from which the weight can be written as :

$$\left.\begin{aligned} \frac{1}{P_{\beta_1'}} &= \frac{1}{P_1 + P_2} + \frac{P_2^2}{(P_1 + P_2)^2 P_\beta} \\ \frac{1}{P_{\beta_2'}} &= \frac{1}{P_1 + P_2} + \frac{P_1^2}{(P_1 + P_2)^2 P_\beta} \end{aligned}\right\} \qquad \text{(II.10)}$$

(II.3) Simple Adjustment Method

Consider two values for each angle $\overline{X}$ and $\overline{Y}$, in Figure (4), with different weights as follows :

$\overline{X} = \beta_1$ to have weight P_1

or $\overline{X} = \beta - \beta_2$ to have weight P_2

and $\overline{Y} = \beta - \beta_1$ to have weight P_1

or $\overline{Y} = \beta_2$ to have weight P_2

Where we can be obtained the finally value for each $\overline{X}$ and $\overline{Y}$ as follows :

$$\left.\begin{aligned} \overline{X} &= \frac{P_1 \beta_1 + P_2 (\beta - \beta_2)}{P_1 + P_2}, \\ \overline{Y} &= \frac{P_1 (\beta - \beta_1) + P_2 \beta_2}{P_1 + P_2} \end{aligned}\right\} \qquad \text{(II.11)}$$

Estimation of Precision

Standard deviation for unknown $\overline{X}$ and $\overline{Y}$, are obtained by the following formulae :

$$\left.\begin{aligned} \sigma^2_{\overline{x}} &= \sigma^2 \frac{1}{P_{\overline{x}}} = \sigma^2\left(\frac{1}{P_1 + P_2} + \frac{P_2^2}{(P_1 + P_2)^2 P_\beta}\right), \\ \sigma^2_{\overline{Y}} &= \sigma^2 \frac{1}{P_{\overline{Y}}} = \sigma^2\left(\frac{1}{P_1 + P_2} + \frac{P_1^2}{(P_1 + P_2)^2 P_\beta}\right) \end{aligned}\right\} \qquad \text{(II.12)}$$

where

$$\frac{1}{P_{\overline{x}}} = Q_{\overline{x}\,\overline{x}} = \frac{1}{P_1 + P_2} + \frac{P_2^2}{(P_1 + P_2)^2 P_\beta},$$

$$\frac{1}{P_{\overline{Y}}} = Q_{\overline{Y}\,\overline{Y}} = \frac{1}{P_1 + P_2} + \frac{P_1^2}{(P_1 + P_2)^2 P_\beta}.$$

Consider $\overline{v}_1$ and $\overline{v}_2$ are corrections for each $\overline{X}$ and

$\overline{Y}$ respectively, where

$$\left.\begin{aligned}\bar{v}_1 = \overline{X} - \beta_1 = \frac{P_2}{P_1 + P_2}(\beta - \beta_1 - \beta_2),\\ \bar{v}_2 = \overline{Y} - \beta_2 = \frac{P_1}{P_1 + P_2}(\beta - \beta_1 - \beta_2).\end{aligned}\right\} \quad (II.13)$$

Using values from Equation (II.13), standard deviation for unit weight can be calculated by formula

$$\overline{\sigma}^2 = \frac{\bar{v}^T P \bar{v}}{n - 1} \quad (II.14)$$

Now, value σ can be obtained as follows

$$\overline{\sigma}^2 = \sigma^2 \left(1 + \frac{P_1 P_2}{(P_1 + P_2) P_\beta} \right),$$

$$\text{i.e., } \sigma^2 = \overline{\sigma}^2 / \left(1 + \frac{P_1 P_2}{(P_1 + P_2) P_\beta} \right) \quad (II.15)$$

Or

$$\sigma = \frac{\overline{\sigma}}{\sqrt{\left(1 + \frac{P_1 P_2}{(P_1 + P_2) P_\beta} \right)}}$$

$$= \sqrt{\frac{\frac{\bar{v}^T P \bar{v}}{n-1}}{1 + \frac{P_1 P_2}{(P_1 + P_2) P_\beta}}} \quad (II.16)$$

CONCLUSION

The accuracy of observing angles for different case of observations can be readily calculated through the formulae of the priories even in case of unequal weights.

Also a proper choice for types of observations can be easily selected.

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