PARAMETRIC REPRESENTATION OF CIRCLES IN CENTRAL AND PARALLEL PROJECTIONS

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ABSTRACT

This paper presents a mathematical analysis of the projections of a circle in both central and parallel projections. The parametric equations of a circle as a space curve are derived. The curves (conics) of projections of this circle in central and parallel projections are represented parametrically by applying the parametric equations of the circle to the general transformation equations which are derived in [1]. As a result, all types of these conics are represented by general parametric equations.

INTRODUCTION

The projection of a circle in central and parallel projections are conic sections. The mathematical representations of these projections (conics) in some spacial cases are presented in [2,3,4]. In [2] all types of these conics (hyperbola, parabola and ellipse) are analysed mathematically in central projection for the special case where the picture plane is vertical. In [3] the case where the conics are only ellipses in central projection is considered. Also, this case (the conics are only ellipses) is treated in [4] for both central and parallel projections.

In this paper the general case is considered, namely, when the picture plane is in general position with respect to the coordinate system of the object (circle). In this case the conics which represent the projections of the circle on this plane are represented parametrically by general equations which are applicable for both central and parallel projections. Moreover, these general parametric equations are proved to be valid for all types of conics.

NOTATIONS

The following notations are referred to Figures (1,2):

O-XYZ	rectangular coordinate system of the space object.
$Q(X_q,Y_q,Z_q)$	coordinates of the centre Q of the circle ψ.
$S(X_s,Y_s,Z_s)$	coordinates of the point of sight S.
o-uv	rectangular coordinate system of the picture plane π.
h	vertical distance between o and XY-plane.
d	principal distance.
γ,γ_o,γ_1	zenith angles of the projecting line, the normal of π and the normal of plane α of ψ.
θ_1	azimuth angles of the normal of the plane α.
ϕ	angle between the normal of π and the projecting line SO.
ϕ_1	angle between α and π.
l,m,n	direction cosines of the projecting line SO.
l_o,m_o,n_o	direction cosines of the normal of π.
l_1,m_1,n_1	direction cosines of the normal of α.

In central projection there are following relationships between (X_s,Y_s,Z_s), (l,m,n) h, d and ϕ:

$$X_s = l\,(nd + h\cos\phi)\,/\,n\cos\phi,$$

$$Y_s = m\,(nd + h\cos\phi)\,/\,n\cos\phi, \text{ and}$$

$$Z_s = n\,(nd + h\cos\phi)\,/\,n\cos\phi.$$

PARAMETRIC REPRESENTATION OF A CIRCLE

In space a circle ψ may be defined by its plane α, radius r and centre Q, Figure (1). Let P be a point on ψ, then its position on α is determined by an angle t measured from a horizontal line q passing through Q and lying in α. Let in Figure (1), $(Q;\xi,\eta)$ be a system of cartesian coordinates lying in α, where the ξ-axis coincides with q.

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Then the coordinates (ξ,η) of point P with respect to this system are $\xi = r \cos t$, $\eta = r \sin t$, Then, the coordinates (X,Y,Z) of P with respect to the system (O;X,Y,Z) are

$$X = \xi \sin \theta_1 - \eta \cos \gamma_1 \cos \theta_1 + X_q$$
$$Y = \xi \cos \theta_1 - \eta \cos \gamma_1 \sin \theta_1 + Y_q$$
$$Z = \eta \sin \gamma_1 + Z_q$$

or

$$\left.\begin{aligned} X &= r(\sin\theta_1\cos t - \cos\gamma_1\cos\theta_1\sin t) + X_q \\ Y &= -r(\cos\theta_1\cos t + \cos\gamma_1\sin\theta_1\sin t) + Y_q \\ Z &= r(\sin\gamma_1\sin t) + Z_q. \end{aligned}\right\} \quad (1)$$

where θ_1 is the azimuth of the normal of α.
Since $l_1 = \cos\theta_1\sin\gamma_1$, $m_1 = \sin\theta_1\sin\gamma_1$ and $n_1 = \cos\gamma_1$, then

$$\left.\begin{aligned} X &= r[m_1\cos t - n_1 l_1 \sin t]/[\sin\gamma_1] + X_q \\ Y &= -r[l_1\cos t + n_1 m_1 \sin t]/[\sin\gamma_1] + Y_q \quad 0 \le t \le 2\pi \\ Z &= r[(1-n_1^2)\sin t]/[\sin\gamma_1] + Z_q \end{aligned}\right\} \quad (2)$$

These equation are a regular parametric representation of a circle lying in the space, where t is the parameter. The plane α of this circle is given by

$$l_1 (X - X_q) + m_1 (Y - Y_q) + n_1 (Z - Z_q) = 0 \quad (3)$$

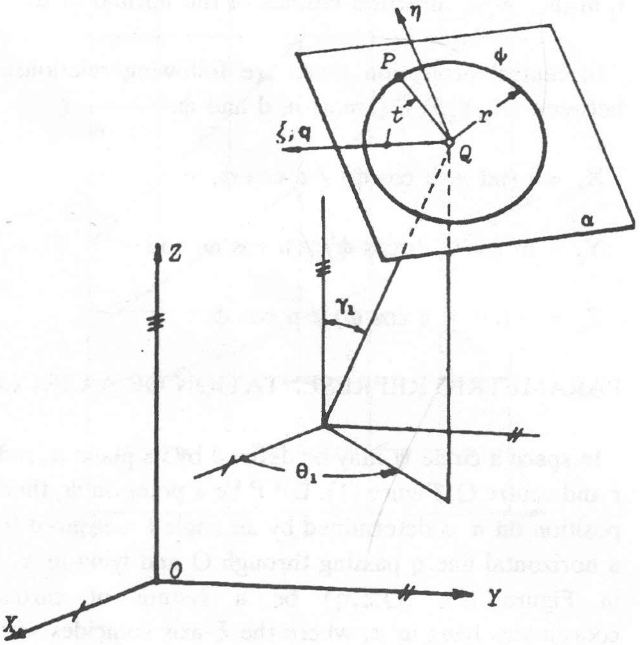

Figure 1.

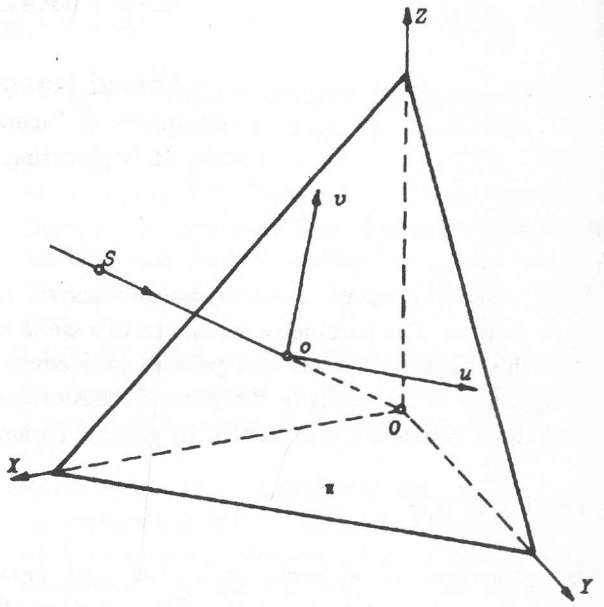

Figure 2.

PARAMETRIC REPRESENTATION OF THE PROJECTION OF A CIRCLE

According to [1], the coordinates (u,v) of the projection P_c of a space point P(X,Y,Z) projected centrally from S onto the picture plane π are given by

$$\left.\begin{aligned} u &= K[a_1X + b_1 Y + c_1 Z]/[a_3 X + b_3 Y + c_3 Z + d_3] \\ v &= K[a_2X + b_2 Y + c_2 Z]/[a_3 X + b_3 Y + c_3 Z + d_3] \end{aligned}\right\} \quad (4)$$

where a_i, b_i, c_i (i=1,2,3), d_3 and K are listed in Tables 1,2 for central and parallel projections respectively.
Substitution from Equations (2) into (4) leads to

$$\left.\begin{aligned} u &= K[rA_1\cos t + rB_1\sin t + C_1]/[rA_3\cos t + rB_3\sin t + C_3] \\ v &= K[rA_2\cos t + rB_2\sin t + C_2]/[rA_3\cos t + rB_3\sin t + C_3] \end{aligned}\right\} \quad (5)$$

where A_i, B_i, C_i (i=1,2,3) are listed in Table 3.
Equation (5) are the parametric equations of the central projection ψ_c of the circle ψ from S into π, where t is the parameter. To prove that Equations (5) represent a conic section, one eliminates the parameter t between these equations, then the result is a second degree equation in u,v. This equation as well known in analytic geometry represents a conic section. In this paper another approach is applied. Let $Lu + Mv + N = 0$ be the equation of any arbitrary line in π, then, by substitution from Equations (5) into this equation one gets

$$L_o \cos t + M_o \sin t + N_o = 0 \tag{6}$$

where

$$L_o = r[K(LA_1+MA_2)+NA_3],$$
$$M_o = r[K(LB_1+MB_2)+NB_3],$$
$$N_o = [K(LC_1+MC_2)+NC_3].$$

Let in Equation (6) $\tan t/2 = s$, $\sin t = 2s/(1+s^2)$, $\cos t = (1-s^2)/(1+s^2)$, hence $(N_o-L_o)s^2+2M_o s+(M_o+L_o) = 0$, this is a quadratic equation in s, having two roots. This means that any arbitrary line in π meets the curve ψ_c in two points. Therefore ψ_c is a conic [5], and in turn Equation (5) are the parametric equations of a conic section.

TYPES OF CONICS IN CENTRAL PROJECTION

As mentioned before, when eliminating the parameter t from Equations (5) the result is a second degree equation, and according to the coefficients of this second degree equation one can deduce the type of the conic ψ_c. In this paper the type of ψ_c will be determined directly from equation (5) as follows. In protective geometry one known that the type of a conic section ψ_c is determined according to the points of intersection of ψ_c with the line at infinity L_∞. If these points are real distinct, real coincident or imaginary, then ψ_c is respectively a hyperbola, a parabola or an ellipse. For this reason the homogeneous coordinates (x,y,z) will be introduced into Equation (5) by letting $u=Kx/z$ and $v = Ky/z$ where

$$\left.\begin{aligned} x &= rA_1 \cos t + rB_1 \sin t + C_1 \\ y &= rA_2 \cos t + rB_2 \sin t + C_2 \\ z &= rA_3 \cos t + rB_3 \sin t + C_3 \end{aligned}\right\} \tag{7}$$

provided that

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \neq 0 \tag{8}$$

It is to be noted that the point (x,y,O) corresponds to a point at infinity and the locus of points at infinity is the line L_∞. Therefore, let in Equation (7) $z = 0$, $\cos t = (1-s^2)/(1+s^2)$ and $\sin t = (2s)/(1+s^2)$. hence

$$(C_3-rA_3)s^2 + 2(rB_3)\, s + (C_3 + rA_3) = 0 \tag{9}$$

This is a quadratic equation its discriminant $\delta = [r^2 B_3^2 - (C_3^2 - r^2A_3^2)]$ shows that the line L_∞ intersects ψ_c in two real distinct, real coincident or imaginary points according respectively to $\delta > 0$, $\delta = 0$ or $\delta < 0$.

Then, when $\Delta \neq 0$, and according to the magnitudes of δ one can decide that the type of ψ_c in central projection are

$$\left.\begin{aligned} &\text{Hyperbola when } \delta>0, \text{ i.e., } r^2>C_3^2/(A_3^2+B_3^2), \\ &\text{Parabola when } \delta=0, \text{ i.e., } r^2=C_3^2/(A_3^2+B_3^2), \\ &\text{Ellipse when } \delta<0, \text{ i.e., } r^2<C_3^2/(A_3^2+B_3^2). \end{aligned}\right\} \tag{10}$$

Now, the case when $\Delta=0$ will be considered. Substitutions from Tables 1 and 3 (for central projection) into Determinant (8) yields

$$\Delta = -n(1-n_o^2)\cos^2\phi[l_1(X_s-X_q)+m_1(Y_s-Y_q)+n_1(Z_s-Z_q)]$$

In this equation the cases when $n=\cos\phi=0$ and $n_o=1$ will be excluded, since these values cause K tends to zero or infinity. Then, $\Delta=0$ when $l_1(X_s-X_q)+m_1(Y_s-Y_q)+n_1(Z_s-Z_q) = 0$. This equation shows that $\Delta=0$ when the plane α which is given by Equation (3), passes through S. In this case it is evident that ψ_c becomes a straight line.

A special case arises when $l_o=l_1$, $m_o=m_1$ and $n_o=n_1$, i.e., when α is parallel to π. In this case by substituting from Table 1 into Table 3 (for central projection) one can prove that $A_1=-B_2$ and $A_2=A_3=0$. Then, Equations (5) become

$$u = K\,[r(A_1/C_3) \cos t + C_1/C_3]$$
$$v = K\,[-r(A_1/C_3) \sin t + C_2/C_3]$$

These equations are the parametric equations of a circle.

TYPES OF CONICS IN PARALLEL PROJECTION

Table 3 shows that in parallel projection $A_3 = B_3 = 0$ and $C_3 = 1$. Then, Equation (9) becomes $s^2 + 1 = 0$. It is obvious that this equation has two imaginary roots. Hence, in parallel projection the line L_∞ intersects ψ_c in two imaginary points. Therefore, ψ_c is an ellipse on the condition that $\Delta \neq 0$. In this case ψ_c is given by

$$\left.\begin{aligned} u &= K\,(rA_1 \cos t + rB_1 \,\text{sint} + C_1) \\ v &= K\,(rA_2\cos t + rB_2 \sin t + C_2) \end{aligned}\right\} \tag{11}$$

To determine the value of Δ substuting from Tables 2 and 3 (for parallel projection) into Determinant (8) one gets $\Delta = -(1-n_o^2)\cos\phi\cos\phi_1$. This equation shows that $\Delta = 0$ when $\phi_1 = \pi/2$ (the case when $n_o = 1$ and $\phi = \pi/2$ are excluded for the same reasons which were mentioned before in the case of central projection). It is evident that when $\phi_1=\pi/2$ the two planes α and π are perpendicular and then ψ_c is a straight line. As mentioned before in central projection, the special case $l_o=l_1$, $m_o=m_1$ and $n_o=n_1$ causes $A_1=-B_2$ and $B_1=A_2=0$. Then, Equation (11) become

$$u = K(rA_1\text{cost}+C_1), \quad v = K(-rA_1\text{sint}+C_2)$$

and this is the parametric equations of a circle.

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These equations are the parametric equations of a circle.

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Table 3 shows that in parallel projection $A_3 = B_3 = 0$ and $C_3 = 1$. Then, Equation (9) becomes $s^2 + 1 = 0$. It is obvious that this equation has two imaginary roots. Hence, in parallel projection the line L_∞ intersects ψ_c in two imaginary points. Therefore, ψ_c is an ellipse on the condition that $\Delta \neq 0$. In this case ψ_c is given by

$$\left.\begin{aligned} u &= K\,(rA_1 \cos t + rB_1 \sin t + C_1) \\ v &= K\,(rA_2 \cos t + rB_2 \sin t + C_2) \end{aligned}\right\} \qquad (11)$$

To determine the value of Δ substuting from Tables 2 and 3 (for parallel projection) into Determinant (8) one gets $\Delta = -(1-n_o^2)\cos\phi\cos\phi_1$. This equation shows that $\Delta = 0$ when $\phi_1 = \pi/2$ (the case when $n_o = 1$ and $\phi = \pi/2$ are excluded for the same reasons which were mentioned before in the case of central projection). It is evident that when $\phi_1=\pi/2$ the two planes α and π are perpendicular and then ψ_c is a straight line. As mentioned before in central projection, the special case $l_o=l_1$, $m_o=m_1$ and $n_o=n_1$ causes $A_1=-B_2$ and $B_1=A_2=0$. Then, Equation (11) become

$$u = K(rA_1\cos t+C_1),\quad v = K(-rA_1\sin t+C_2)$$

and this is the parametric equations of a circle.

CONCLUSIONS

The mathematical analysis which is presented in this paper results in:

(1) Equations (5) are a general parametric representation of all types of conic sections.
(2) Relations (10) can be used to predict the type of the conic which is the projection ψ_c of a circle.
(3) In central projection; the projection ψ_c of ψ is either a hyperbola, a parabola, an ellipse, a circle or a straight line.
(4) In parallel projection; ψ_c is either an ellipse, a circle or a straight line.
(5) The latter two results agree with the known graphical constructions of the projections of a circle in central and parallel projections.
(6) Tables 1, 2 and 3 are used to calculate the coefficients of Equations (5) and (11) for parallel, angular and oblique perspective as well as orthogonal and oblique parallel projections. A computer program can be written to carry out these calculations besides plotting point by point the projection ψ_c in both perspective and parallel projections.

REFERENCES

[1] Ahmed H.M. El Sherif, "A Mathematical Approach to central and Parallel Projections", *Alexandria Engineering Journal*, Vol. 28, No. 4, October 1989, pp 83-95.

[2] Ahmed M.F. Abou Aly, "Perspective on an Inclined Picture Plane", *MSc. Thesis*, Faculty of Eng., Alex., Univ., 1987.

[3] John H. Mauldin, *Perspective Design*, Van Nostrand Reinhold Company, New York, 1985.

[4] Ming H. Land., "An Algorithm and Basic Program for Generating Ellipses in Perspective Projection with the Microcomputer", *Proceeding ICEGDG*, Vienna 1988, Vol. 1, pp 311-316.

[5] E.A. Maxwell, *Plane Protective Geometry*, Cambridge University Press, 1963.

Table 1. Perspective Projection.

	Parallel	Angular	Oblique
a_1	m	m	$(ml^2_o + mm^2_o + nn_o m_o)$
b_1	-l	-l	$-(ll^2_o + lm^2_o + nn_o l_o)$
c_1	0	0	$-n_o (lm_o - ml_o)$
a_2	0	nl_o	nl_o
b_2	n	nm_o	nm_o
c_2	-m	$-\cos\varphi$	$-(ll_o + mm_o)$
a_3	0	nl_o	nl_o
b_3	n	nm_o	nm_o
c_3	0	0	nn_o
d_3	$-(nd + mh)$	$-(nd + h\cos\varphi)$	$-(nd + h\cos\varphi)$
K	nd/m	$nd/\cos\varphi$	$nd/\sin\gamma_o \cos\varphi$

Table 2. Parallel Projection.

	Orthogonal	Oblique		
		Cavalier	Military	Clinographic
a_1	m_o	m	m	$(ml^2_o + mm^2_o + nn_o m_o)$
b_1	$-l_o$	-l	-l	$-(ll^2_o + lm^2_o + nn_o l_o)$
c_1	0	0	0	$-n_o(lm_o - ml_o)$
a_2	$n_o l_o$	n	1	nl_o
b_2	$m_o l_o$	0	-m	nm_o
c_2	$-\sin^2\gamma_o$	-l	$-\tan\gamma \sin\gamma$	$-(ll_o + mm_o)$
a_3	0	0	0	0
b_3	0	0	0	0
c_3	0	0	0	0
d_3	1	1	1	1
K	$-1/\sin\gamma_o$	$-1/l$	$-1/\sin\gamma$	$-1/\sin\gamma_o \cos\varphi$

Table 3.

	Central projection	Parallel Projection
A_1	$(a_1m_1 - b_1l_1)/\sin\gamma_1$	$(a_1m_1 - b_1l_1)/\sin\gamma_1$
B_1	$-[a_1n_1l_1 + b_1n_1m_1 - c_1(1-n^2_1)]/\sin\gamma_1$	$-[a_1n_1l_1 + b_1n_1m_1 - c_1(1-n^2_1)]/\sin\gamma_1$
C_1	$a_1X_q + b_1Y_q + c_1Z_q$	$a_1X_q + b_1Y_q + c_1Z_q$
A_2	$(a_2m_1 - b_2l_1)/\sin\gamma_1$	$(a_2m_1 - b_2l_1)/\sin\gamma_1$
B_2	$-[a_2n_1l_1 + b_2n_1m_1 - c_2(1-n^2_1)]/\sin\gamma_1$	$-[a_2n_1l_1 + b_2n_1m_1 - c_2(1-n^2_1)]/\sin\gamma_1$
C_2	$a_2X_q + b_2Y_q + c_2Z_q$	$a_2X_q + b_2Y_q + c_2Z_q$
A_3	$(a_3m_1 - b_3l_1)/\sin\gamma_1$	0
B_3	$-[a_3n_1l_1 + b_3n_1m_1 - c_3(1-n^2_1)]/\sin\gamma_1$	0
C_3	$a_3X_q + b_3Y_q + c_3Z_q$	1