MARKOVIAN-MODEL FOR MULTIPLEX SYSTEMS

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ABSTRACT

This work presents a generalized Markovian model for a system of n unidentical-repairable components. Transition probabilities matrix and the set of 2^n differential equations are fully described. The case of identical nonrepairable components are treated as special cases. Deviation from constant failure model is discussed through a suggested procedure to be followed in such cases. Sample of results are presented for testing the consistency of the model.

1. INTRODUCTION AND BASIC ASSUMPTIONS

Reliability concepts for different configurations of components can be better understood through the continuous Markov process [1]. The process structure discusses all possible transitions between system states allowing repair or not.

The number of states for n components configuration is given by $2^n = \sum_{m=o}^{n} \binom{n}{m}$ where m is the number of components failed in any state, and $\binom{n}{m}$ is the combinational formula $\frac{n!}{(n-m)!\,m!}$.

The following notations and definitions are usually utilized [2]:

- $P_{i,j}\{t\}$ probability that a system in state i at time t well be at state j at time $t + \delta t$.
- System states: S_j , $j = 0,1,2,\ldots,2^2-1$.
- x: component success state, $\overline{x}$: component failure state.
- $S_o = [x_1\, x_2 \ldots x_{n-1}\, x_n\}$

  $S_1 = [\overline{x_1}\, x_2 \ldots x_{n-1}\, x_n\}$

  $S_2 = [x_1\, \overline{x_1} \ldots x_{n-1}\, x_n\}$

$S_{2^n-2} = \{x_1\, \overline{x}_2 \ldots \overline{x}_{n-1}\, \overline{x_n}\}$.

$S_{2^n-1} = \{\overline{x_1 x_2} \ldots \overline{x}_{n-1} \overline{x_n}\}$

Following the construction of transition matrix, a set of linear differential equations can be whose solutions are the system state probabilities as function of time. These probabilities are utili-zed to derive an expression for the system reliability according to the configuration under concern.

2. GENERALIZED MARKOVIAN MODEL FOR A SYSTEM OF N UN-IDENTICAL REPAIRABLE COMPONENTS

A. Transition probability matrix

The transition probability $P_{i,j}$ is defined as the probability of system to move form state i at time t, " S_i (t)" to state j at time $t+\delta t$, S_j $(t+\delta t)$". To determine the transition probability matrix, the following postulates are helpful:

i) $P_{i,j} = \lambda_i\ (t)\ \delta t$, when the transition occurring between t and $t + \delta t$ includes only the failure of the i^{th} component with a constant failure rate λ_i.

ii) $P_{i,j} = \mu_i\ (t)\ \delta t$, when the transition occurring between t and $t + \delta t$ includes only the repair of the i^{th} component with a constant repair rate μ_i.

iii) $P_{i,j}$ = zero, when the transition occurring between t and t + δt includes more than failure or repair.

Under these postulates,

The transition probability matrix for a system of n-unidentical repairable components is given by:

$$
\begin{array}{c} \\ 0 \\ 1 \\ p=2 \\ \vdots \\ 2^n-2 \\ 2^n-1 \end{array}
\left[\begin{array}{cccccc}
0 & 1 & 2 & \cdots & 2^n-2 & 2^n-1 \\
1-\sum_{i=1}^{n}\lambda_i & \lambda_1 & \lambda_2 & \cdots & 0 & 0 \\
\mu_1 & 1-(\sum_{i=2}^{n}\lambda_i+\mu_1) & 0 & \cdots & 0 & 0 \\
\mu_2 & 0 & 1-(\sum_{i=1}^{n}\lambda_i-\lambda_2+\mu_2) & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1-(\lambda_1+\sum_{i=2}^{n}\mu_i) & \lambda_1 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{array}\right]
$$

As an illustration on how the matrix is constructed consider the state S_1 (t):

As one repair completion takes place in δt:

$$P_{1,0} = \mu_1\,\delta t$$

As one additional failure takes place in δt:

$$P_{1,n+1} = \lambda_2\,\delta t, P_{1,n+2} = \lambda_3\,\delta t,..., P_{1,2n-1} = \lambda_n\,\delta t,$$

No more than one failure and repair completion in δt are allowed and hence

$$P_{1,2n} = P_{1,2n+1} = ... = P_{1}, \sum_{i=1}^{n}\binom{n}{i} = 0$$

As no state change has taken place

$$P_{1,1} = 1 - (\sum_{i=2}^{n}\lambda_i + \mu_1)\,\delta t$$

B. State probability differential equations

The state probability P_j (t), j = 0,1,2...,2^n-1, is defined as the probability for the system to be in state j at time t. This probability can be viewed as the result of solving a set of 2^n first-order linear differential equation given by:

$$\frac{d}{dt}P(t) = P(t)\,V \qquad (2)$$

where

P(t) = System-state probability vector at time t whose entries are the system state probabilities at time t.

V = differential transition matrix, [as given by expression (P-I) where I is Kronecher delta matrix [3]], whose entries are the component failure and repair rates.

The j^{th} element read from identity (2) is:

$$\frac{d}{dt}P_j(t) = \sum_{i=0}^{2^{-1}} P_i(t)\,P_{i,j} \quad ,j = 0,1,2,...2^n - 1 \qquad (3)$$

with the tremendous progress in computer programming, there are many software packages that can solve the set of differential equations given by identity (2). For easier fed to the computer, the following recurrence relation can be utilized:

For j=0 the resultant differential equation is :

$$\frac{d}{dt}P_o(t) = (-\sum_{i=1}^{n}\lambda_i)P_o(t) + \sum_{i=1}^{n}\mu_i P_i(t) \qquad (4.a)$$

For 0 < j ≤ n the resultant differential equations are:

$$\frac{d}{dt}P_j(t) = \lambda_j P_o(t) - (\sum_{i=1}^{n}\lambda_i - \lambda_j + \mu_j)P_j(t) + \sum_{i=1}^{n-1}\mu_{j,1}P_{j,\bar{1}}(t) \qquad (4.b)$$

For n < j ≤ 2^n -1 the resultant differential equations are:

$$\frac{d}{dt}P_j(t) = \sum_{k=1}^{m}\lambda_{j,\bar{k}}P_{j,\bar{k}}(t) - [\sum_{1-1}^{n-m}\lambda_{j,1} + \sum_{k=1}^{m}\mu_{j,\bar{k}}]P_j(t)$$

$$+ \sum_{1=1}^{n-m}\mu_{j,1}P_{j,\bar{1}}(t) \ \text{for}, m = 2,3,4,...,2^n - 1$$

$$j = [\sum_{i=1}^{m-1}\binom{n}{i}] + 2,...\sum_{i=1}^{m}\binom{n}{i} \qquad (4.c)$$

where

$\lambda_{j,\bar{k}}, \mu_{j,\bar{k}}$ = failure rate, repair rate of k^{th} bad component in state S_j.

$P_{j,k}(t)$ = State probability S_j when the k^{th} bad component is replaced by a good one.

$\lambda_{j,l}\,\mu_{j,l}$ = Failure rate, repair rate of ℓ^{th} good component is replaced by a bad one.

n = Total number of components.

m = Number of bad components in the state S_j.

3. SPECIAL CASES

3.1 Un-identical/nonrepairable components

One can simply get the relevant set of differential equations by substituting $\mu_i = 0$ in equations (4.a)-(4.c). The result is:

$$\frac{d}{dt}P_o(t) = (-\sum_{i=1}^{n} \lambda_i)P_o(t) \qquad (5.a)$$

$$\frac{d}{dt}P_j(t) = \lambda_j P_o(t) - (\sum_{i=1}^{n} \lambda_i - \lambda_j)P_j(t), j = 1,2,...,n \qquad (5.b)$$

$$\frac{d}{dt}P_j(t) = \sum_{k=1}^{m} \lambda_{j,\bar{k}} P_{j,k}(t) - (\sum_{l=1}^{n-m} \lambda_{j,l})P_j(t) \qquad (5.c)$$

for, m = 2,3,...,n

$$j = [\sum_{i=1}^{m-1} \binom{2}{i}] + 1, [\sum_{i=1}^{m-1} \binom{n}{i}] + 2, ... \sum_{i=1}^{m} \binom{n}{i}.$$

As this set contains uncoupled differential equations, the set can be solved analytically using Laplace transform technique with following boundary conditions:

$$P_0(0) = 1, \text{ and } P_j(0) = 0 \text{ for } j > 0 \qquad (6.a)$$

$$\sum_{j=0}^{2^n-1} P_j(t) = 1 \qquad (6.b)$$

Depending on the value of j, the solution can be derived as:

<u>j = 0:</u>

$$P_o(t) = \exp[-\sum_{i=1}^{n} \lambda_i t] \qquad (7.a)$$

<u>$0 < j \le n$:</u>

$$P_j(t) = \exp[(-\sum_{i=1}^{n} \lambda_I + \lambda_j)t] - \exp[-\sum_{i=1}^{n} \lambda_i t] \qquad (7.b)$$

$n < j \le \sum_{i=1}^{2} \binom{n}{i}$:

$$p_j(t) = \exp[-\sum_{l=1}^{n-2} \lambda_{j,l} t] - \sum_{k=1}^{2} \exp[-(\sum_{l=1}^{n-2} \lambda_{j,l} + \lambda_{j,\bar{k}})t]$$

$$+ \exp[-(\sum_{i=1}^{n} \lambda_i)t] \qquad (7.c)$$

m = 3,5,7,..., $\{{}^{n}_{n-1}\ {}^{n\ odd}_{n\ even}; j > \sum_{i=1}^{2} \binom{2}{i}$, odd :

$$P_j(t) = \exp[\sum_{l=1}^{n-m} \lambda_{j,l}) + \sum_{s=1}^{\binom{m}{T}} \{(-1)^T \exp[-(\sum_{l=1}^{n-m} \lambda_{j,l} + \sum_{k=1}^{T} \lambda_{j,\bar{k},s})t]$$

$$+ (-1)^{T+1} \exp[-(\sum_{i=1}^{n} \lambda_i - \sum_{k=1}^{T} \lambda_{j,\bar{k},l)} t]\} - \exp[-(\sum_{i=1}^{n} \lambda_i)t]. \qquad (7.d)$$

m = 4,6,8,..., $\{{}^{n}_{n-1}\ {}^{n\ even}_{n\ odd}; j > \sum_{i=1}^{2} \binom{n}{i}$, even :

$$P_j(t) = \exp[\sum_{l=1}^{n-m} \lambda_{j,l} t] + \sum_{s=1}^{\binom{m}{T}} (-1)^T \exp[-(\sum_{l=1}^{n-m} \lambda_{j,l} + \sum_{k=1}^{T} \lambda_{j,\bar{k},s})t]$$

$$+ \exp[(\sum_{i=1}^{n} \lambda_i - \sum_{k=1}^{T} \lambda_{j,\bar{k},l)} t]\} + \sum_{l=1}^{2^{(m/2)}} (-1)^m/2 \exp[-(\sum_{i=1}^{n} \lambda_i \sum_{k=1}^{m/2} \lambda_{j,\bar{k}s})t].$$

$$+ \exp[-(\sum_{i=1}^{n} \lambda_i)t]. \qquad (7.e)$$

It should be noticed that the evaluation $\lambda_{j,\bar{k}}$, the failure rate of k^{th} bad component in state S_j, depends on which components that have been bad. The suffix s is used to assign certain set of those probabilities. Shortly we can define $\lambda_{j,k,\bar{s}}$ as the failure rate of the s^{th} set of "k" components.

3.2 Identical repairable components

When λ_i and μ_i are replaced by λ and μ respectively, the differential equations (4.a) - (4.c) becomes:

j = 0:

$$\frac{d}{dt}P_o(t) = -n\lambda p_o(t) + n\mu p_1(t), \tag{8.a}$$

0 < j ≤ n:

$$\frac{d}{dt}p_j(t) = \lambda p_o(t) - [(n-1)\lambda + \mu]P_j(t) + (n-1)\mu P_{n+1}, \tag{8.b}$$

n < j ≤ 2^n-1:

$$\frac{d}{dt}p_j(t) = m\lambda P_{j-1} - [(n-m)\lambda + m\mu]P_j(t) + (n-m)\mu P_l(t)$$

$$m=2,3,\ldots,n; j = \sum_{i=1}^{m-1}\binom{n}{i} + 1; l = [\sum_{i=1}^{m}\binom{n}{i}] + 1 \tag{8.c}$$

where

$$P_j(t) = P_{j+1}(t) = \ldots = P_k(t),\ k = \sum_{i=1}^{m}\binom{n}{i} \tag{8.d}$$

3.3 IDENTICAL NON-REPAIRABLE COMPONENTS

This is the simplest case in which letting μ equal zero, the differential equations (8.a)-(8.c) becomes:

j = 0:

$$\frac{d}{dt}P_o(t) = -n\lambda p_o(t) \tag{9.a}$$

0 < j ≤ n:

$$\frac{d}{dt}P_j(t) = \lambda P_o(t) - (n-1)\lambda P_j(t)$$

n < j ≤ c^n -1:

$$\frac{d}{dt}P_j(t) = m\lambda P_{j-1}(t) - (n-m)\lambda P_j(t) \tag{9.c}$$

Solution using Laplace transform technique with the boundary conditions (6.a) and (6.b) gives:

j = 0:

$$P_0(t) = \exp[-n\lambda t] \tag{10.a}$$

0 < j ≤ n:

$$P_j(t) = \exp[-(n-1)\lambda t] - \exp[-n\lambda t] \tag{10.b}$$

$n < j \le \sum_{i=1}^{n}\binom{n}{i}$:

$$p_j(t) = \exp[-(n-2)\lambda t] - 2\exp[-(n-1)\lambda t + \exp[-n\lambda t] \tag{10.c}$$

$m = 3,5,7,\ldots, \{^{n\ \ n\ odd}_{n-1\ \ n\ even}; j > \sum_{i=1}^{2}\binom{n}{i}$, odd:

$$P_j(t) = \exp[-(n-m)\lambda t] + \sum_{i=1}^{(m-1)\backslash 2}\binom{m}{i}\{(-1)^i \exp[-(n-m+i)\lambda t]$$

$$+ (1)^{i+1}\exp[-(n-i)\lambda t]\} - \exp[-n\lambda t]. \tag{10.d}$$

$m = 4,6,8,\ldots, \{^{n\ \ n\ even}_{n-1\ \ n\ odd}; j > \sum_{i=1}^{2}\binom{n}{i}$, even:

$$P_j(t) = \exp[-(n-m)\lambda t + \sum_{i=1}^{(m-2)\backslash 2}\binom{m}{i}(-1)^i\{\exp[-(n-m+i)\lambda t]$$

$$+\exp[-(n-i)\lambda t]\}$$

$$+ \binom{m}{m/2}(-1)^{m/2}\exp[-(n-\frac{m}{2})\lambda t] + \exp[-n\lambda t] \tag{10.e}$$

4. RELIABILITY CALCULATIONS

Following the evaluation of system state probabilities one can easily derive the expression for system reliability

"R(t)". The expression mainly dependent on the configuration under considerations. Figures (1) and (2) show different configurations for 3-unidentical repairable.non-repairable component and 10-identical non-repairable component systems respectively. Formulae for R (t) are indicated under each configuration.

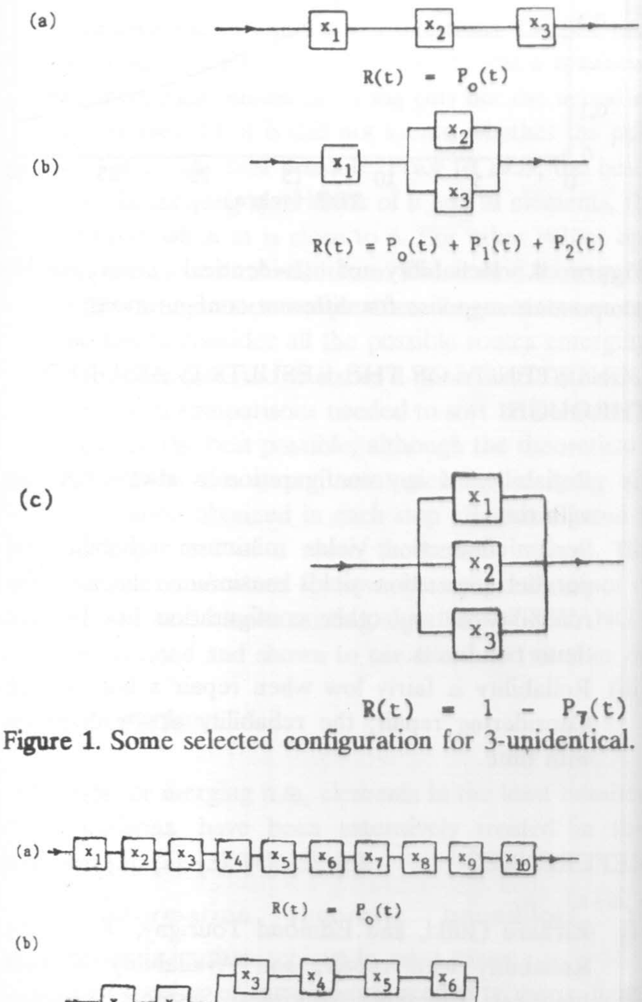

Figure 1. Some selected configuration for 3-unidentical.

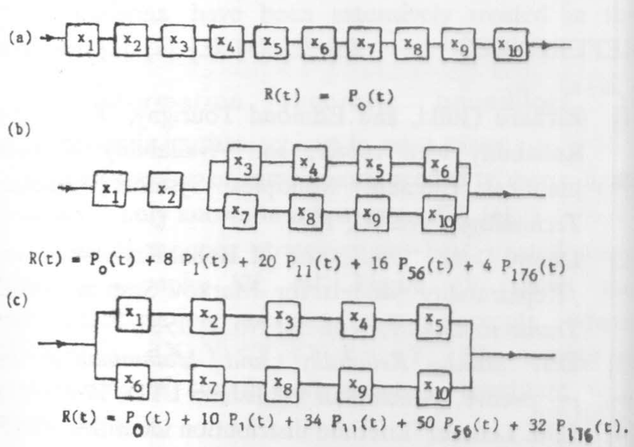

Figure 2. Some selected configurations for 10-identical components.

5.DEVIATION FROM THE CONSTANT FAILURE RATE MODEL

It is practically convenient that the hazard function not to be constant over the period during which the component failure rate was estimated. To make the generalized markovian model presented in section (2) applicable in that case, the following procedure is suggested:

(i) Assume the hazard to obey an exponential power model [4] defined as

$$h(t) = \alpha \beta t^{\beta - 1} \exp[\alpha t^{\beta}] \qquad (11)$$

where, α and β are scale and shape parameters respectively,
h(t) has a minimum value at t_c where t_c is given by

$$t_c = \left(\frac{1-\beta}{\alpha\beta}\right)^{1/\beta} \qquad (12)$$

(ii) Use the estimated failure rate extracted from actual operational experience of over a period of time equals t_f-t_c as:

$$\hat{\lambda} = \frac{1}{t_f - t_c} \int_{t_c}^{t_f} h(t)dt$$

$$= \frac{1}{T_f - t_c} \{\exp[\alpha t_f^{\beta}] - \exp[\frac{1-\beta}{\beta}]\} \qquad (13)$$

(iii) Knowing t_c,t_f and $\hat{\lambda}$, use equations (11), (13) to get the parameters α and β.

(iv) Subdivide the whole interval t_f-t_c into a reasonable number of intervals of equal or not equal widths. For the first subdivision find the average hazard function and consider it as λ. Use an appropriate value of μ/λ to get μ.

(v) Construct the transition probability matrix and solve the differential equations for system state probabilities.

(vi) According to the configuration under concern, write down the appropriate formulae for **R** (t) to be used in this subdivision.

(vii) Repeat steps from iv to vi for subsequent subdivisions to evaluate R (t) for each subdivision keeping in mind the continuity requirements between subdivisions.

6. NUMERICAL ILLUSTRATIONS

For the configurations shown in Figure (1), with $\lambda_1 = 1 \cdot 10^{-4}$, $\lambda_2 = 1.10^{-5}$, $\lambda_3 = 1.10^{-6}$ and $\mu_i = 5\ \lambda_i$, i=1,2,3. Figures (3-a) and (3-b) illustrate R (t) in the case of repairable and unrepairable components. Reliability responses for the 10-identical component systems shown in Figure (2), with $\lambda = 1.10^{-6}$, are plotted in Figure (4).

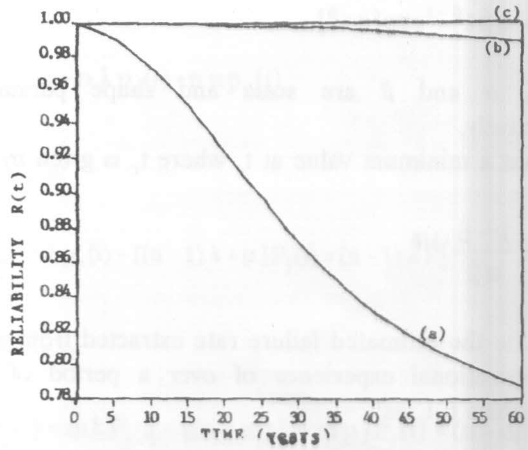

Figure 3-a. Reliability of 3-unidentical repairable components response for different configurations.

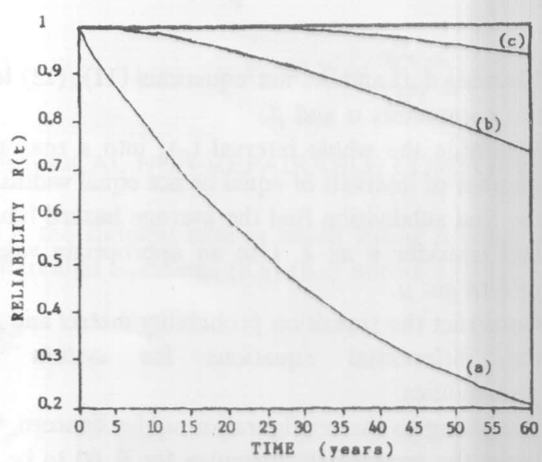

Figure 3-b. Reliability of 3-unidentical non-repairable components response for different configurations.

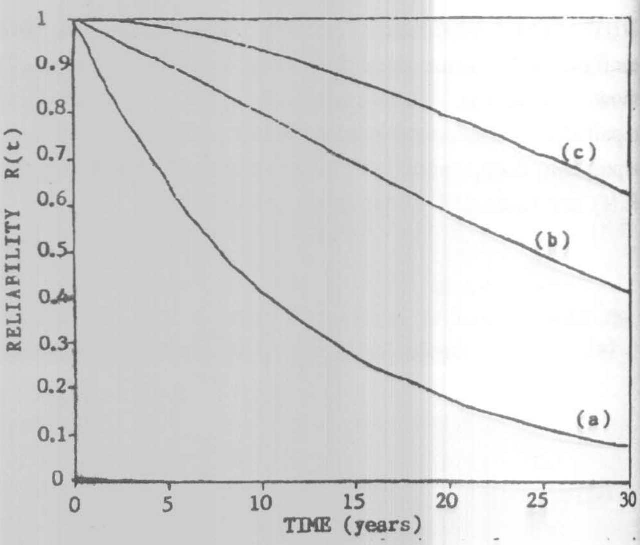

Figure 4. Reliability of 10-identical non-repairable components response for different configurations.

CONSISTENCY OF THE RESULTS IS ASSURED THROUGH:

(i) Reliability of any configuration is always decaying with time.

(ii) Series connection yields minimum reliability and parallel connection yields maximum reliability. The reliability of any other configuration lies between these two limits.

(iii) Reliability is fairly low when repair s not allowed. Considering repair, the reliability slowly decreases with time.

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