EFFECT OF ANGLE OF REPOSE ON THE CRITICAL SHEAR STRESS OF INDIVIDUAL PARTICLES

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ABSTRACT

The critical shear stress of nonuniform mixture is theoretically studied. The effect of angle of repose on the value of critical shear stress is introduced. Shields diagram for incipient motion is modified to adopt the condition of a widely distributed grain size mixture. The suggested modification of Shields criteria for incipient motion is examined using the available data.

NOTATIONS

C_D	drag coefficient of a particle
d	grain diameter
d_{av}	mean diameter of a mixture
F	drag force
g	gravitational acceleration
R	resistance force
Re	Reynolds number
R_*	Grain Reynolds number
T_{*c}	dimensionless critical shear stress of individual particle
T_{*u}	dimensionless critical shear stress of uniform mixture
u_d	local velocity at a particle level
u_*	shear velocity
α	parameter incorporating the effects of shape and roundness
β	parameter incorporate the effects of sorting of grain mixture
γ	specific weight of water
γ_s	specific weight of grain
Φ	angle of repose of individual particle of size d
Φ_u	angle of repose of a uniform mixture of size d
τ	critical shear stress of individual particle of size d
τ_{oc}	critical shear stress of uniform mixture
ν	kinematic viscosity

INTRODUCTION

Most of natural sediment mixture forming alluvial canals have a nonuniform grain size distribution. The hydraulic conditions of incipient motion of nonuniform mixture differs from that of uniform mixture. Critical shear stress of nonuniform mixture is affected by its specific weight, mean diameter, grain size distribution, shape factor and compaction of bed sediments. The effect of grain size distribution, compaction of bed material and shape factor of grains is embedded in the value of angle of repose of a mixture. In almost all formulas, which express the incipient motion of nonuniform mixture, the effect of the angle of repose of a nonuniform mixture is considered to be equal to the angle of repose of uniform mixture. However, it is apparent that a better distribution of a particle sizes produces a better interlocking and a higher value of angle of repose. It would be expected that angular particles would interlock more thoroughly than rounded particles, and hence that sand composed of angular particles would have larger angle of repose. This trend is shown in Table (1) which confirm this prediction.

Table 1. Effect of Angularity and grading on angle of repose after Sowers & Sowers

Shape & Grading	Loose	Dense
Rounded, uniform	30°	37°
Rounded, well graded	34°	40°
Angular, uniform	35°	43°
Angular, well graded	39°	45°

Eaglson and Dean[3] determined experimentally the value of the angle of repose, tan Φ for various "d/k" combinations, where k is the bed-particles diameter, and d is the size of the single particle resting on the bed. The single particles were spheres of various sizes and densities resting on two different bottom roughnesses of natural sand, the first of a diameter 1.83 mm, the second with diameter 0.79 mm. Their results are shown in Figure (1). They indicated that their experimental values of angle of repose departed significantly from the theoretical expectations.

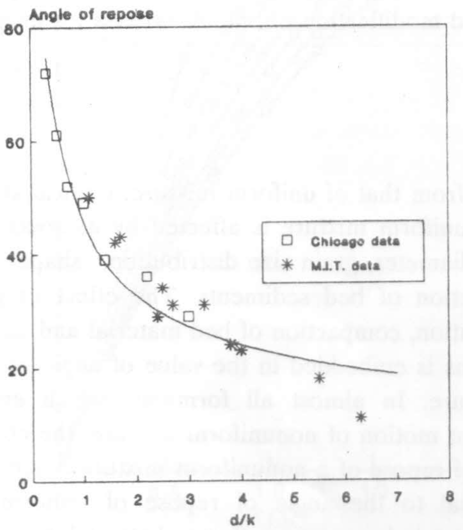

Figure 1. Relationship between angle of repose and d/k after Eaglson and Dean.

Miller and Byrne[5] determined experimentally the angle of repose for a single particle on a fixed rough bed. They studied experimentally the effect of grain shape, and ratio of particle diameter to the mean diameter of bed mixture on the value of angle of repose. They developed an empirical equation for predicting angle of repose as a function of the above factors. The equation is written as:

$$\phi = \alpha\left(\frac{d}{d_{av}}\right)^{-\beta} \tag{1}$$

In which α is a parameter incorporating the effects of shape and roundness in both particle and bed and ß is a parameter incorporating the effects of sorting of the bed grains.

They found out that the value of α equal to 50 for spheres, 61.5 for intermediate sphericity and roundness, and 70 for intermediate sphericity and low roundness. The results of their experiments is shown Figure (2).

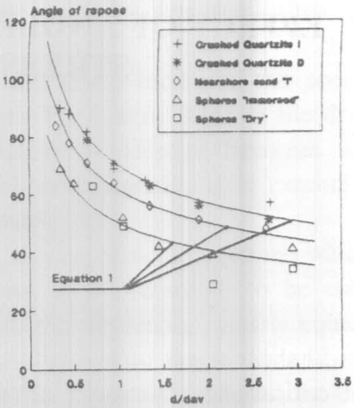

Figure 2. Relationship between angle of repose and d/d_{av}.

They concluded that the angle of repose increases with the decrease of size, increase of sphericity and decrease in sorting. The aim of this study is to express the critical shear stress of nonuniform mixture in term of angle of repose and grain diameter.

THEORETICAL ANALYSIS

The critical shear stress of individual particles in a nonuniform mixture is affected by the angle of internal friction. The angle of internal friction is assumed to be equal to the value of angle of repose of a particle size resting on rough bed having mean diameter "d_{av}" which can be expressed using equation (1). Using the same procedure made by Shields[7] in driving his criteria for incipient motion for uniform mixture, the critical shear stress for individual particle in a nonuniform mixture can be derived as follows:-

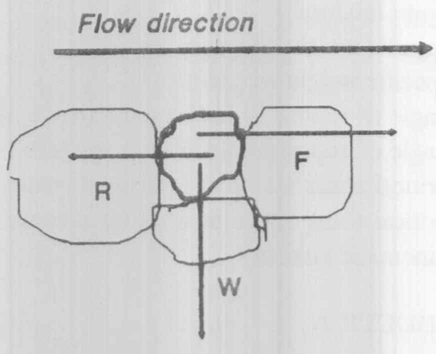

Figure 3.

For a nonuniform mixture consisting a grain fraction of size "d", (Figure 3), the force required to move the particle is given by

$$R = \alpha_1(\gamma_s - \gamma)d^3\tan(\phi) \quad (2)$$

In which "α_1" is a coefficient depends on the shape of the particle.
The drag force exerted by the fluid on the particle is

$$F = C_D\gamma\frac{u_d^2}{2g}\alpha_2 d^2 \quad (3)$$

Where "α_2" is a coefficient such that "$\alpha_2 d^2$" gives the projected area of the particle, and "u_d" is the velocity at the top of the particle. Using Karman-Prandtl equation for velocity distribution, "u_d" can be expressed as follows

$$\frac{u_d}{u_*} = \varphi\left(\frac{u_* d}{\nu}\right) \quad (4)$$

Since "C_D" is the drag coefficient of the particle and can be expressed as

$$C_D = \varphi_1\left(\frac{u_d d}{\nu}\right) = \varphi_2\left(\frac{u_* d}{\nu}\right) \quad (5)$$

The term "$u_* d/\nu$" is defined as grain Reynolds number "R_*"
Substituting equation (4) and equation (5) into equation (3) it becomes

$$F = \varphi_2(R_*)\gamma\frac{u_*^2}{2g}\varphi_1^2(R_*)\alpha_2 d^2 \quad (6)$$

Equating R and F for the condition of incipient motion and introducing the subscript "c", one gets

$$\alpha_1(\gamma_s - \gamma)d^3\tan(\phi) = \varphi_2(R_{*c})\gamma\frac{u_{*c}^2}{2g}\varphi_1^2(R_{*c})\alpha_2 d^2$$

$$\frac{\gamma u_{*c}^2}{(\gamma_s - \gamma)gd} = \frac{2\alpha_1}{\alpha_2}f(R_{*c})\tan(\phi) \quad (7)$$

Where "u_{*c}" is the critical shear velocity which is defined

$$u_{*c} = \sqrt{\frac{\tau_c}{\rho}} \quad (8)$$

Equation (7) can be written in term of the critical shear stress as follows

$$\frac{\tau_c}{(\gamma_s - \gamma)d} = f(R_{*c})\tan(\phi) \quad (9)$$

According to equation (9), the critical shear stress of individual particles is directly proportion to the function of grain Reynolds number. The constant of the proportion is the friction coefficient "tan(Φ)".
Shields criterion for incipient motion which is almost universally adopted for the estimation of critical shear stress for uniform mixture, can be expressed in the following equation

$$\frac{\tau_{oc}}{(\gamma_s - \gamma)d} = f_1(R_{*c}) = \frac{f_1(R_{*c})}{\tan(\phi_u)}\tan(\phi_u) \quad (10)$$

In which "Φ_u" is the angle of repose of the uniform mixture of size "d". The derivation of equation (9) is exactly the same as the derivation of Shields criterion. Equation (9) can be written in term of Shields parameter as follows:

$$\frac{\tau_c}{(\gamma_s - \gamma)d} = f_1(R_{*c})\frac{\tan(\phi)}{\tan(\phi_u)} \quad (11)$$

The term "$\tau_c/(\gamma_s-\gamma)d$" is defined as the dimensionless critical shear stress "T_{*c}".
The function in the right hand side of equation (11) can be described by using Yassin's[8] empirical equations which express Shields criteria for incipient motion. Equation (11) can be written as follows:

$$\text{For } 0.3 > R_* \geq 3: \quad \frac{\tau_c}{(\gamma_s - \gamma)d} = 0.118(R_{*c})^{-0.973}\frac{\tan(\phi)}{\tan(\phi_u)} \quad (12\text{-a})$$

$$\text{For } 2.0>R_* \geq 4: \frac{\tau_c}{(\gamma_s-\gamma)d} = 0.09(R_{*c})^{-0.585}\frac{\tan(\phi)}{\tan(\phi_u)} \quad \text{(12-b)}$$

$$\text{For } 4.0>R_* \geq 10: \frac{\tau_c}{(\gamma_s-\gamma)d} = 0.0434(R_{*c})^{-0.119}\frac{\tan(\phi)}{\tan(\phi_u)} \quad \text{(12-c)}$$

$$\text{For } 10>R_* \geq 30: \frac{\tau_c}{(\gamma_s-\gamma)d} = 0.0275(R_{*c})^{-0.0792}\frac{\tan(\phi)}{\tan(\phi_u)} \quad \text{(12-d)}$$

$$\text{For } 30>R_* \geq 400: \frac{\tau_c}{(\gamma_s-\gamma)d} = 0.0194(R_{*c})^{0.181}\frac{\tan(\phi)}{\tan(\phi_u)} \quad \text{(12-e)}$$

$$\text{For } R_*>500: \frac{\tau_c}{(\gamma_s-\gamma)d} = 0.06\frac{\tan(\phi)}{\tan(\phi_u)} \quad \text{(12-f)}$$

Using equations (12), Shields diagram expressing incipient motion condition is modified so that it can describe the critical shear stress of individual rounded particles in a nonuniform mixture as shown in Figure (4).

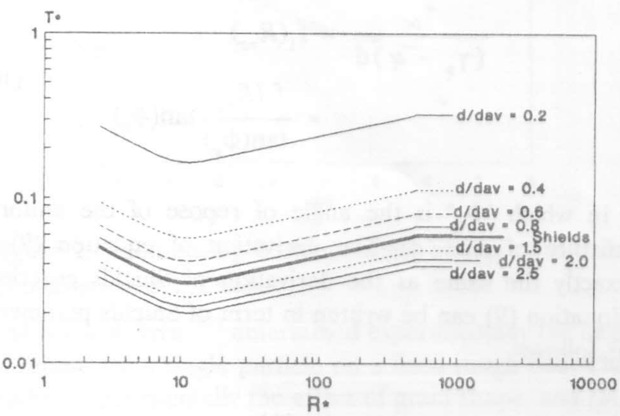

Figure 4. Modified Shields diagram.

The value of the critical shear stress of individual particles of size "d" differs from mixture to mixture according to the ratio "d/d_{av}". This relation can be described by dividing equation (12) by the critical shear stress of the uniform mixture of size "d" as follows

$$\frac{\tau_c}{\tau_{oc}} = \frac{\tan(\alpha\,[d/d_{av}]^{-\beta})}{\tan(\alpha)} \quad (13)$$

For fine grain fraction having size less than the mean diameter of mixture, the critical shear stress is greater than that of uniform mixture. This condition occurs because fine grain fractions are prevented to move by the coarser grain fractions. On the other hand, grain fractions coarser than the mean diameter have critical shear stress less than that of uniform mixture Figure (5). This trend is recognized by Chin[2], Profit[6], Little & Mayer[7], and Baghdadi[1].

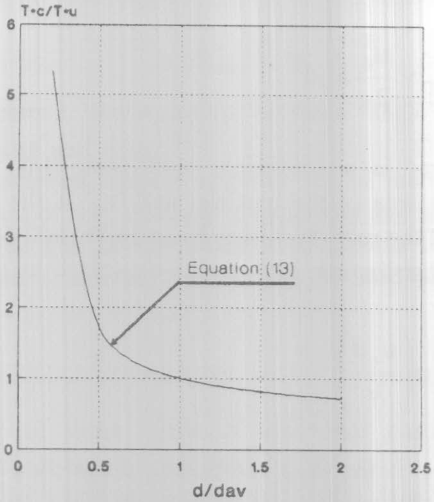

Figure 5. Relationship between the ratio of τ/τ_{oc} and d/d_{av}.

VERIFICATION OF THE THEORETICAL DETERMINATION OF THE CRITICAL SHEAR STRESS

The critical shear stress of individual particles in a nonuniform mixture can not be experimentally determined because it is impossible to determine the hydraulic condition in which a particle size under consideration starts to move. Therefore, actual values of critical shear stress of individual grain fraction are not found. The available data of critical shear stress are only for the whole mixture. The theoretical formula for incipient motion is verified by determining the minimum shear stress of individual particles in a mixture using equation (12). The minimum shear stress is considered to be the theoretical critical shear stress of the whole mixture. Comparison between the actual and the theoretical critical shear stress of 64 experiments made by Chin[2], Little & Mayer[4] and Profit[6] is done (Figure 6).

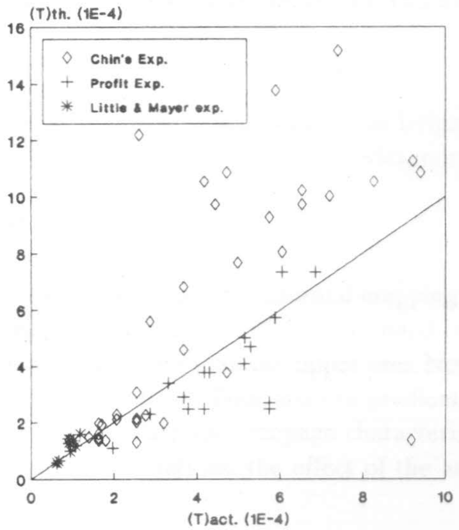

Figure 6. Relationship between the actual and the theoretical critical shear stress.

It is found that for values of shear stress less than $0.0004 t/m^2$, equation (12) gives good results and the standard error of estimate is equal to 0.00001. The discrepancy between the actual and the critical shear stress may come from the accuracy of determination of the value of α and ß in equation (1).

CONCLUSION

The value of the angle of repose is affected by the grain size distribution of the mixture and the grain shape factor. The critical shear stress of individual particles in a wide distributed grain size mixture differs from that of uniform mixture. For grain fractions finer than a mean diameter of the mixture, the critical shear stress is greater than that determined by Shields. For grain particles coarser than the mean diameter, the critical shear stress is less than that of uniform mixture. The value of angle of repose of individual particles in a nonuniform mixture is responsible of changing the value of the critical shear stress. The incipient motion condition of individual particles can be determined from the modified Shields diagram (Figure 4) or from equation (12). Equation (9) gives satisfactory results for actual critical shear stress less than 0.0004 ton/m^2. More investigations is required to determine accurately the value of angle of repose of individual particles.

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