A GRADIENT METHOD IN HILBERT SPACE

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ABSTRACT

In this paper, a new gradient method for solving systems of linear simultaneous equations is suggested. The method is shown to converge, and the rate of convergence is investigated. An algorithm for implementing the method is presented. The results of applying the algorithm to various numerical problems illustrate the effectiveness and usefulness of the new technique.

1. INTRODUCTION

Several algorithms for solving linear systems of equations are given in the literature [1-13]. Consider the general linear system described by the equation:

$$A\,x = f \tag{1}$$

where the linear operator A, mapping a Hilbert space H into itself, is bounded, self adjoint and positive definite, i.e.,

$$\gamma \|x\|_2 \le (Ax,x) \le \delta \|x\|_2,\ \delta > \gamma > 0 \quad \forall x \in H \tag{2}$$

Most algorithms can be expressed as follows:

i) Select an initial approximation x_o to the solution x^* of (1).
ii) For k:= 1 until n, compute

$$x_k = x_{k-1} + \tau_k d_k$$

where d_k is chosen such that $(v_{k-1},r_k)=0$, $r_{k-1}=Ax_{k-1}-f$, and τ_{k-1} and v_{k-1} are determined based on the particular algorithm and whose properties are to be determined.

This class of methods is essentially that described in [3]. A well known algorithm of this type is the method of conjugate gradient first discovered by Hestenes and Stiefel [4], which is applicable if A is bounded, selfadjoint and positive definite. They showed that the conjugation algorithm could be regarded as a variant of Gaussian elimination on the operator A. Householder [12] described a class of iterations which is called orthogonalization methods. An orthogonalization method steps along a set of specially generated directions until, after a finite number of steps, a solution is reached.

A closely related approach for solving the system of linear equations is the well known steepest descent method [10]. The rate of convergence of this method is not worse than that of a geometric progression with common ratio ρ given by:

$$\rho = \frac{(\delta - \gamma)}{(\delta + \gamma)}$$

However, when ρ is close to unity, the convergence is very slow.

In this paper, we propose a new gradient method based on the steepest descent. An acceleration scheme is used to improve the rate of convergence of the new method. Numerical results show that the new method is far superior than the original steepest descent and outperform the conjugate gradient method.

2. IMPROVED GRADIENT METHOD

We consider the case where H is a real Hilbert space. The solution of the system (1) is equivalent to minimizing the functional [7]:

$$F(x) = (Ax,x) - 2\,(f,x) \tag{3}$$

Starting from an initial guess $x_o \in H$, we construct a sequence of points x_k from the expression

$$x_k = x_{k-1} + \tau_k d_k + w_k, \quad k = 1, 2, ... \tag{4}$$

where

$$d_k = f - Ax_{k-1}, \quad w_k = \sum_{i=1}^{n} a_i^{\,k} \phi_i \tag{5}$$

where $\{\phi_i\}_{i=1,2,...,n}$ is a system of linearly independent elements. The unknown parameters τ_k and a_i^k are obtained from the minimization of the functional

$$F(x_k) = (Ax_k,x_k) - 2(f,x_k) \tag{6}$$

This is accomplished by obtaining the solution of the system of equations:

$$\frac{\partial F(x_k)}{\partial a_i^k} = 0, \quad \frac{\partial F(x_k)}{\partial \tau_k} = 0, \quad k = 1,2,..., \; i = 1,2,...,n \tag{7}$$

From equations (4) through (7), we can obtain, via a simple transformation, the system of equations:

$$(Aw_k,\phi_i) + \tau_k(Ad_k,\phi_i) = (d_k,\phi_i), \; i = 1, 2, ..., n \tag{8-a}$$

$$(Aw_k,d_k) + \tau_k(Ad_k,d_k) = (d_k,d_k). \tag{8-b}$$

Using the properties given in equation (2), this system is uniquely solvable for any n, and hence the approximation x_k given by equation (4) is unique.

It is noted that when $w_k = 0$, the method degenerates into the original steepest descent method [10]. By introducing the correction w_k, the aim is to accelerate the convergence; however, extra computations will be required at each iteration.

Consider a correction w_k in the form:

$$w_k = y_k - \tau z_k = \sum_{i=1}^{n} (b_i^k - \tau c_i^k) = (Ad_k, \phi_i), \; i = 1,2,...,n \tag{9}$$

where the parameters b_i^k and c_i^k are found from the systems of equations

$$(Ay_k,\phi_i) = (d_k,\phi_i), \; (Az_k,\phi_i) = (Ad_k,\phi_i), \; i=1,2,...,n \tag{10}$$

If we substitute equation (9) into the system of equations (8), we can easily see that the first n equations of system (8) become identities, while the last becomes:

$$\tau_k(Ad_k,d_k-z_k) = (d_k,d_k-Ay_k) \tag{11}$$

Hence the solution of (8) amounts to solving the two systems of linear equations (10) with the same system matrix, and equation (11) for τ_k.

We notice that, after the first iteration, the first system (10) has the trivial solution, i.e., $y_k=0$, $k=2,3,...$, since by virtue of (3) and (4) we obtain:

$$d_{k+1} = f-Ax_k = d_k-\tau_k Ad_k-Aw_k$$

Moreover, as is obvious from the system (8)

$$(d_{k+1},\phi_i) = 0, \; i=1,2,...,n, \; k=1,2,... \tag{12}$$

Assuming the initial approximation is given by:

$$x_0 = \sum_{i=1}^{n} b_i\phi_i, \quad (Ax_0,\phi_i) = (f,\phi_i) \quad i=1,2,...,n \tag{13}$$

then, $y_k=0$ for all iterations since

$$(d_i,\phi_i) = 0, \; i = 1, 2, ..., n \tag{14}$$

From this logic, the algorithm becomes,

$$x_k = x_{k-1} + \tau_k u_k, \quad u_k = d_k - z_k \tag{15-a}$$

$$d_k = f-Ax_{k-1}, \; z_k = \sum_{i=1}^{n} c_i^k\phi_i, \; k = 1, 2, ... \tag{15-b}$$

where the parameters c_i^k and H_k are uniquely given by the system:

$$(Az_k,\phi_i) = (Ad_k,\phi_i), \; i = 1, 2, ..., n \tag{16-a}$$

$$\tau(Ad_k,u_k) = (d_k,d_k), \; k = 1, 2, ... \tag{16-b}$$

3. RATE OF CONVERGENCE

To prove the convergence of the proposed algorithm, we introduce an operator Z, mapping H into itself and given by:

$$Zg = g + h, \; g \in H \tag{17}$$

where $h \in H_n \subset H$ is the solution of the equation:

$$PA(g + h) = \theta \tag{18}$$

here, P is the projector onto the subspace H_n, generated by the system of elements $\phi_1, ..., \phi_n$.

From equations (17) and (18), it follows that the operator Z is linear and has the properties:

$$PAZ = 0,\ Z^2 = Z,\ QZ = Q,\ ZQ = Z, \tag{19}$$

where Q is the projector onto the subspace $H \ominus H_n$. Since the system described by equation (1) has a unique solution $x^* \subset H$ for any $f \in H$, and since $PAx^* = Pf$, then, when $g = x^*$, in equation (18), we have $h = -x_o$, where x_o is the initial approximation (13). Hence, we have:

$$x^* = x_o + Zx^* \tag{20}$$

Consider the equation:

$$WV = g \tag{21}$$

in which $v \in H \ominus H_n$ and

$$W = AZ,\ g = f - Ax_o \in H \ominus H_n \tag{22}$$

Recalling properties (19), and the expressions (17) and (18), we can easily see that the operator W acts in the subspace $H \ominus H_n$ and is selfadjoint. Hence, there exist constants σ, η satisfying the inequality

$$\gamma \le \sigma \le \eta \le \delta \tag{23}$$

and such that

$$\sigma\|v\|^2 \le (Wv,v) \le \eta\|v\|^2 \quad \forall\, v \in H \ominus H_n \tag{24}$$

From the preceding discussion, it is easily seen that the systems (1) and (21) are equivalent and their solutions $x^* \in H$, $v^* \in H \ominus H_n$ are related by:

$$x^* = x_o + Zv^*,\ v^* = Qx^* \tag{25}$$

Then, it follows that the problem of minimizing the functional (3) in the Hilbert space H is equivalent to that of minimizing the functional

$$\Phi(v) = (Wv,v) - 2(g,v) \tag{26}$$

in the subspace $H \ominus H_n$.

Theorem:

If the operator A in (1) is bounded, selfadjoint and positive definite, then the proposed gradient method described by (15) and (16), converges. Moreover, the rate of convergence is given by,

$$\|x^* - x_k\| \le \frac{1}{\sqrt{\gamma\delta}}\, q^k \|f - Ax_o\| \tag{27}$$

where $q = (\eta-\sigma)/(\eta+\sigma)$.

Proof:

From the relation,

$$x_k = x_o + Zv_k \tag{28}$$

the sequence $\{x_k\}$, constructed by (15) and (16) is connected with the minimizing sequence $\{v_k\}$ of functional (26), found by the method of steepest descent ($v_o = 0$).

From equations (25), (28), (22) and (19), we have

$$(Ax_k - Ax^*,\ x_k - x^*) = (Wv_k - Wv^*,\ v_k - v^*) \tag{29}$$

Moreover, from the method of steepest descent, we have

$$(Wv^* - Wv_k,\ v^* - v_k) \le q^{2k}(Wv^*, v^*) \tag{30}$$

Apart from the apriori estimate (27), which follows from (29), (30), (19) and (21), we have the posteriori estimate

$$\|x^* - x_k\| \le \frac{1}{\sqrt{\gamma\delta}} \|f - Ax_k\|$$

which is obtained from the same relations and the equation

$$f - Ax_k = Wv^* - Wv_k$$

Notice that, from equation (23), we have

$$q = (\eta-\sigma)/(\eta+\sigma) \le (\delta-\gamma)/(\delta+\gamma) = \rho$$

which emphasizes that the rate of convergence of the new gradient method is not worse than that of the method of steepest descent. Numerical results show that in practice the new method is far superior than the original steepest descent. Moreover, the new method outperforms several conjugate gradient methods for solving systems of linear equations. This is illustrated in section 5.

4. COMPUTATIONAL SCHEME

In the previous two sections, we presented the new gradient method. This method may be used to solve systems of linear algebraic equations as well as linear integral equations and differential equations. The computational scheme has some specific features that depend on the class of problems to be solved. In this section we give the scheme suitable for the solution of linear integral equations. Other schemes can by gathered very easily. The major steps are given below.

Step 1. Specify a system of linearly independent functions $\{\phi_i(\xi)\}_{i=1,2,...,n}$ and compute their values at the points $\{\xi_i\}_{i=1,2,...,m}$, $\phi_i = \{\phi_1^{\,i}, ..., \phi_m^{\,i}\}$.

Step 2. Construct the matrix

$$\Lambda = \{A\phi_i, \phi_j\}_{i,j=1,2,...,n},$$

$$\lambda_{ij} = \sum_{l=1}^{m}\sum_{p=1}^{m} \rho_l a_{lp} \rho_p \phi_l^{\,i} \phi_p^{\,i}, \quad i,j = 1,2,...,n$$

where $\rho = \{\rho_i\}_{i=1,2,...,m}$ are the coefficients of the chosen quadrature formula, and $\{a_{lp}\}_{l,p=1,2,...,m}$ are the values of the kernel of the integral operator A at the sampling points.

Step 3. Invert the matrix Λ and obtain $\{\lambda_{ij}^{-1}\}_{i,j=1,2,...,n}$.

Step 4. Find the initial approximation

$$x_p^o = \sum_{i=1}^{n} \alpha_i^o \phi_p^{\,i}, \quad p = 1, 2, ..., n \text{ where}$$

$$\alpha_i^o = \sum_{j=1}^{n} \lambda_{ij}^{-1} \sum_{p=1}^{m} \rho_p f_p \phi_p^{\,j}, \quad i,j = 1, 2, ..., n.$$

Step 5. Evaluate the following quantities

$$d_l^k = f_l - \sum_{p=1}^{m} \rho_p a_{lp} x_p^{k-1}, \quad s_l^k = \sum_{p=1}^{m} \rho_p a_{lp} d_p^k, \quad l = 1,2,...,m$$

$$z_p^k = \sum_{i=1}^{n} \phi_p^{\,i} \sum_{j=1}^{n} \lambda_{ij} \sum_{l=1}^{m} \rho_l s_l^k \phi_l^{\,j}, \quad p = 1,2,...,m$$

$$\tau_k = \left[\sum_{p=1}^{m} \rho_p (d_p^k - z_p^k)\right]^{-1} \sum_{p=1}^{m} \rho_p d_p^k d_p^k,$$

$$x_p^k = x_p^{k-1} + \tau_k (d_p^k - z_p^k), \quad p = 1,2,...,m, \; k = 1,2,...$$

5. NUMERICAL RESULTS

In this section, we illustrate by some examples the effectiveness of the proposed gradient method. A comparison with the steepest descent method [10] as well as the conjugate gradient method [9] is given. The computations were carried out on a PC-AT383 microcomputer with a processor running at 33MHz, and equipped with a 80387 co-processor for speeding up numerical operations. The notation in the tables below is as follows: k is the number of iterations to obtain the approximate solution with the given accuracy ε; t is the time taken to solve the problem in seconds, (if the time is put in parentheses, this means that computations stopped at the time indicated.

Example 1: The integral equation

$$\int_{-1}^{1} \frac{u(\xi)}{1 + (x - \xi)^2} \, d\xi = f(x)$$

where

$$f(x) = \ln\left(\frac{4 + x^2}{(1 + x^2)^2}\right) + 2\tan^{-1}\left(\frac{2x}{x^4 + x^2 + 2}\right)$$

has the exact solution $u(x) = 2|x|$.

The functions $\{\phi_i(x)\}_{i=1,2,...,5}$ are taken to be Chebyshev polynomials. This problem was solved using the new proposed gradient method, the steepest descent method and the conjugate gradient method. Simpson's rule was used with m base points, $\varepsilon = 10^{-6}$. The discrete norm of the error vector $\|e\|$ was used to estimate the results. Table 1 shows the results of the computation.

Example 2. To find the solution of the boundary value problem

$$x'' - 25\,x + 20\,y = 0, \quad x(0) = x(1) = 0.$$

Table 1. Results of computations for example 1.

Method	m=21			m=41			m=81		
	‖e‖	k	CPU time	‖e‖	k	CPU time	‖e‖	k	CPU time
proposed method	0.02694	3	3.7	0.02417	3	7.1	0.02181	3	18.2
steepest descent	0.03548	4699	349.7	0.03206	4703	1275.8	0.03075	4791	(3600)
conjugate gradient	0.03111	44	40.3	0.03005	43	98.4	0.02889	46	201.4

Table 2. Results of computations for example 2.

Method	m	n	k	t	‖e‖
proposed method	25	4	31	2.5	0.001464
	50	7	95	13.5	0.000519
	100	11	167	78.6	0.000184
steepest descent	25	-	404	35.6	0.001473
	50	-	1638	302.7	0.000532
	100	-	6600	2610.3	0.000203
conjugate gradient	25	-	54	4.2	0.001465
	50	-	142	21.3	0.000522
	100	-	710	142.4	0.000187

Table 3. Results of the computations for example 3.

Method	k	t	x_1	x_8	x_{16}	x_{24}	x_{32}	x_{40}
exact solution	-	-	23.40000	19.20000	14.40000	9.60000	4.80000	0.20000
proposed method	48	78	23.39999	19.19999	14.39999	9.60001	4.79999	0.20018
conjugate gradient	113	221	23.40001	19.19999	14.40002	9.59999	4.80000	0.20022
steepest descent	2400	(2700)	23.39997	19.20006	14.40000	9.60004	4.80000	0.20033

Using the finite difference method, we obtain the system of linear equations:

$$\frac{x_{i-1} - 2x_i + x_{i+1}}{h^2} - 25x_i\, 20y_i = 0,\ i = 1,2,\dots,m-1$$

$$x_o = x_m = 0, \quad h = \frac{m-1}{m}.$$

The vector ϕ_i was taken to be

$$\phi_i = \{\phi_j^i\}_{j=1,2,\dots,m-1,\ i=1,2,\dots,n}$$

$$\phi_j^i = 1, \text{ if } j=(i\text{-}1)l+1,\dots,il,\quad l=(m\text{-}1)/m$$

$$= 0 \text{ otherwise.}$$

The results are summarized in Table 2.

Example 3. Consider the system of linear equations

$$\sum_{j=1}^{m-1} G_{ij} x_j = f_i, \quad i = 1,2,\ldots,m-1$$

where

$$G_{ij} = i(m-j),\ i \leq j,\ f_i = m^3 - |2i-m|^3,\ m = 80,\ i = 1,2,\ldots,m-1$$

$$= j(m-i),\ i \geq j.$$

The exact solution is $x_i = -(f_{i+1} - 2f_i + f_{i-1})/m$. Table 3 shows the results of solving this system. The accuracy is $\varepsilon = 10^{-1}$. The functions ϕ_i's, $i = 1,2,\ldots,15$ are Chebyshev polynomials. We notice here the symmetry of the solution with respect to x_{40}.

6. CONCLUSION

By introducing a correction element to the steepest descent type of method for solving systems of linear equations, the convergence was accelerated dramatically. It has been shown that the proposed gradient method is exceptionally superior in terms of numerical stability. A proof for convergence of the method has been given. The method is applicable to solve linear integral equations and differential equations. The proposed acceleration scheme can be extended to conjugate gradient algorithms. Similar ideas are now under investigation to accelerate the convergence of Quasi-Newton methods for minimizing nonlinear functionals.

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