ON TRANSFORMATIONS OF CARTESIAN COORDINATES

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ABSTRACT

This paper is devoted to investigate some kinds of Transformation of cartesian coordinates. It also presents a brief study of errors propagation which are arising in these transformations based on least squares. The analysis of each method of transformation is discussed and some conclusions and recommendations in that phase are stated.

INTRODUCTION

The coordinate system with its various kinds and varieties is considered to be a convenient method for recording position in both plane or space, especially for geodetic purposes such as monitoring of spatial deformation for large constructions like towers and dames as well as monitoring of crustal movements.

The coordinates of control points to be used in orienting the behaviour and amount of the monitored movement have to be transferred along and transverse to axes of the structure to facilitate such orientations.

In most cases, however, the geodetic monitoring of deformations involves coordinates of control points located on the construction, related to coordinates of established micro-net in a transformative form to serve the purpose of monitoring.

1. Spatial Transformation in X, Y Coordinates

Regarding general transformations, all points of the first system are transformed to the second system by considering the following items:

(a) Scale factor.
(b) Rotation through space.
(c) Translation through space.

This can be expressed in the following relation (Figure (1)):

$$\left.\begin{aligned} y_i' &= f_1(y,x) = a_1 y_i' + b_1 x_i + c_1 \\ x_i' &= f_2(y,x) = a_2 y_i + b_2 x_i + c_2 \end{aligned}\right\} \quad (I.1)$$

where
a_1, b_1, c_1, a_2, b_2 and c_2 = transformation parameters
x_i, y_i = coordinates point (i) in first system
x_i', y_i' = coordinates same point in second system.

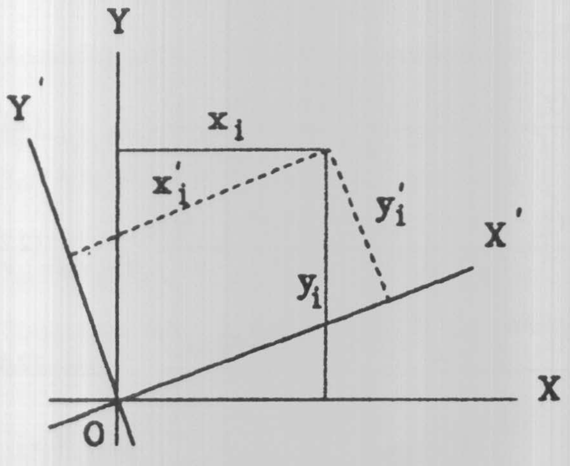

Figure 1.

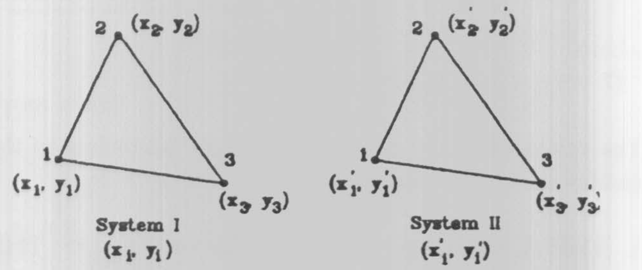

Figure 2.

Now, consider three points 1, 2, and 3 which are common to two coordinate systems (Figure (2)).

The transformation formulae from system I to system II from Equation I.1 can be in the form:

$$\left.\begin{aligned} y_1' &= a_1y_1+b_1x_1c_1 \quad x_1' = a_2y_1+b_2x_1+c_2 \\ y_2' &= a_1y_2+b_1x_2c_1 \quad x_2' = a_2y_2+b_2x_2+c_2 \\ y_3' &= a_1y_3+b_1x_3c_1 \quad x_3' = a_2y_3+b_2x_3+c_2 \end{aligned}\right\} \quad (I.2)$$

2. DETERMINATION OF THE TRANSFORMATION PARAMETERS

Providing that there are at least two points which are common in use for both systems, these terms can be calculated and adopted to convert as many additional points as required. The method for determining these parameters can be performed as follows:

$$y_1' - y_2' = a_1(y_1-y_2)+b_1(x_1-x_2), x_1' - x_2' = a_2(y_1-y_2)+b_2(x_1-x_2)$$

$$y_2' - y_3' = a_1(y_2-y_3)+b_1(x_3-x_3), x_2' - x_3' = a_2(y_2-y_3)+b_2(x_2-x_3)$$

and hence,

$$a_1 = \frac{(y_1' - y_2')(x_2 - x_3) - (y_2' - y_3')(x_1 - x_2)}{D} \tag{2.1}$$

$$b_1 = \frac{(y_1 - y_2)(x_2' - x_3') - (y_2 - y_3)(y_1' - y_2')}{D} \tag{2.2}$$

$$a_2 = \frac{(x_1' - x_2')(x_2 - x_3) - (x_2' - x_3')(x_1 - x_2)}{D} \tag{2.3}$$

$$b_2 = \frac{(y_1 - y_2)(x_2' - x_3') - (x_1' - x_2')(y_2 - y_3)}{D} \tag{2.4}$$

where

$D = (y_1-y_2)(x_2-x_3)-(y_2-y_4)(x_1-x_2)$

The values of a_2, b_2 can be determined according to the conditioned conformality ($b_1 = - a_2$ and $a_1 = b_2$).

3. THE CALCULATED COORDINATES OF THE NEW SYSTEM

After calculating the transformation parameters, let us consider several points which are common related to two coordinates.

Assuming initial point (x_o, y_o) in the first system, according to Equation (1.2) the coordinates transformed in the second system are:

$$\left.\begin{aligned} y_o' &= a_1 y_o + b_1 x_o + c_1 \\ x_o' &= a_2 y_2 + b_2 x_o + c_2 \end{aligned}\right\} \tag{3.1}$$

where:

$(y_{i\,min} \rightarrow y_o$ and $x_{i\,min} \rightarrow x_o)$

From Equation (1.2) and (3.1), we have

$$\left.\begin{aligned} y_i' - y_o' &= a_1(y_i-y_o)+b_1(x_i-x_o) \\ x_i' - x_o' &= a_2(y_i-y_o)+b_2(x_i-x_o) \end{aligned}\right\} \tag{3.2}$$

from which:

$$\left.\begin{aligned} y_o' &= y_i' - a_1(y_i-y_o) - b_1(x_i-x_o) \\ x_o' &= x_i' - a_2(y_i-y_o) - b_2(x_i-x_o) \end{aligned}\right\} \tag{3.3}$$

Now consider three points which are common to two coordinate system. From Equation (3.3) we obtain three values for both y_o' and x_o'. Hence, the average value is:

$$\left.\begin{aligned} y'_{oavr} &= \frac{y'_{o1} + y'_{o2} + y'_{o3}}{3} \\ x'_{oavr} &= \frac{x'_{o1} + x'_{o2} + x'_{o3}}{3} \end{aligned}\right\} \tag{3.4}$$

Asuming four points, hence the average value is:

$$y'_{oavr} = \frac{y'_{o1} + y'_{o2} + y'_{o3} + y'_{o4}}{4}$$

$$x'_{oavr} = \frac{x'_{o1} + x'_{o2} + x'_{o3} + x'_{o4}}{4}$$

We can transform from system (x_i,y_i) to another (x'_i,y'_i) according to Equation (3.2).

For the translation of origin, generally all available common known points between the two systems are used, the average being considered as the origin namely:

$$x'_o = \frac{\sum x'}{n} \qquad y'_o = \frac{\sum y'}{n}$$

Where n is the number of such known points.

Generally, the transformation form can be written as follows:

$$\left.\begin{aligned} y'_i &= y'_o + a_1(y_i - y_o) + b_1(x_i - x_o) \\ x'_i &= x'_o + a_2(y_i - y_o) + b_2(x_i - x_o) \end{aligned}\right\} \quad (3.5)$$

From the above mentioned treatment, it may be concluded that the transformation of points can be easily performed using the above mentioned method provided that number of points not exceed than four.

4. TRANSFORMATIONS OF SEVERAL STATIONS USING LEAST SQUARES

If there are more than three common points, such as occurs in cartographic digitising and in the adjustment of aerial triangulation to several round control points, the determination of the parameters from only two or three of the points is unsatisfactory. The coordinates of any of these points may contain small errors and their use will accordingly introduce some sort of errors into the transformation operation for all other points. Under these circumstances, all the data which would be available for determination of a_1, b_1, a_2 and b_2 ought to be taken into consideration. This involves a solution of the parameters by the methods of least squares, which is a more sophisticated numerical solution based upon statistical theory of errors [2].

Assuming v_x and v_y are error values of coordinates, according to the basic principle of the method of least squares:

$$[v_x^2 + v_y^2] = \min \quad (4.1)$$

Consider several points "n" which are common to two coordinate systems.

The transformation formulae, in the general case, from system I (x_i, y_i) to system II (x'_i, Y'_i) can be easily performed in the form:

$$\left.\begin{aligned} y'_1 &= a_1 y_1 + b_1 x_1 + c_1 & x'_1 &= a_2 y_1 + b_2 x_1 + c_2 \\ y'_2 &= a_1 y_2 + b_1 x_2 + c_1 & x'_2 &= a_2 y_2 + b_2 x_2 + c_2 \\ &\cdots\cdots & &\cdots\cdots \\ &\cdots\cdots & &\cdots\cdots \\ y'_n &= a_1 y_n + b_1 x_n + c_1 & x'_n &= a_2 y_n + b_2 x_n + c_2 \end{aligned}\right\} \quad (4.2)$$

4.1 *Determination of the Transformation Prameters*

From Equation (4.2) we can obtain the values of parameters (unknowns) of number (m = 6). Where number of equations (n). Where n > m, therefore, using parametric method in solution.

Assuming coordinates x'_i and y'_i would be determined with error $v_{x\,i}$, $v_{y\,i}$ respectively, where:

$$\left.\begin{aligned} y'_1 + v_{y1} &= a_1 y_1 + b_1 x_1 + c_1 & x'_1 + v_{x1} &= a_2 y_1 + b_2 x_1 + c_2 \\ y'_2 + v_{y2} &= a_1 y_2 + b_1 x_2 + c_1 & x'_2 + v_{x2} &= a_2 y_2 + b_2 x_2 + c_2 \\ &\cdots\cdots & &\cdots\cdots \\ &\cdots\cdots & &\cdots\cdots \\ y'_n + v_{yn} &= a_1 y_n + b_1 x_n + c_1 & x'_n + v_{xn} &= a_2 y_n + b_2 x_n + c_2 \end{aligned}\right\} \quad (4.1.1)$$

According to (4.1.1), these equations lead to:

$$\left.\begin{aligned} v_{y1} &= a_1 y_1 + b_1 x_1 + c_1 - y'_1 & v_{x1} &= a_2 y_1 + b_2 x_1 + c_2 - x'_1 \\ v_{y2} &= a_1 y_2 + b_1 x_2 + c_1 - y'_2 & v_{x2} &= a_2 y_2 + b_2 x_2 + c_2 - x'_2 \\ &\cdots\cdots & &\cdots\cdots \\ &\cdots\cdots & &\cdots\cdots \\ v_{yn} &= a_1 y_n + b_1 x_n + c_1 - y'_n & v_{xn} &= a_2 y_n + b_2 x_n + c_2 - x'_n \end{aligned}\right\} \quad (4.1.2)$$

Summation values in Equation (4.1.2) and dividing by n, we obtain:

$$\left.\begin{aligned} \frac{[v_y]}{n} &= \frac{[y]}{n}a_1 + \frac{[x]}{n}b_1 + c_1 - \frac{[y']}{n} = 0 \\ \frac{[v_x]}{n} &= \frac{[y]}{n}a_2 + \frac{[x]}{n}b_2 + c_2 - \frac{[x']}{n} = 0 \end{aligned}\right\} \quad (4.1.3)$$

From which

$$\left.\begin{aligned} c_1 &= y'_o - y_o\, a_1 - x_o b_1 \\ c_2 &= x'_o - y_o\, a_2 - x_o b_2 \end{aligned}\right\} \quad (4.1.4)$$

where:

$$y'_o = \frac{[y']}{n}, x'_o = \frac{[x']}{n}, y_o = \frac{[y]}{n}, x_o = \frac{[x]}{n} \quad (4.1.5)$$

Substituting Equation (4.1.3) from (4.1.2) we get:

$$\left.\begin{aligned} v_{y1} &= a_1\bar{y}_1 + b_1\bar{x}_1 - \bar{y}'_1 & v_{x1} &= a_2\bar{y}_1 + b_2\bar{x}_1 - \bar{x}'_1 \\ v_{y2} &= a_1\bar{y}_2 + b_1\bar{x}_2 - \bar{y}'_2 & v_{x2} &= a_2\bar{y}_2 + b_2\bar{x}_2 - \bar{x}'_2 \\ &\cdots\cdots & &\cdots\cdots \\ &\cdots\cdots & &\cdots\cdots \\ v_{yn} &= a_1\bar{y}_n + b_1\bar{x}_n - \bar{y}'_n & v_{x2} &= a_2\bar{y}_2 + b_2\bar{x}_2 - \bar{x}'_2 \end{aligned}\right\} \quad (4.1.6)$$

where :

$$\bar{y}_i = y_{i\,Red} = y_i - \frac{[y]}{n} = y_i - y_o,\ [\bar{y}] = 0,$$

$$\bar{x}_i = x_{i\,Red} = x_i - \frac{[x]}{n} = x_i - x_o,\ [\bar{x}] = 0,$$

$$\bar{y}_i' = y_{i\,Red}' = y_i' - \frac{[y']}{n} = y_i' - y_o',\ [\bar{y}'] = 0,$$

$$\bar{x}_i' = x_{i\,Red}' = x_i' - \frac{[x']}{n} = x_i' - x_o',\ [\bar{x}'] = 0,$$

According to (4.1) and from Equations (4.1.6), the set of normal equations can be stated :

<u>Group 1</u>

$$\left.\begin{aligned} \frac{\partial[vv]}{\partial a_1} &: [\bar{y}^2]a_1 + [\bar{y}\,\bar{x}]b_1 - [\bar{y}\,\bar{y}'] = 0, \\ \frac{\partial[vv]}{\partial b_1} &: [\bar{y}\,\bar{x}]a_1 + [\bar{x}^2]b_1 - [\bar{x}\,\bar{y}'] = 0, \end{aligned}\right\} \quad (4.1.8)$$

and <u>Group 2</u>

$$\left.\begin{aligned} \frac{\partial[vv]}{\partial a_2} &: [\bar{y}^2]a_2 + [\bar{y}\,\bar{x}]b_2 - [\bar{y}\,\bar{y}'] = 0, \\ \frac{\partial[vv]}{\partial b_2} &: [\bar{y}\,\bar{x}]a_2 + [\bar{x}^2]b_2 - [\bar{x}\,\bar{x}] = 0, \end{aligned}\right\} \quad (4.1.9)$$

Directly from Equations (4.1.8), (4.1.9), the coefficients (parameters) of these equations will be in the form of:

$$\left.\begin{aligned} a_1 &= 1/D\left[[\bar{y}\,\bar{y}'][\bar{x}^2] - [\bar{y}\,\bar{x}][\bar{x}\,\bar{y}']\right] \\ b_1 &= 1/D\left[[\bar{y}^2][\bar{x}\,\bar{y}'] - [\bar{y}\,\bar{x}][\bar{y}\,\bar{y}']\right], \\ a_2 &= 1/D\left[[\bar{y}\,\bar{y}'][\bar{x}^2] - [\bar{y}\,\bar{x}][\bar{x}\,\bar{x}']\right] \\ b_2 &= 1/D\left[[\bar{y}^2][\bar{x}\,\bar{x}'] - [\bar{y}\,\bar{x}][\bar{y}\,\bar{x}']\right], \end{aligned}\right\} \quad (4.1.10)$$

where:

$$D = [\bar{y}^2][\bar{x}^2] - [\bar{y}\,\bar{x}]^2$$

Moreover, the translation formula becomes:

$$\left.\begin{aligned} y_i' &= y_o' + a_1(y_i - y_o) + b_1(x_i - x_o) \\ x_i' &= x_o' + a_2(y_i - y_o) + b_2(x_i - x_o) \end{aligned}\right\} \quad (4.1.1$$

This last technique is used for any number of statio and sets.

4.2 Estimate of Accuracy

Mean square error for unit weight can be calculate from the formula [1]:

$$\sigma_o = \sqrt{\frac{v\,v}{2n-6}}$$

where:

$$[v\,v] = [v_x^2 + v_y^2] = [\bar{y}'^2 + \bar{x}'^2] - [\bar{y}\,\bar{y}']a_1 - [\bar{x}\,\bar{y}']b_1 - [\bar{y}\,\bar{x}']a_2[\bar{x}\,\bar{x}']b_2$$

and the mean square error for coefficient (priori) is given by:

$$\left.\begin{aligned} \sigma_{a1} &= \sigma_{a2} = \sigma_o\sqrt{\frac{[\bar{x}^2]}{D}} \\ \sigma_{b1} &= \sigma_{b2} = \sigma_o\sqrt{\frac{[\bar{x}^2]}{D}} \end{aligned}\right\} \quad (4.2.2)$$

5. TRANSORTATION USING TWO POINTS ARE COMMON TO BOTH GRIDS

Provided that there are at least two points which ar common to both systems, these terms can be calculate and used to convert as many additional points as required The method of solving the unknowns may be carried ou as follows:

Consider two points A and B are common to bot systems. We use the following notation to describe eac point:

Point	1st System	2nd System
A	x_1, y_1	x_1', y_1'
B	x_2, y_2	x_2', y_2'

According to Equation (1.1) we have:

$$\left.\begin{aligned} y_1' &= a_1 y_1 + b_1 x_1 + c_1, \quad x_1' = a_2 y_1 + b_2 x_1 + c_2 \\ y_2' &= a_1 y_2 + b_1 2_1 + c_1, \quad x_2' = a_2 y_2 + b_2 x_2 + c_2 \end{aligned}\right\} \quad (5.1)$$

From Equation (5.1) we obtain:

$$\left.\begin{aligned} y_1' - y_2' &= = a_1(y_1 - y_2) + b_1(x_1 - x_2) \\ x_1' - x_2' &= = a_2(y_1 - y_2) + b_2(x_1 - x_2) \end{aligned}\right\} \quad (5.2)$$

According to condition conformality:

$$b = -a_2, \quad a_1 = b_2 \quad (5.3)$$

using condition (5.3), Equation (5.2) can be written in the following form:

$$\left.\begin{aligned} y_1' - y_2' &= = b_1(x_1 - x_2) + a_1(y_1 - y_2) \\ x_1' - x_2' &= = b_1(y_1 - y_2) + a_2(x_1 - x_2) \end{aligned}\right\} \quad (5.4)$$

from which it follows that:

$$a_1 = \frac{(x_1 - x_2)(x_1' - x_2') + (y_1 - y_2)(y_1' - y_2')}{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \frac{\Delta x \Delta x' + \Delta y \Delta y'}{S^2} \quad (5.5)$$

and

$$b_1 = \frac{(y_1' - y_2')(x_1 - x_2) - (x_1' - x_2')(y_1 - y_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \frac{\Delta y' \Delta x - \Delta x' \Delta y}{S^2} \quad (5.6)$$

where: Δx, Δy, $\Delta x'$ and $\Delta y'$ = different coordinates in both systems

$$S^2 = \Delta x^2 + \Delta y^2$$

from Equation (5.5) and (5.6), the evaluation of coefficients of translations can be performed.

Lastly, in practice, however, many if not most geodetic problems yield non-linear conditions. Direct non-linear least squares adjustment is so complex that is very rarely used if at all. Consequently, the original non-linear conditions are linearized using Taylor's Series, then least squares is iterated in order to eliminate the effect of higher order terms neglected in the linearization.

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