SPHERICAL PERSPECTIVE

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ABSTRACT

In this paper the gonomonic and the stereographic projections are used to develop a graphical method to construct the spherical perspective of points, straight lines, planes, circles and spheres. Although the straight lines are represented in this mode of perspective as circular arcs, but this gives a comprehensive and undistorted images of reality. The problems related to planes are treated. These are the coincidence, the parallelism and the intersection problems. The spherical perspective of a circle having its plane in different positions with respect to the picture plane is considered. Among these positions one only yields the spherical perspective of circle as circles. The spherical perspective of spheres is always circles.

NOTATIONS

These notations are shown in Figure (2):

$\varphi(O,R)$	Picture sphere with O as centre and R as radius.
π,π_0	Picture plane and horizon plane.
π_1	Ground plane or horizontal plane.
$A(A_1,A_2)$	Space point A,Where A_1 ,A_2 its horizontal and vertical projections.
$A[A',A'_1]$	Space point A,where A its spherical perspective and A'_1 the spherical perspective of A_1.
A''	Gonomonic projection of A from O into φ.
E	Centre of the stereographic projection.
P	Principal point.
$c(Q,r,\alpha)$	Circle c with Q as centre, r as radius and α as its plane.
e	The equator.
f	Principal meridian of φ.
g, h	Ground line and horizon line.
t_1	Trace of π on π_1, the horizontal trace.

INTRODUCTION

The spherical perspective is a mode of perspective known as Fish-eye perspective [5]. It is known that true perspective of an object corresponds to what the human eye perceives. This is not the case in linear perspective in which the perspective of the object is constructed on a picture plane. Some examples show that this linear perspective is not only theoretically wrong but also leads to considerable distortions.

In linear perspective two parallel straight lines have one vanishing point, but in true perspective these lines have two vanishing points [3, 5]. Since, if these lines are infinitely long and located horizontally in front of a human eye, and when this eye is rolled (without moving the human head) in the right and left directions, then a vanishing point for these lines will be seen in each direction.

A serious distortion arises in linear perspective when repeated similar objects, say cubes, are located parallel to the picture plane, Figure (1).

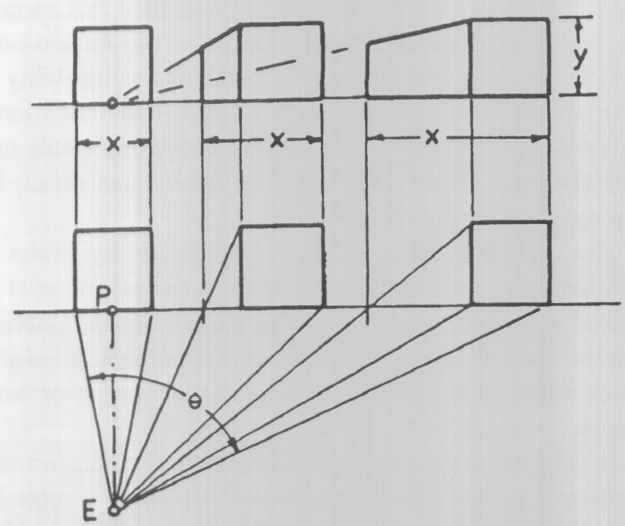

Figure 1.

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In this case the over-all perspective dimensions (x, y) increase as the cubes recede. But in true perspective the most distance cube is the smallest one. The distortion caused to the over-all perspective dimensions in linear perspective is due to the limitation of the angle of vision θ which should not exceed than 60° [1].

In linear perspective the perspective of a circle is always a circle as long as its plane is parallel to the picture plane. This is not the case in true perspective where the human eye sees this circle as a circle if and only if its plane is perpendicular to the line joining its centre to the eye. Moreover, in linear perspective the perspective of a sphere is a circle whenever the picture plane is perpendicular to the line joining its centre to the centre of projection . In true perspective this sphere is always seems to the human eye as a circle regardless of its position with respect to the human eye.

To avoid these wrongs and distortions caused by linear perspective; the picture plane must be curved surface such that the distance between the centre of projection and the different points of this surface is stationary [3,5]. This curved surface is known as the picture surface and the perspective is known as the curvilinear perspective [3]. The cylindrical and spherical surfaces are used as picture surfaces. For these two surfaces the curvilinear perspective is known as panoramic [7] and spherical perspective [1,3,5].

In curvilinear perspective the perspective of an object is constructed on the picture surface, and then this surface is developed onto a plane to obtain a two-dimensional perspective representation. Due to the undevelopability of the sphere, the problem of obtaining two-dimensional spherical perspective representation is still unsolvable one [3]. However, in [5] this problem is treated analytically by an approximated method.

The spherical perspective of an object resembles a photograph of this object taken by a camera of what is called wide-angle lens as shown in the photograph inserted at the end of this paper. This mode of photography is used when it is required to take a photograph of wide scope.

In this paper the problem of obtaining two-dimensional spherical perspective representation is treated graphically by two successive projections. The gonomonic projection of a space point A from the centre of the picture sphere into this sphere gives point A″, Figure (2). The stereographic projection of A″ into a picture plane gives the spherical perspective A′ of A.

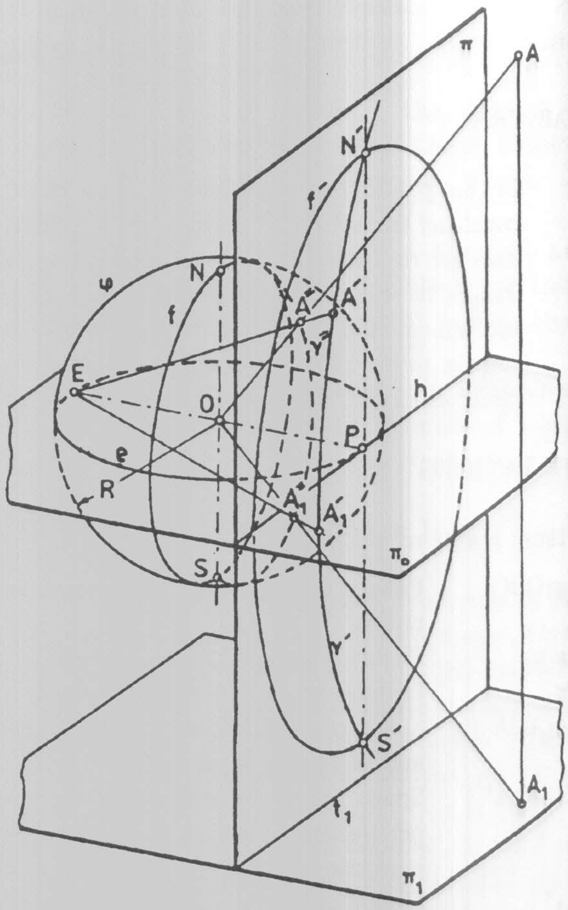

Figure 2.

SPHERICAL PERSPECTIVE OF POINTS

In Figure (2) the picture plane π is tangent to the picture sphere $\varphi(O, R)$ at the principal point P which lies on the equator e. This equator lies on the horizon plane π_0 which is parallel to the ground plane π_1. This picture plane intersects π_0 and π_1 in the horizon line h and the trace t_1. The spherical perspective A′ of a space point A is obtained as follows: The gonomonic projection of A from O into φ gives point A″. Since the ray OA intersects φ in two points, hence A″ is taken as the one between O and A. The stereographic projection of A″ into π from the antipode E of P gives A′. To ensure one-to-one correspondence of A and A′, the spherical perspective A'_1 of the horizontal projection A_1 of A is determined on π. The gonomonic projection of the line AA_1 into φ is a great circle γ'' passing through the north pole N and the south pole S. The stereographic projection of γ'' from E into π is a circle γ passing through A', A'_1, N' and S'. The circular arc $A'A'_1$ of γ' will be called the arc of correspondence. The plane which passes through O and being parallel to π cuts φ in the principal meridian f. The stereographic projection of f into π is a circle f". Figure (3) illustrates the construction of the spherical perspective [A', A'_1] of A which is given by its orthogonal projections (A_1, A_2). The picture sphere is represented by its orthogonal projections, and the picture plane is taken as a vertical plane and represented by its horizontal trace t_1. Points (A''_1, A''_2) and (A''_{11}, A''_{12}) are the orthogonal projections of the gonomonic projections A", A''_1 of points A, A_1 into φ. Points (A''_1, A''_2), (A''_{11}, A''_{12}) are obtained by the well known method of descriptive geometry which determines the intersection points of the rays OA and OA_1 with φ. The points A'_1 and A'_{11}, Figure (3), are the horizontal projections of the stereographic projections of A" and A''_1. From A'_1 and A'_{11} the spherical perspective [A', A'_1] is located on the two rays $O_2A''_2$ and O_2 A''_{12} on the vertical plane as shown in Figure (3). In Figure (3) it is to be noted that: (1) The circle f' is the stereographic projection of the principal meridian f of φ which is parallel to π, (2) The stereographic projection e' of e coincides with the horizon line h.

In this mode of representation it will be considered that the positions of the geometric elements should be located in front of both E and the plane of the principal meridian. This will locate the spherical perspective of these geometric elements inside the circle f'.

SPHERICAL PERSPECTIVE OF STRAIGHT LINES

The gonomonic projections of a line m(A, B) and of its horizontal projection $m_1(A_1, B_1)$ from O into φ are two circular arcs m"[A", B"] and $m''_1[A''_1, B''_1]$ of two great circles. The stereographic projections of m" and m''_1 are two circular arcs m'{A', B'} and $m'_1\{A'_1, B'_1\}$ which are the spherical perspective of m.

It is known in spherical geometry [2,4] that the role played by lines in plane geometry is taken over by the great circles of a sphere. This is due to :(1) Among all curves joining any two points A",B" on a sphere φ the shortest is the arc of great circle γ''. (2) Through any two points A",B" on φ, not being antipodal points, there passes a unique great circle γ''. These yield the following fact; through any two points A',B' in π there passes a unique circle γ' which is the stereographic projection of γ''. Then, throughout this paper it is meant by the notation $\gamma'\{A',B'\}$ a circle γ' passes through two points A',B'.

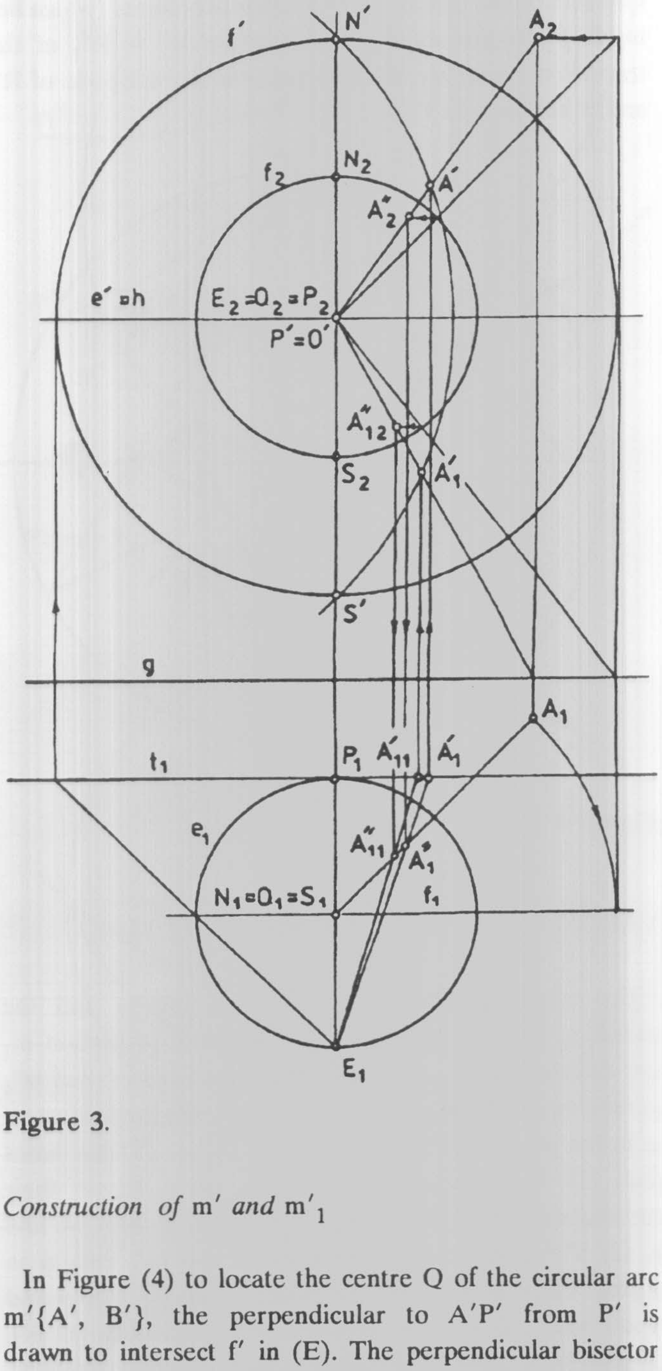

Figure 3.

Construction of m' *and* m'_1

In Figure (4) to locate the centre Q of the circular arc m'{A', B'}, the perpendicular to A'P' from P' is drawn to intersect f' in (E). The perpendicular bisector of (E)A' intersects A'P' at D. The line d through D and perpendicular to A'P' is the line of centres of point

A′, [6], that is the locus of centres of all circles which are the stereographic projections of all great circles on φ passing through A″. Similarly, the line k of centres of B′ is constructed. The centre Q is the intersection of d and k. By the same method the circular arc $m'_1\{A'_1, B'_1\}$ can be constructed. It is to be noted that m′ and m'_1 intersect in two points. One of these two points (which lies inside f′) is the spherical perspective $R' = R'_1$ of the trace R of m on π_1, and the other is the antipode of R″ and it lies outside f′.

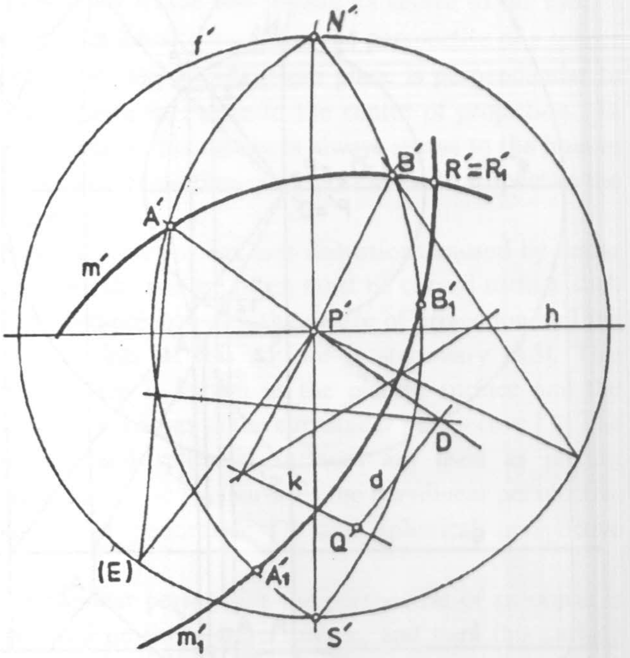

Figure 4.

VANISHING POINTS OF A STRAIGHT LINE

The vanishing points [I′, J′] of a line m and the vanishing points $[I'_1, J'_1]$ of its horizontal projection m_1 are determined by drawing from O two lines m° and $m°_1$ parallel to m and m_1, Figure (5). These two lines intersect φ in the points (I, J) and (I_1, J_1). In Figure (5) one notes that: (1) The two points (I_1, J_1) lie on e. Hence their stereographic projections $[I'_1, J'_1]$ lie on the horizon line h. (2) The plane of the two lines m° and $m°_1$ cuts φ in a great circle k on which the points (I, J), (I_1, J_1), N and S are lying. Hence, the stereographic projections [I′, J′], $[I'_1, J'_1]$, N′ and S′ of these points lie on a circle k′ which is the stereographic projection of k. In Figure (6) the line m is represented by its orthogonal projections (m_1, m_2). To locate [I′, J′] on m′ and $[I'_1, J'_1]$ on m'_1, two lines $m°_1$ and $m°_2$ are drawn from O_1 and P′ parallel to m_1 and m_2. The line $m°_1$ intersects e_1 in the points (I_1, J_1) from which one determines $[I'_{11}, J'_{11}]$ on t_1 and then $[I'_1, J'_1]$ on h. The circle k′ is drawn with I'_1 J'_1 as diameter, and it must pass through N′ and S′. The intersection points of k′ and $m°_2$ are [I′,J′].

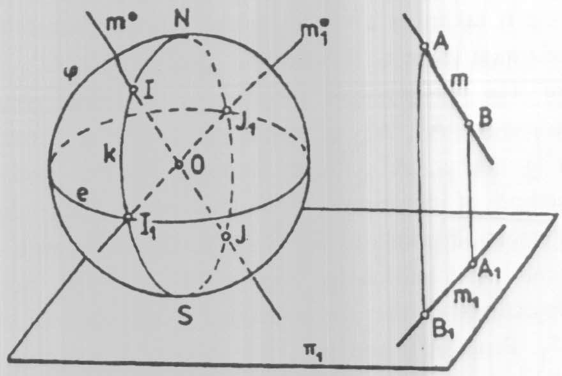

Figure 5.

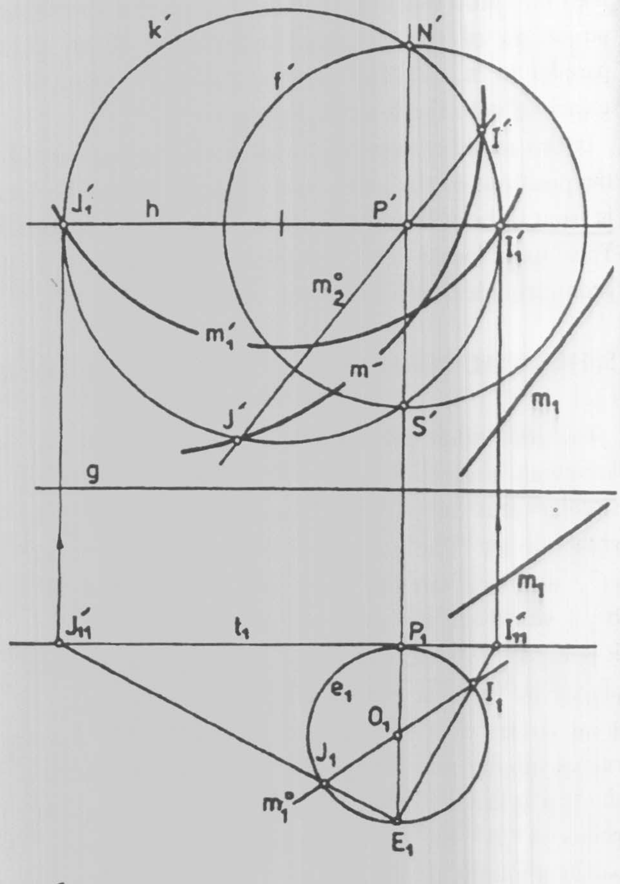

Figure 6.

In case when m is represented by its spherical perspective [m', m'_1], the vanishing points are located on m' and m'_1 as follows: The intersection points of m'_1 and h are [I'_1, J'_1]. The circle k' is drawn with $I'_1J'_1$ as diameter to intersect m' in I', J'.

PARTICULAR POSITIONS OF A LINE

i) Line m perpendicular to π_1: m'_1 is a point, and m' is a circular arc passing through N' and S', Figure (7). The vanishing points of m are S' and N'.

ii) Line n perpendicular to π: n' and n'_1 are two straight lines passing through P', Figure (7). The vanishing points of n coincide with P'.

iii) Line r parallel to π_1: r' and r'_1 are two circular arcs, Figure (7). The vanishing points [F', V'] of r and the vanishing points [F'_1, V'_1] of r_1 coincide as shown in Figure (7).

iv) Line q parallel to π: q' and q'_1 are two circular arcs, Figure (7). The vanishing points of q lie on f'.

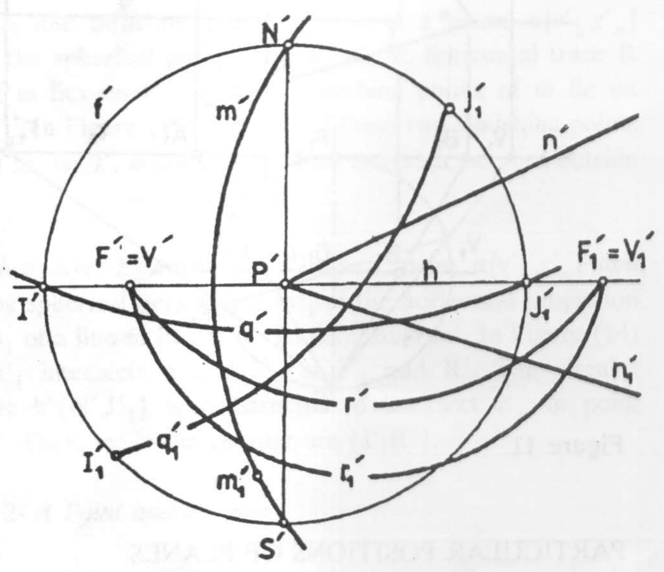

Figure 7.

RELATIVE POSITIONS OF TWO LINES

Intersecting Lines

If two lines m and n have a common point M, then the common point M' of m' and n', and the common point M'_1 of m'_1 and n'_1 must lie on a single arc of correspondence as in Figure (8).

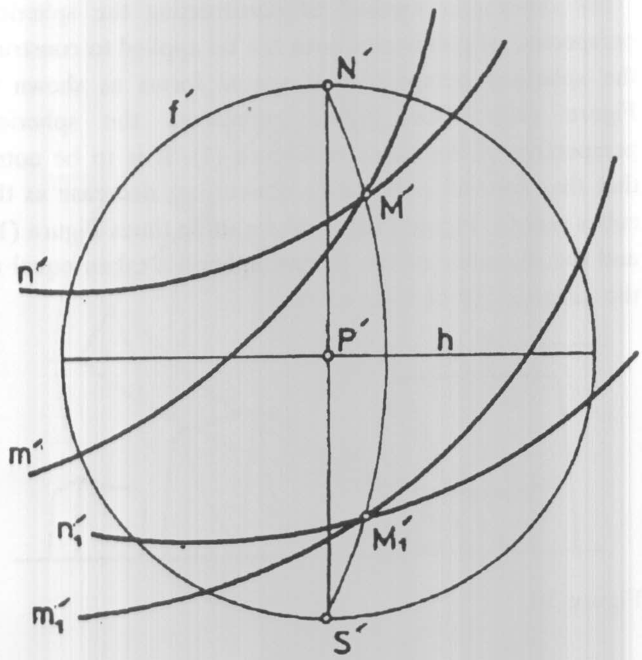

Figure 8.

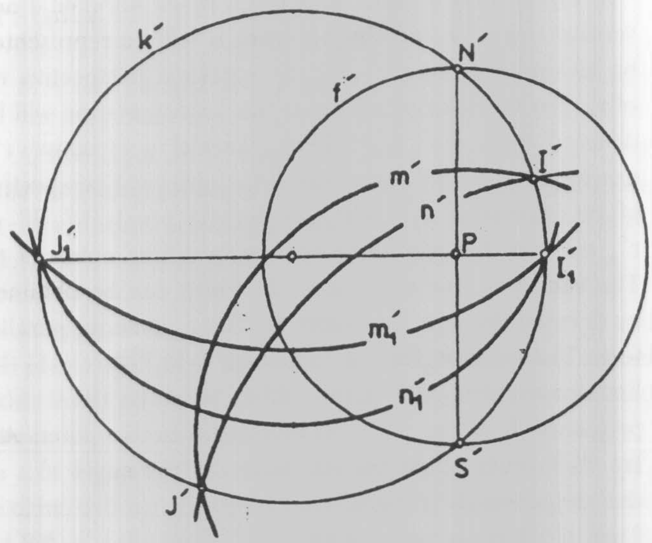

Figure 9.

Parallel Lines

Two lines m and n are parallel if m' and n' have two common vanishing points [I', J'], Figure (9), and m'_1 and n'_1 have two common vanishing points [I'_1, J'_1].

APPLICATION

The developed method of constructing the spherical perspective of points and lines can be applied to construct the spherical perspective of spatial forms as shown in Figure (10). This Figure represents the spherical perspective of the cubes of Figure (1). It is to be noted that the over-all perspective dimensions decrease as the cubes recede. Figure (10) is drawn three times Figure (1), and the diameter of the picture sphere is taken equal to the distance EP of Figure (1).

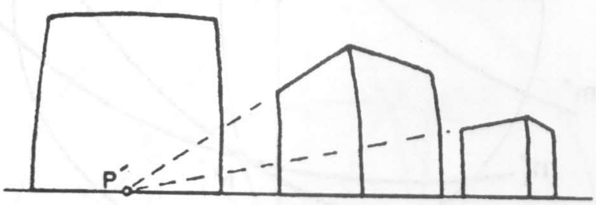

Figure 10.

REPRESENTATION OF PLANES

In Figure (11) a plane α is given by its horizontal and vertical traces (r_1, s_2). In this work α will be represented by its vanishing line v' and the spherical perspective r'_1 of r_1. For the sake of reference this representation will be denoted by $\alpha[v'_\alpha, r'_\alpha]$. The spherical perspective r'_α can be obtained by constructing the spherical perspective of two points of r_1 . The intersection points V', F' of r'_α with the horizon line h are the vanishing points of r_1. The vanishing line v'_α is a circle which can be obtained by drawing through the center O of φ a plane α_0 parallel to α. The plane α_0 cuts φ in a circle v. In Figure (11) the line drawn through P' parallel to s_2 intersects the vertical projection f_2 of the principal meridian f in two points A_2, B_2. These two points are extremities of the major axis of the ellipse which is the vertical projection of the circle v. Then the stereographic projections A', B' of A and B lie simultaneously on both f' and v'_α. Similarly the line drawn through O_1 parallel to r_1 intersects e_1 in two points V_1, F_1. These two points are the extremities of the major axis of the ellipse which is the horizontal projection of the circle v. The stereographic projection of V_1, F_1 are V', F' which are the vanishing points of r_1 as mentioned before. These points V', F' lie on v'_α. Therefore, v'_α is the circle which is drawn to pass through three points of the four points A', B', V', F' as shown in Figure (11).

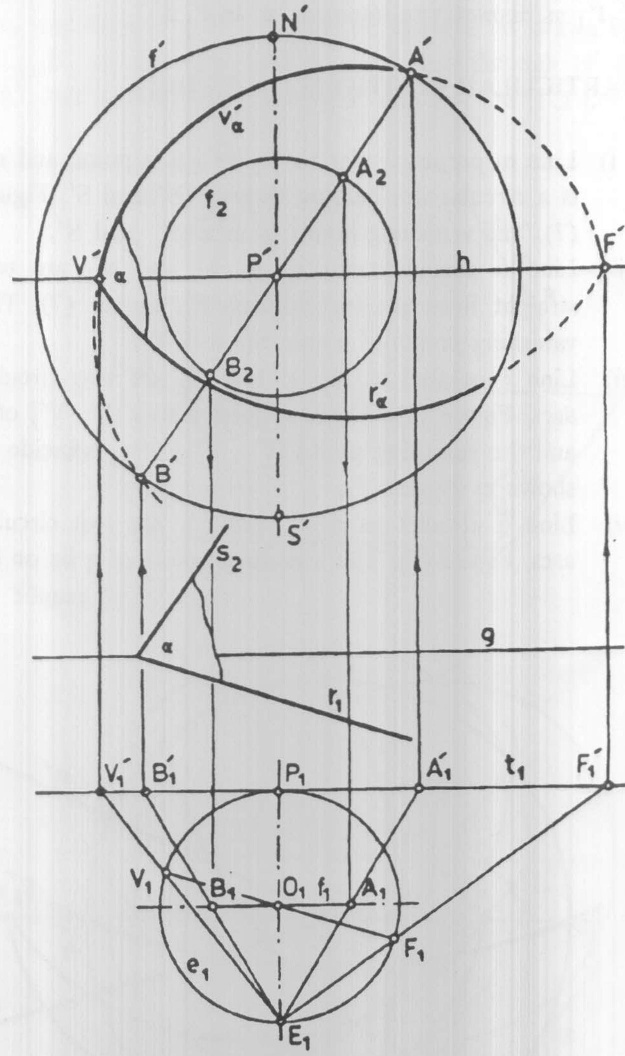

Figure 11.

PARTICULAR POSITIONS OF PLANES

(i) A plane α parallel to π, Figure (12): v'_α coincides with f' and r'_α is a circular arc.

(ii) A plane β parallel to π_1, Figure (12): v'_β and r'_β are two straight lines coincide with h.

(iii) A plane γ perpendicular to both π and π_1, Figure (12): v'_γ and r'_γ are two straight lines, where v'_γ coincides with the line P'N', and r'_α passes through P'.

(iv) A plane α perpendicular to π, Figure (13): v'_α and

r'_α are two straight lines pass through P'.

(v) A plane β perpendicular to π_1, Figure (13): v'_β and r'_β are two circular arcs, where v'_β passes through N'.

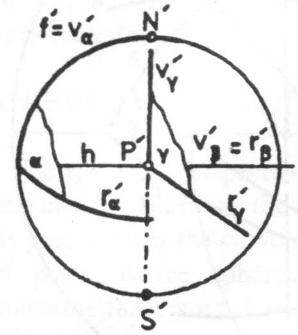

Figure 12.

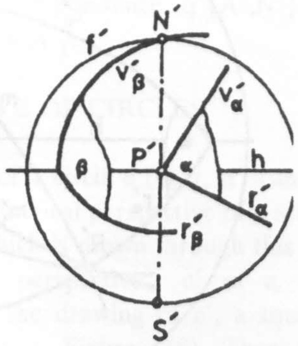

Figure 13.

PROBLEMS OF COINCIDENCE

1- *A line and a Plane*

A line $m[m',m'_1]$ is contained in a plane $\alpha[v'_\alpha,r'_\alpha]$ if the spherical perspective R' of the horizontal trace R of m lies on r'_α, and the vanishing points of m lie on v'_α. In Figure (14) one only of these two vanishing points of m, say J', is considered since the other one lies outside f'.

Illustrative Example (1): Given a plane $\alpha[v'_\alpha,r'_\alpha]$ and the spherical perspective m'_1 of the horizontal projection m_1 of a line m lies in α. To determine m'. In Figure (14) m'_1 intersects h and r'_α at J'_1 and R'. The circular arc $k'\{N',J'_1\}$ is constructed to intersect v'_α in point J'. Then, m' is the circular arc {J',R'}.

2- *A Point and a Plane*

A point $A[A',A'_1]$ is contained in a plane $\alpha[v'_\alpha,r'_\alpha]$ if there exists a line $m[m',m'_1]$ passing through A and lying in α, Figure (14).

Illustrative Example (2): Given a plane $\alpha[v'_\alpha,r'_\alpha]$ and A'_1 of a point A lies in α. To determine A'. In Figure (14) the spherical perspective R' of the horizontal trace of a line m is chosen on r'_α. Then, a circular arc $m'_1\{A'_1,R'\}$ is constructed to intersect h in J'_1. The spherical perspective m', is determined as in the illustrative example (1). The arc of correspondence $q'\{N',A'_1\}$ is constructed to intersect m' in A'.

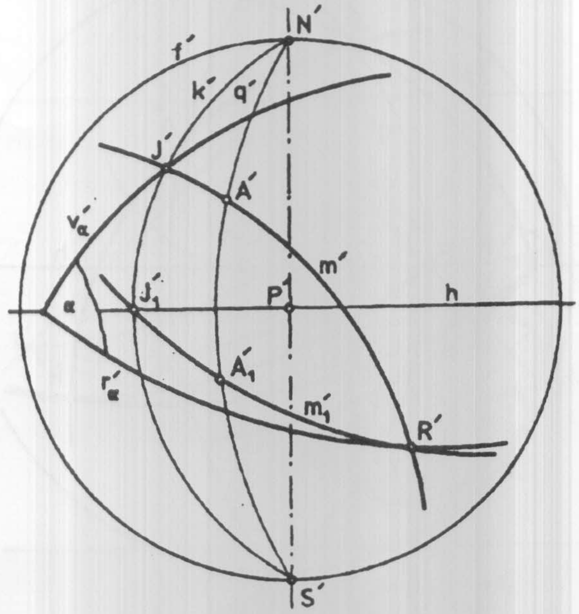

Figure 14

PROBLEMS OF PARALLELISM

1- *A line parallel to a Plane*

A line $m[m',m'_1]$ is parallel to a plane $\alpha[v'_\alpha,r'_\alpha]$ if the vanishing points of m lie on v'_α. In Figure (15) one only of these vanishing points are considered, which is J', for the same reason mentioned before.

Illustrative Example (3): Given a plane $\alpha[v'_\alpha,r'_\alpha]$ and a point $A[A',A'_1]$ not lying in α. To determine one of the lines, say $m[m',m'_1]$, which passes through A and being parallel to α.

In Figure (15) a point J'_1 is chosen on h. Then m'_1 is the circular arc $\{J'_1,A'_1\}$. The arc of correspondence $q'\{N',J'_1\}$ is constructed to intersect v'_α in J'. Hence m' is the circular arc {J',A'}.

2- *Two Parallel Planes*

Two planes $\alpha[v'_\alpha,r'_\alpha]$ and $\beta[v'_\beta,r'_\beta]$ are parallel if they have a common vanishing line $v'_\alpha=v'_\beta$ as shown in Figure (15).

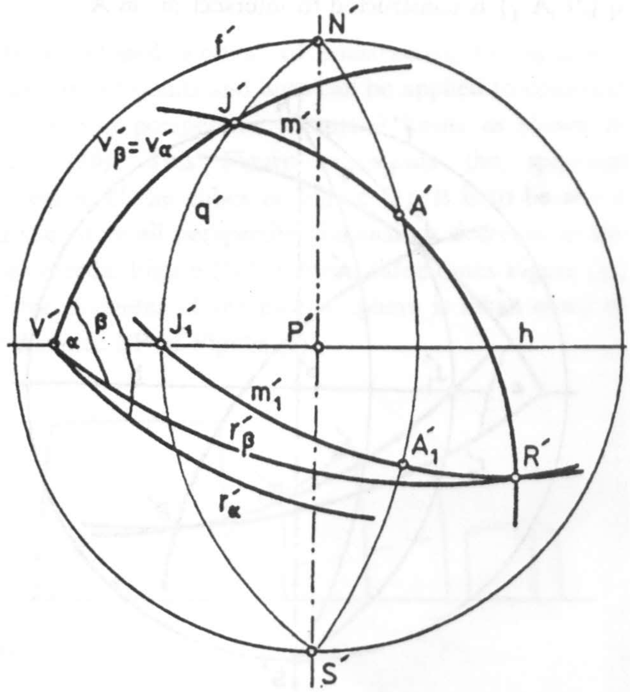

Figure 15.

Illustrative Example (4): Given a plane $\alpha[v'_\alpha, r'_\alpha]$ and a point $A[A', A'_1]$ not lying in α. To determine the plane $\beta[v'_\beta, r'_\beta]$ which passes through A and being parallel to α.

In Figure (15) a line $m[m', m'_1]$ passes through A and being parallel to α is constructed as in the illustrative example (3). Then r'_β is the circular arc $\{V', R'\}$ where R' is the intersection point of m' and m'_1.

PROBLEMS OF INTERSECTION

1- Two Intersecting Planes

The common line $m[m', m'_1]$ of two planes $\alpha[v'_\alpha, r'_\alpha]$ and $\beta[v'_\beta, r'_\beta]$ is constructed by determining the intersection points J', R' of v'_α, v'_β and r'_α, r'_β as shown in Figure (16). Then the arc of correspondence $q'\{N', J'\}$ is constructed to intersect h in J'_1. Hence m'_1 is the circular arc $\{J'_1, R'\}$ and m' is the circular arc $\{J', R'\}$.

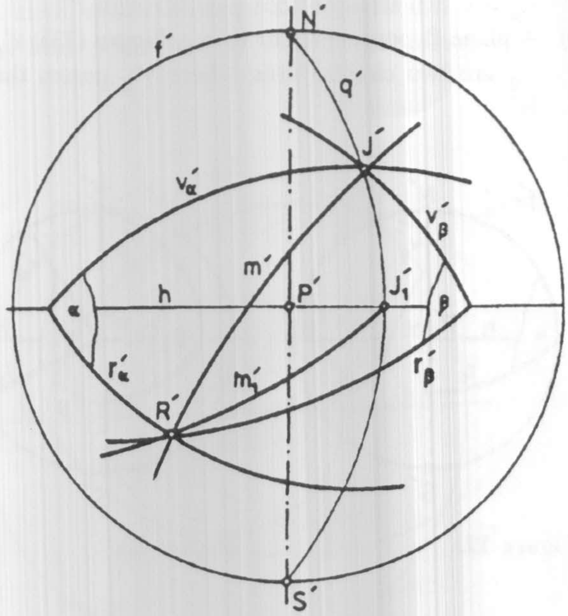

Figure 16

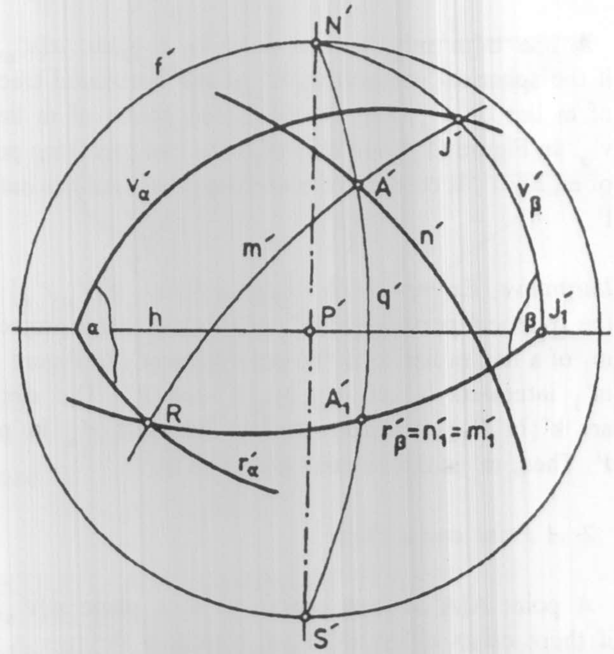

Figure 17

2- Intersection Point of a Line and a Plane

Given in Figure (17) a line $n[n', n'_1]$ and a plan $\alpha[v'_\alpha, r'_\alpha]$. To determine the common point A[A',A' of n and α. An auxiliary plane $\beta[v'_\beta, r'_\beta]$ passin through n is constructed. For simplicity β is taken perpendicular to π_1 where r'_β coincides with n'_1 and v'_β is the circular arc $\{N',J'_1\}$ as shown in Figure (17). The common line $m[m',m'_1]$ of α and β is constructed as given in Figure (16). Then, A′ is the intersection point of m′, n′. The arc of correspondence q′{A′,N′} is constructed to intersect n'_1 in A'_1.

SPHERICAL PERSPECTIVE OF CIRCLES

The spherical perspective of a circle c (Q, r, α) can be obtained by determine the spherical perspective of a set of its points. Then the curve which is drawn through this set of points is the spherical perspective c′ of c. To minimize the errors during the drawing of c′, a square ABCD is drawn tangent to c, Figure (18). Then the spherical perspective A′, B′, C′, D′, of the vertices of this square and the spherical perspective 1′, 2′, 3′, 4′, 5′, 6′, 7′, 8′, of the set of points 1, 2, 3, 4, 5, 6, 7, 8, are constructed. Hence, c′ is the curve drawn tangent to A′B′, B′C′, C′D′, D′A′ and passing through the points 1′ ,2′, 3′, 4′, 5′, 6′, 7′, 8′.

It is preferred that one of the side of the square is parallel to the picture plane.

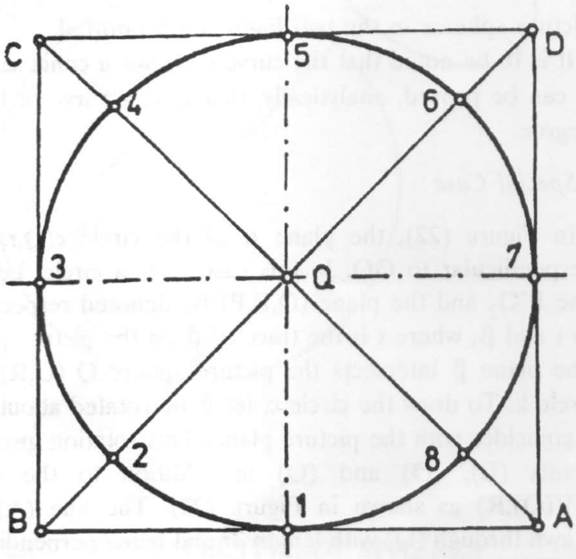

Figure 18.

Construction of c′

In Figure (19). the circle c is represented by its orthogonal projections (c_1, c_2). The plane α of c is in a general position with respect to the picture plane.

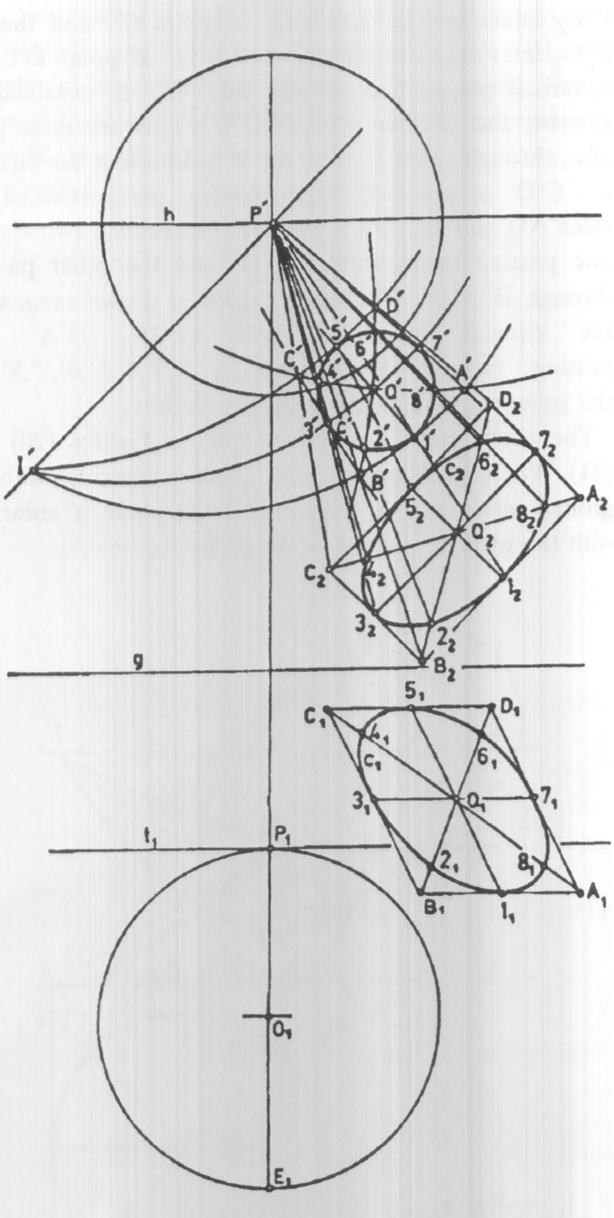

Figure 19.

To draw c′, the spherical perspective A′, Q′ of A, Q are determined. One of the vanishing points, say I′, of the parallel lines AB,37 and CD is determined. The spherical perspectives of lines AB,37 are constructed by drawing the circular arcs {I′,A′} and {I′,Q′}. The rays $P'1_2$, $P'B_2$ intersect the circular arc I′A′ at points 1′ and B′. The rays $P'3_2$ and $P'7_2$ intersect the circular arc I′Q′ in the points 3′ and 7′. The spherical perspective of the diagonals AC and DB are drawn by constructing the two circular arc {A′,Q′} and {B′,Q′} . The ray

$P'C_2$ intersects the arc $A'Q'$ at point C', and the ray $P'D_2$ intersects the circular arc $B'Q'$ at point D'. The spherical perspective of the side CD is obtained by drawing the circular arc $\{C',D'\}$. This arc must pass, also, through point I'. The ray $P'5_2$ intersect the circular arc $C'D'$ at point $5'$. The spherical perspective of the sides AD and BC are drawn by constructing two circles, one passing through A', $7'$, D' and the other passing through B',$3'$,C'. The curve which is drawn tangent to the circular arcs $A'B'$,$B'C'$, $C'D'$, $D'A'$ and passing through the points $1'$,$2'$,$3'$,$4'$,$5'$,$6'$,$7'$,$8'$ is the spherical perspective c' of the circle c.

The above steps are carried out on Figures (20) and (21). Where in Figure (20) the plane α coincides with the ground plane, and in Figure (21) the plane α coincides with the picture plane. For simplicity the

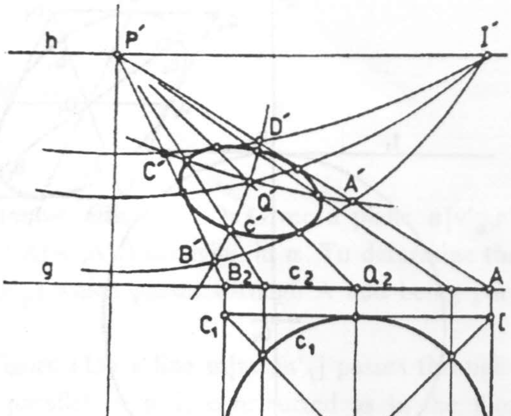

Figure 20.

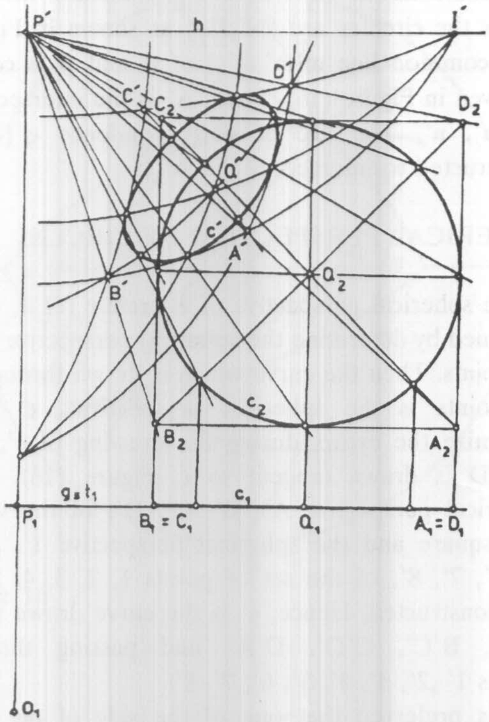

Figure 21.

picture spheres in the two Figures are omitted.

It is to be noted that the curve c' is not a conic section. It can be proved, analytically, that c' is a curve of fourth degree.

Special Case

In Figure (22), the plane α of the circle $c(Q,r,\alpha)$ is perpendicular to OQ. In this case c' is a circle. Let the line $P'Q_2$ and the plane (Q,E,P) be denoted respectively by t and β, where t is the trace of β on the picture plane. The plane β intersects the picture sphere Q (O,R) in a circle k. To draw the circle c, let β be rotated about t till it coincides with the picture plane. This rotation gives the points (E), (O) and (Q) in addition to the circle (k)((O),R) as shown in Figure (22). The line (A) (B) drawn through (Q) with length 2r and being perpendicular to (O)(Q) represents the circle c after the rotation of β. The two rays (O)(A) and (O)(B) intersect (k) at the points [A] and [B]. The two rays (E)[A] and (E)[B] intersect t at the two point A' and B'. Where these two points are extremities of a diameter of the required circle c'.

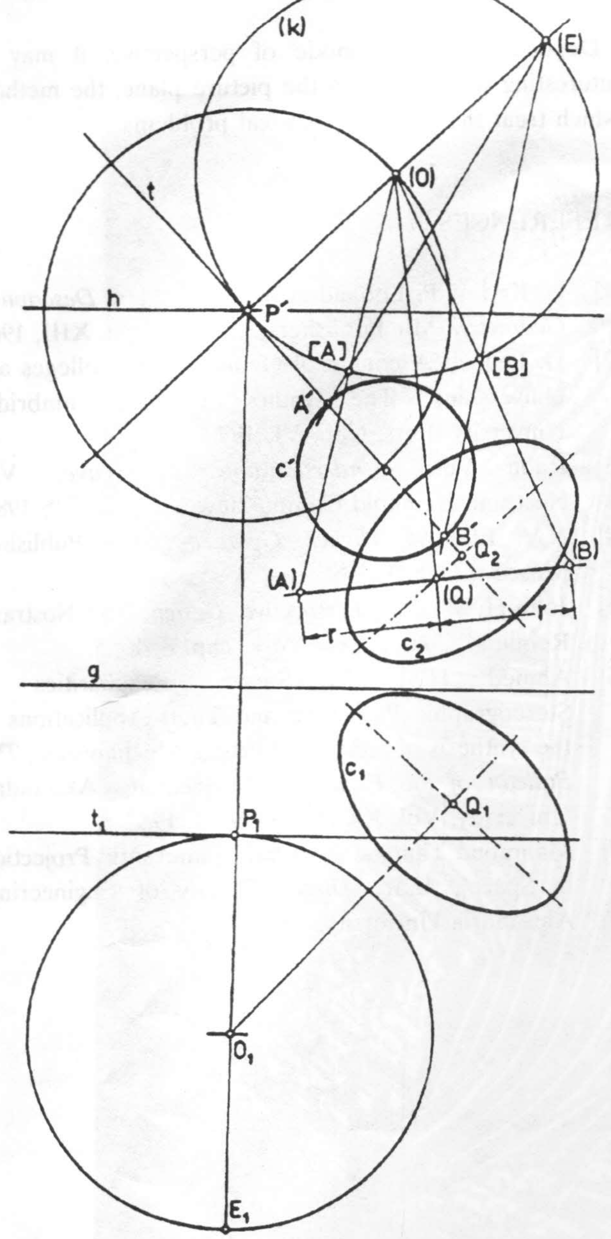

Figure 22.

the coincidence with the picture plane gives the two circles (k)((O),R) and (q)((Q),r).The tangents from (O) to the circle (q) intersect (k) at the points [A] and [B]. The two rays (O)[A] and (O)[B] intersect t at the extremities A′ and B′ of a diameter of c′.

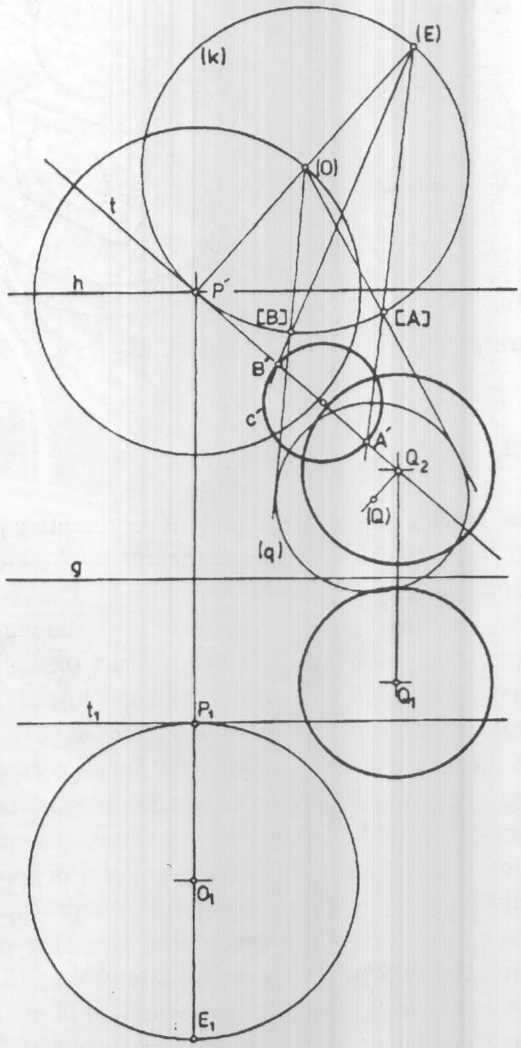

Figure 23.

SPHERICAL PERSPECTIVE OF SPHERES

The spherical perspective of a sphere Ψ(Q,r) is a circle c′. The sphere Ψ is represented in Figure (23) by its orthogonal projections. To draw the circle c′. let the circles of intersection of the plane β (Q,E,P) with the picture sphere φ (O,R) and the sphere Ψ be donated by k and q. The rotation of β about its trace t (P′,Q_2) till

APPLICATION

The methods of constructing the spherical perspective of circles and spheres are used to construct the spherical perspective of the object which is given in Figure (24) by its orthogonal projections. This spherical perspective is shown in Figure (25).

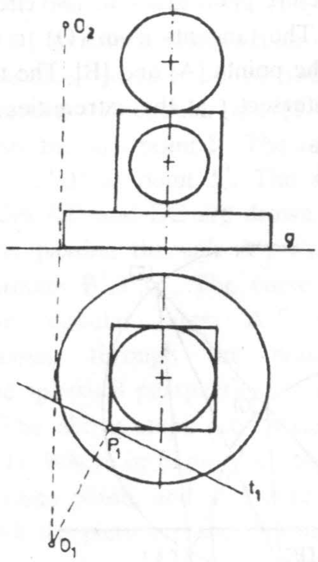

Figure 24.

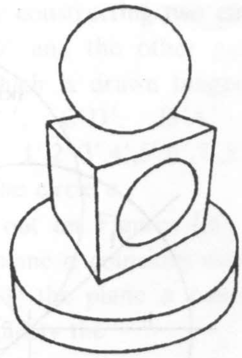

Figure 25.

CONCLUSION

In this paper a graphical method for representing points, straight lines, planes, circles and spheres by the mode of the spherical perspective is developed. The method depends on the gonomonic and the stereographic projections and it can be used to construct the spherical perspective of spatial forms as in Figures (10), (25). The spherical perspective can be considered as a general answer for any distortion caused by linear perspective. Although the straight lines are represented as curves , but this curvature can be minimized by choosing a picture sphere of large radius. One of the privileged of spherical perspective is that, all lines in any position with respect to the picture plane have two accessible vanishing points. Moreover, the angle of vision can be extended up to 180^{o}.

It have been shown that the representation of planes by their vanishing lines and the spherical perspective of their horizontal traces is convenient to solve the problems related to these planes.

The spherical perspective of a circle is not a conic section but a curve of fourth degree. This curve is only a circle when the plane of the circle is perpendicular to the line joining its center to the center of the picture sphere. The distortion caused by the spherical perspective of a circle increases when the angle between the plane of this circle and the line joining its center to the center of the picture sphere decreases as shown in Figure (21).

On the other hand, the spherical perspective of a sphere is always a circle.

Depending on this mode of perspective, it may be interesting to develop, on the picture plane, the methods which treat the so-called metrical problems.

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