# THE PERIODIC SOLUTION OF CERTAIN HIGHLY NONLINEAR OSCILLATORS

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ABSTRACT

The harmonic balance-perturbation method given by van Dooren [23] for the equation $\ddot{X} = f(x, t)$, $(. = d/dt)$ is modified to suit the equations $\ddot{X} = f(x, \dot{x}, t)$. The method is illustrated by considering a specific example previously studied in [24, 25]. Comparison of the present results and previous ones reveals the effectiveness of the suggested technique.

## 1. INTRODUCTION

In a series of trial-blazing papers Urabe [1-4] introduced a complete criterion for the numerical and the theoretical study of periodic solutions to certain periodic non-linear ordinary differential equations. His method made use of a high order Galerkins approximation together with the Newton's iterative procedure for the solution of the resulting non-linear algebraic equations. The broad applicability of Urabe's results especially to applications in mechanical vibrations such as harmonic and subharmonic oscillations and combinations tones, has been disseminated by Bouc [5, 6], Shinohara [7-9], Defilippi and Latil [10], and by Van Dooren [11-17].

Since it is well known that the Fourier series representation of the solution to various problems of practical importance contains a dominant part, Fontenot and Burrus [18, 19], and Van Dooren [20 - 22] introduced a modification of Urabe's Galerkin procedure by taking into account the advantages of this property. These methods are essentially based on an appropriate combination of the harmonic balance method and the perturbation method. Although these analytical methods cannot fully complete with Urabe's complete numerical method, very good results are obtained. The main advantages of these analytical methods are: (1) the determining equations in the subsequent approximations are solved by a very simple procedure; (2) the order of smallness for the coefficients of the Galerkins approximations is predicted; and (3) the effect of the problem parameters is easily studied.

Van dooren [23] suggested an analytical approximate method based on finite Chebysheve series and a harmonic balance perturbation technique to reduce the difficulty in the algebraic manipulations involved in Urabe's method. His method deals with the highly non-linear ordinary differential equation excluding the first derivatives.

In this present work we modify the method of Van Dooren [23] to suit the highly non-linear ordinary differential equations that contain the first derivatives. Illustrative example is solved and comparisons of the results with others are given.

## 2. THE MODIFIED HARMONIC BALANCE PERTURBATION METHOD

In this section, we describe the application of the method proposed by Van Dooren [23] to suit the highly nonlinear ordinary differential equations that contain first derivatives.

### *2.1. The First Approximation*

Let us consider the general highly nonlinear periodic differential equation of the form

$$\ddot{X} + A\dot{X} + BX = f(X, \dot{X}, t) \tag{1}$$

where the dot over a variable indicates differentiation with respect to time t. The above equation can be written as

$$L(t) = f(X, \dot{X}, t) \tag{1.a}$$

where

$$L(t) = \ddot{X} + A\dot{X} + BX$$

We consider the class of equation (1) when the right hand side is a nonlinear polynomial of X, $\dot{X}$ and furthermore at t. The dependence on the time t is sinusoidal with coefficients that cannot be neglected.

The method suggests that the approximate solution X(t) of equation (1) be represented in the form of the following finite series.

$$X^{m}(t, \varepsilon) = a_o + a_1 \cos t + b_1 \sin t + \varepsilon \sum_{n=2}^{\iota} (a_n \cos nt + b_n \sin nt) \tag{2}$$

Where $\varepsilon$ is an artificial small parameter that its role will be clear in due course. The order of approximation $\iota$ will depend on the balance of higher harmonics upon substitution of equation (2) into equation (1).

From the differentiation of equation (2) with respect to t we get

$$\dot{X}^m(t,\varepsilon) = b_1\cos t - a_1\sin t + \varepsilon \sum_{n=2}^{\iota} n(b_n\cos nt - a_n\sin nt) \quad (3.a)$$

also

$$\ddot{X}^m(t,\varepsilon) = -b_1\sin t - a_1\cos t - \varepsilon \sum_{n=2}^{\iota} n^2(b_n\sin nt + a_n\cos nt) \quad (3.b)$$

When $\varepsilon$ is set equal to zero, the following predominant expressions result:

$$X^m(t,0) = a_o + b_1\sin t + a_1\cos t \quad (4.a)$$

$$\dot{X}^m(t,0) = b_1\cos t - a_1\sin t \quad (4.b)$$

The unknown coefficients in expression of equation (2) are determined by a harmonic balance of the corresponding coefficients of sin nt and cos nt for n = 0, 1, 2, ..., $\iota$.

This balance is supplemented by a perturbation approach. This can seen by applying the above combination to the following equation.

$$L_m(t,\varepsilon) = f(X^m(t,0), \dot{X}^m(t,0), t) \quad (5)$$

This means that in the linear left hand side of equation (1) the full expression, equation (2) is substituted. While in the nonlinear right hand side only the predominant part equations (4.a) and (4.b) are used.

The application of the above technique results in a system of algebraic equation in the unknown coefficients $a_n$, $b_n$, n = 0, 1, 2, ..., $\iota$.

These equations are linear except for the leading coefficients which are determined from a system of non-linear algebraic equation.

This will be explained in the example at the end of this section.

### 2.2. *The Second Approximation*

Following the outlines of [23], we assume that the first approximation can be improved by considering higher approximations. To obtain the second approximation we set

$$X(t) = X^m(t,\varepsilon) + \varepsilon y(t,\varepsilon) \quad (6)$$

where $X^m(t,\varepsilon)$ represents the first approximation obtained above, and $y(t,\varepsilon)$ represents the correction. Substituting equation (6) into equation (1) we get

$$\ddot{X}^m(t,\varepsilon) + A\dot{X}^m(t,\varepsilon) + BX^m(t,\varepsilon) + \varepsilon\ddot{y}(t,\varepsilon) + \varepsilon A\dot{y}(t,\varepsilon) + \varepsilon By(t,\varepsilon) = f(X^m + \varepsilon y, \dot{X}^m + \varepsilon\dot{y}, t) \quad (7)$$

But from equation (6), it is evident that

$$\ddot{X}^m(t,\varepsilon) + A\dot{X}^m(t,\varepsilon) + BX^m(t,\varepsilon) = f(X^m(t,0), \dot{X}^m(t,0), t) \quad (8)$$

Combination of equations (7) and (8) we get

$$\ddot{y}(t,\varepsilon) + A\dot{y}(t,\varepsilon) + By(t,\varepsilon) = \varepsilon^{-1}G(y,\dot{y},t) \quad (9)$$

Where

$$G(y,\dot{y},t) = f(X^m + \varepsilon y, \dot{X}^m + \varepsilon\dot{y}, t) - f(X^m(t,0), \dot{X}^m(t,0), t) \quad (10)$$

To obtain the analytical expression for the correction y(t), we follow exactly the same procedure outlined in the previous section. We propose to have

$$y^m(t,\varepsilon) = C_o + C_1\cos t + d_1\sin t + \varepsilon \sum_{n=2}^{\iota} (C_n\cos nt + d_n\sin nt) \quad (11)$$

with the leading terms

$$\left.\begin{aligned} y^m(t,0) &= C_o + C_1\cos t + d_1\sin t \\ \dot{y}^m(t,0) &= -C_1\sin t + d_1\cos t \end{aligned}\right\} \quad (12)$$

Again the full expression (11) is substituted in the linear part of equation (9) and only the leading (predominant) part is used in the non-linear part of equation (9). The unknown coefficients can be obtained by a harmonic balance of coefficients of sin nt and cos nt for n = 0, 1, 2, ..., $\iota$.

Finally the solution in the second approximation is written as

$$X(t) = (a_o + \varepsilon C_o) + (a_1 + \varepsilon C_1)\cos t + (b_1 + \varepsilon d_1)\sin t + \varepsilon \sum_{n=2}^{\iota} [(a_n + \varepsilon C_n)\cos nt + (b_n + \varepsilon d_n)\sin nt] \quad (13)$$

Higher approximations can be obtained in exactly similar fashion.

## 3. ILLUSTRATIVE EXAMPLE

In this section we consider the following nonlinear problem

$$\ddot{X} + 0.2\dot{X} + X^3 = 0.3\cos t \qquad (14)$$

that previously solved numerically by Hayashi [24] who gave also the results for its solution by the perturbation method and the harmonic balance method. Bernard [25] solved equation (14) using a combination of the orthogonal polynomial series and equivalent linearization method.

In this section, we solve the equation (14) applying the modified harmonic balance-perturbation method in section 2. Using series (2), choosing $\iota = 15$ the determining equations in the first approximation are readily found to be

$$0 = -\left(\frac{a_0^3}{8} + \frac{3}{4}a_0 a_1^2 + \frac{3}{4}b_1^2 a_0\right) \qquad (15)$$

$$-a_1 = -0.2b_1 - \left(-0.3 + \frac{3}{4}a_0^2 a_1 + \frac{3}{4}a_1^3 + \frac{3}{4}b_1^2 a_1\right) \qquad (16)$$

$$-b_1 = -0.2a_1 - \frac{3}{4}b_1(a_0^2 + a_1^2 + b_1^2) \qquad (17)$$

$$-4\varepsilon a_2 = -0.2(2\varepsilon)b_2 - \frac{3}{4}a_0(a_1^2 - b_1^2) \qquad (18)$$

$$-4\varepsilon b_2 = 0.2(2\varepsilon)a_2 - \frac{3}{2}a_0 a_1 b_1 \qquad (19)$$

$$-9\varepsilon a_3 = -0.2(3\varepsilon)b_3 - \frac{a_1}{4}(a_1^2 - 3b_1^2) \qquad (20)$$

$$-9\varepsilon b_3 = 0.2(3\varepsilon)a_3 - \frac{1}{4}b_1(3a_1^2 - b_1^2) \qquad (21)$$

$$-n^2\varepsilon a_n = 0, \; -n^2\varepsilon b_n = 0, \; (n = 4, 5, \ldots) \qquad (22)$$

Equation (15) is satisfied by $a_o = 0$, then equations (16) and (17) are solved numerically using the Newton Raphson iterative method for the unknowns $a_1 = b_1$. From equations (18), (22), the remaining coefficients are obtained. We get in the first approximation the following values.

$a_1 = -0.3099719293 \quad a_3 = -6.729415351 \times 10^{-4}$

$b_1 = 0.06705241561 \quad b_3 = 5.733694702 \times 10^{-4}$

Hence the first order approximation of the solution corresponding to (14) is given by

$$X^{(1)}(t) = a_1\cos t + b_1\sin t + a_3\cos 3t + b_3\sin 3t \qquad (23)$$

The results in the second approximation are briefly reported below:

$$X^{(2)}(t) = (a_1 + \varepsilon C_1)\cos t + (b_1 + \varepsilon d_1)\sin t + (\varepsilon a_3 + \varepsilon^2 C_3)\cos 3t + (\varepsilon b_3 + \varepsilon^2 d_3)\sin 3t; \ldots \qquad (24)$$

$$-\varepsilon C_1 + 0.2\varepsilon d_1 = -\frac{6}{4}a_1 b_1 \varepsilon d_1 - \frac{3}{4}b_1^2 \varepsilon C_1 - \frac{3}{4}a_1^2 \varepsilon a_3 - \frac{3}{2}a_1 b_1 \varepsilon b_3 + \frac{3}{4}b_1^2 a_3 - \frac{9}{4}a_1^2 \varepsilon C_1 \qquad (25)$$

$$-\varepsilon d_1 - 0.2\varepsilon C_1 = -\frac{6}{4}a_1 b_1 \varepsilon C_1 - \frac{3}{4}a_1^2 \varepsilon d_1 - \frac{3}{4}a_1^2 \varepsilon b_3 + \frac{3}{2}a_1 b_1 \varepsilon a_3 - \frac{3}{4}b_1^2 \varepsilon b_3 - \frac{9}{4}b_1^2 \varepsilon d_1 \qquad (26)$$

$$-9\varepsilon^2 C_3 + 0.6\varepsilon^2 d_3 = \frac{6}{4}a_1 b_1 \varepsilon d_1 + \frac{3}{4}b_1^2 \varepsilon C_1 - \frac{3}{2}a_1^2 \varepsilon a_3 - \frac{3}{2}b_1^2 \varepsilon a_3 - \frac{3}{4}a_1^2 \varepsilon C_1 \qquad (27)$$

$$-9\varepsilon^2 d_3 - 0.6\varepsilon^2 C_3 = -\frac{6}{4}a_1 b_1 \varepsilon C_1 - \frac{3}{4}a_1^2 \varepsilon d_1 - \frac{3}{2}a_1^2 \varepsilon b_3 - \frac{3}{2}b_1^2 \varepsilon b_3 + \frac{3}{4}b_1^2 \varepsilon d_1 \qquad (28)$$

$$-25\varepsilon^2 C_5 + \varepsilon^2 d_5 = -\frac{3}{4}a_1^2 \varepsilon a_3 + \frac{3}{2}a_1 b_1 \varepsilon b_3 + \frac{3}{4}b_1^2 \varepsilon a_3 \qquad (29)$$

$$-25\varepsilon^2 d_5 - \varepsilon^2 C_5 = -\frac{3}{4}a_1^2 \varepsilon b_3 - \frac{3}{2}a_1 b_1 \varepsilon a_3 + \frac{3}{4}b_1^2 \varepsilon b_3 \qquad (30)$$

The numerical values of the coefficients are

$C_I = -7.291125353 \times 10^{-5}$
$d_1 = 4.26301733 \times 10^{-5}$
$C_3 = -1.096138017 \times 10^{-5}$
$d_3 = 1.092014233 \times 10^{-5}$
$C_5 = -1.035704292 \times 10^{-6}$
$d_5 = 2.45601629 \times 10^{-6}$

Finally, we obtain the second approximation

$$X^{(2)}(t) = -0.3100448\cos t + 6.7095046\times 10^{-2}\sin t - 6.8390292\times 10^{-4}\cos 3t + 5.842891\times 10^{-4}\sin 3t - 1.0357043\times 10^{-6}\cos 5t + 2.4560163\times 10^{-6}\sin 5t \qquad (31)$$

The process can be easily repeated in order to obtain higher approximations to any desired accuracy.

## 4. CONCLUSION AND RESULTS

In order to appreciate the efficiency of the method outlined in the previous sections, we substitute the solution obtained in the second approximation (31) into the equation (14) and calculate the corresponding error for different values of t then we calculate the average error. It was found that such error is of the order $1.5\times 10^{-6}$. This example has been previously studied numerically by

Hayashi [24] who gave also its solution by the harmonic method and the perturbation method, the example is also studied in [25], these four methods in the second approximation lead to an average error of order $9.9x10^{-4}$, $-2x10^{-3}$, $-2x10^{-3}$ and $8.3x10^{-6}$ respectively. Hence it can be argued that the present work gives a simple procedure for the solution of nonlinear differential equation of the form (1) that admit periodic solutions.

Further more the implementation of this technique to the specific example studied here reveals that the resulting average error is less than the corresponding error due to other methods and at the same order of approximation we have more harmonics.

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